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A NOTE ON CHEEGER SETS

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ABSTRACT. Starting from the quantitative isoperimetric inequality, we prove a sharp quantitative version of the Cheeger inequality.

A Cheeger set E for an open subset $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is any minimizer of the variational problem

$$c_m(\Omega) = \inf \left\{ \frac{P(E)}{|E|^m} : E \subset \Omega \,, 0 < |E| < \infty \right\} , \tag{1}$$

where $|E|$ is the Lebesgue measure of E, and $P(E)$ denotes its distributional perimeter; see [3, Chapter 3]. In order to avoid trivial situations, it is assumed that Ω has finite measure and that the parameter m satisfies the constraints

$$m > \frac{1}{n'} \,, \qquad \text{where} \quad n' = \frac{n}{n-1} \,. \tag{2}$$

Under these assumptions on Ω and m, it is not difficult to show that Cheeger sets always exist. The study of qualitative properties of Cheeger sets has received particular attention in recent years; see for example [1, 9, 10, 11, 28, 29, 27]. Another interesting question is how to provide lower bounds on $c_m(\Omega)$ in terms of geometric properties of Ω. The basic estimate in this direction is the *Cheeger inequality*,

$$|\Omega|^{m-(1/n')} c_m(\Omega) \geq |B|^{m-(1/n')} c_m(B) \,, \tag{3}$$

where B is the Euclidean unit ball. The bound is sharp, in the sense that equality holds in (3) if and only if $\Omega = x_0 + rB$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$. In this note we strengthen this lower bound in terms of the *Fraenkel asymmetry* of Ω, defined as

$$A(\Omega) = \inf \left\{ \frac{|\Omega \Delta (x_0 + r\,B)|}{|\Omega|} : |r\,B| = |\Omega| \,, x_0 \in \mathbb{R}^n \right\} ,$$

where Δ denotes the symmetric difference between sets. Note that $A(\Omega) = 0$ if and only if Ω is a ball.

Theorem. *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, with $|\Omega| < \infty$, and let m satisfy (2). Then*

$$|\Omega|^{m-(1/n')} c_m(\Omega) \geq |B|^{m-(1/n')} c_m(B) \left\{ 1 + \left(\frac{A(\Omega)}{C(n,m)} \right)^2 \right\} , \tag{4}$$

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where $C(n,m)$ is a constant depending only on n and m.

As will be seen from the proof, a possible value for $C(m,n)$ is given by

$$C(n,m) = \frac{2}{m-(1/n')} + \frac{61\,n^7}{(2-2^{1/n'})^{3/2}}\,.$$

This kind of improvement on a given sharp geometric-functional inequality has been extensively considered in the literature, e.g. concerning the isoperimetric inequality [4, 7, 32, 20, 24, 25, 21, 30, 18, 2], Sobolev inequalities [8, 12, 13, 22, 14], Faber-Krahn and isocapacitary inequalities [31, 26, 5, 6, 23, 18, 19], the Gaussian isoperimetric inequality [15] and the Wulff inequality [16, 17]. In particular, inequality (4) improves an analogous result contained in [23], where the exponent 3 is found in place of the exponent 2; in turn, the exponent 2 is sharp, as we will notice below.

In the proof of the theorem we will use the quantitative isoperimetric inequality

$$P(E) \geq n|B|^{1/n}|E|^{1/n'}\left\{1+\left(\frac{A(E)}{C_0(n)}\right)^2\right\}, \tag{5}$$

where the exponent 2 is sharp; see [21, 17, 30] (here, $C_0(n)$ is a constant depending only on the dimension n, which can be chosen equal to $\frac{61\,n^7}{(2-2^{1/n'})^{3/2}}$; see [17]). The strategy consists in showing that, if E is the Cheeger set of an almost optimal Ω in (3), then, first, $|\Omega\setminus E|$ is correspondingly small and, secondly, E is almost optimal in the isoperimetric inequality (and thus, by (5), it is close to a ball).

To begin with, we notice that $c_m(B) = \frac{P(B)}{|B|^m}$. Indeed, if $F\subset B$ has finite and positive measure, and $r\in(0,1]$ is such that $|r\,B| = |F|$, then $P(F)\geq P(r\,B)$ by the isoperimetric inequality. Therefore,

$$\frac{P(F)}{|F|^m} \geq \frac{P(r\,B)}{|r\,B|^m} = \frac{n|B|r^{n-1}}{|B|^m r^{nm}} \geq n|B|^{1-m} = \frac{P(B)}{|B|^m}\,,$$

where in the last inequality we have used (2) and $r\leq 1$. This ensures that $c_m(B) = \frac{P(B)}{|B|^m}$ and, by the well-known characterization of the equality cases in the isoperimetric inequality, B is the only Cheeger set for B. A similar argument proves in fact the validity of (3). Indeed, assume without loss of generality that $|\Omega| = |B|$ and consider $E\subset\Omega$, with finite and positive measure. If $r\in(0,1]$ is such that $|E| = |r\,B|$, then, again by the isoperimetric inequality,

$$\frac{P(E)}{|E|^m} \geq r^{n-1-nm}\frac{P(B)}{|B|^m} \geq \frac{P(B)}{|B|^m} = c_m(B)\,,$$

and (3) follows.

We notice that inequality (4) is sharp in the decay rate of $A(\Omega)$. Indeed, by (1) we know that $c_m(\Omega)\leq\frac{P(\Omega)}{|\Omega|^m}$, and, from $c_m(B) = \frac{P(B)}{|B|^m} = n|B|^{1-m}$, we immediately get

$$|\Omega|^{m-(1/n')}c_m(\Omega) - |B|^{m-(1/n')}c_m(B) \leq n|B|^{1/n}\left(\frac{P(\Omega)}{n|B|^{1/n}|\Omega|^{1/n'}} - 1\right).$$

Then, the exponent 2 being sharp in (5), it is *a fortiori* sharp in (4).

We can now prove our result.

Proof of the theorem. Without loss of generality, we can assume that $|\Omega| = |B|$. Since we always have $A(\Omega) \le 2$, if $c_m(\Omega) \ge 2\, c_m(B)$, then (4) is verified as soon as we take $C(n,m) \ge 4$. We are therefore going to assume that $c_m(\Omega) \le 2\, c_m(B)$.

Let $E \subset \Omega$ be a Cheeger set for Ω, so that

$$\frac{P(E)}{|E|^m} = c_m(\Omega)\,. \tag{6}$$

Note that, up to a translation of E (and, correspondingly, of Ω), we can also assume that

$$A(E) = \frac{|E \Delta (r\, B)|}{|E|}\,, \tag{7}$$

for some $r \in (0,1]$. We now divide the argument into two steps.

Step 1. We introduce the *isoperimetric deficit* $\delta(E)$ *of* E, defined as

$$\delta(E) = \frac{P(E)}{n|B|^{1/n}|E|^{1/n'}} - 1\,,$$

and prove the following inequalities concerning E:

$$|E| \quad \ge \quad |\Omega| \left(\frac{c_m(B)}{c_m(\Omega)}\right)^{\frac{1}{m-(1/n')}}\,, \tag{8}$$

$$\delta(E) \quad \le \quad \frac{c_m(\Omega) - c_m(B)}{c_m(B)}\,. \tag{9}$$

In order to prove (8), note that, by the isoperimetric inequality,

$$\frac{P(E)}{|E|^m} \ge n|B|^{1/n}|E|^{(1/n')-m}\,.$$

Thus, by (6), recalling that $c_m(B) = n|B|^{1-m}$, we have

$$|E|^{m-(1/n')} \ge \frac{n|B|^{1/n}}{c_m(\Omega)} = |B|^{m-(1/n')}\frac{c_m(B)}{c_m(\Omega)}\,,$$

that is, (8). We now prove (9). On dividing by $n|B|^{1/n}$ the inequality

$$c_m(\Omega) - c_m(B) \ge \frac{P(E)}{|E|^{1/n'}}\,|E|^{(1/n')-m} - n|B|^{1-m}\,,$$

we find that

$$\begin{aligned} \frac{c_m(\Omega) - c_m(B)}{n|B|^{1/n}} \quad &\ge \quad (1+\delta(E))|E|^{(1/n')-m} - |B|^{(1/n')-m} \\ &= \quad \delta(E)|E|^{(1/n')-m} + (|E|^{(1/n')-m} - |B|^{(1/n')-m})\,. \end{aligned}$$

By (2) and $|E| \le |\Omega| = |B|$, the second term on the right-hand side is nonnegative; therefore we have proved that

$$\frac{c_m(\Omega) - c_m(B)}{n|B|^{1/n}} \ge \frac{\delta(E)}{|E|^{m-(1/n')}} \ge \frac{\delta(E)}{|B|^{m-(1/n')}}\,,$$

as desired.

Step 2. Thanks to (7), we can estimate $A(\Omega)$ as follows:

$$\begin{aligned}|\Omega|A(\Omega) &\leq |\Omega\Delta B| \leq |\Omega\Delta E| + |E\Delta(r\,B)| + |B\Delta(r\,B)| \\ &= 2(|\Omega| - |E|) + |E|A(E) \leq 2(|\Omega| - |E|) + |\Omega|A(E)\,.\end{aligned} \tag{10}$$

By (8) we find that

$$|\Omega| - |E| \leq \frac{|\Omega|}{c_m(\Omega)^{\frac{1}{m-(1/n')}}} \left(c_m(\Omega)^{\frac{1}{m-(1/n')}} - c_m(B)^{\frac{1}{m-(1/n')}} \right) .$$

Since $t^a \leq s^a + at^{a-1}(t-s)$ whenever $a \geq 1$ and $0 < s \leq t$, and $t^a \leq s^a + as^{a-1}(t-s)$ whenever $0 < a \leq 1$ and $0 < s \leq t$, and recalling that $c_m(\Omega) \geq c_m(B)$, we get

$$|\Omega| - |E| \leq \frac{|\Omega|}{m - (1/n')} \, \frac{c_m(\Omega) - c_m(B)}{c_m(B)} . \tag{11}$$

On the other hand, by (9) and (5),

$$A(E) \leq C_0(n) \sqrt{\frac{c_m(\Omega) - c_m(B)}{c_m(B)}} , \tag{12}$$

and combining (10), (11) and (12), we find

$$A(\Omega) \leq \frac{2}{m - (1/n')} \left(\frac{c_m(\Omega) - c_m(B)}{c_m(B)} \right) + C_0(n) \sqrt{\frac{c_m(\Omega) - c_m(B)}{c_m(B)}} .$$

Since $c_m(\Omega) \leq 2c_m(B)$, we finally get

$$A(\Omega) \leq C(n,m) \sqrt{\frac{c_m(\Omega) - c_m(B)}{c_m(B)}} ,$$

where $C(n,m)$ is defined as

$$C(n,m) = \frac{2}{m - (1/n')} + C_0(n) .$$

We have thus achieved the proof of the theorem. $\square$

To conclude, let us remark that the above argument may be repeated in the case that the Euclidean perimeter $P(E)$ in (1) is replaced by some anisotropic perimeter

$$P_\psi(E) = \int_{\partial E} \psi(\nu_E(x)) d\mathcal{H}^{n-1}(x)$$

(here E has smooth boundary, ν_E is its outer unit normal vector field, and $\psi : \mathbb{R}^n \to [0, \infty)$ is a convex function with $\psi(t\nu) = t\psi(\nu) > 0$ for every $t > 0$ and $\nu \in \partial B$). The only relevant change consists of replacing (5) with the corresponding quantitative version of the Wulff inequality proved in [17].

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