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BORG-TYPE UNIQUENESS THEOREMS FOR PERIODIC JACOBI OPERATORS WITH MATRIX-VALUED COEFFICIENTS

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Abstract. We give a simple proof of Borg-type uniqueness theorems for periodic Jacobi operators with matrix-valued coefficients.

1. Introduction

Consider a self-adjoint Jacobi operator J acting on $\ell^2(\mathbb{Z})^m, m \geqslant 1$, and given by

(1.1)
$$(Jy)_n = a_n y_{n+1} + b_n y_n + a^*_{n-1} y_{n-1}, \quad n \in \mathbb{Z}, \quad y = (y_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})^m, \quad y_n \in \mathbb{C}^m,$$

where $\det a_n \neq 0$ and $a_n, b_n = b^*_n$, $n \in \mathbb{Z}$ are p-periodic sequences of the complex $m\times m$ matrices. We denote the class of such operators by $\mathfrak{J}_p$. It is well known that the spectrum $\sigma(J)$ of J is absolutely continuous and consists of non-degenerate intervals $[\lambda^+_{n-1}, \lambda^-_n], \lambda^+_{n-1} < \lambda^-_n \leqslant \lambda^+_n, n = 1, ..., N < \infty$. These intervals are separated by the gaps $\gamma_n = (\lambda^-_n, \lambda^+_n), n = 1, \dots, N-1$ with the length > 0. Introduce the fundamental $m \times m$ matrix-valued solutions $\varphi = (\varphi_n(z))_{n\in\mathbb{Z}}, \vartheta = (\vartheta_n(z))_{n\in\mathbb{Z}}$ of the equation

(1.2)
$$a_n y_{n+1} + b_n y_n + a^*_{n-1} y_{n-1} = z y_n, \qquad \varphi_0 \equiv \vartheta_1 \equiv 0,\ \varphi_1 \equiv \vartheta_0 \equiv I_m, \quad (z,n) \in \mathbb{C}\times\mathbb{Z},$$

where I_m is the identity $m \times m$ matrix. We define the $2m \times 2m$ monodromy matrix M_p by

$$M_p(z) = \begin{pmatrix} \vartheta_p(z) & \varphi_p(z) \\ \vartheta_{p+1}(z) & \varphi_{p+1}(z) \end{pmatrix}, \quad z \in \mathbb{C}. \tag{1.3}$$

Let $\tau_1(z), \dots, \tau_{2m}(z)$ be eigenvalues of $M_p(z)$. Recall the well-known fact $\sigma(J) = \bigcup_{j=1}^{2m} \{z \in \mathbb{C} : \ |\tau_j(z)| = 1\}$; see [KKu]. Let J^0 be the unperturbed Jacobi matrix with $a^0_n = I_m$, $b^0_n = 0$ and τ^0_j be the corresponding eigenvalues. Note that $\sigma(J^0) = [-2, 2]$ and $|\tau^0_j(z)| = 1$ for all $(z, j) \in \sigma(J^0)\times\mathbb{N}_{2m}$, where $\mathbb{N}_s = \{1, \dots, s\}$. Introduce

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the following classes of Jacobi operators:

(1.4)
$$\mathrm{Iso}_p(J^0) = \{J \in \mathfrak{J}_p : \ \sigma(J) = [-2,2], \ \ |\tau_j(z)| = 1, \text{ for all } (z,j) \in \sigma(J) \times \mathbb{N}_{2m}\},$$

(1.5)
$$\mathrm{Uni}_p(J^0) = \{J \in \mathfrak{J}_p : \ J \text{ is unitary equivalent to } J^0\},$$

(1.6)
$$\mathfrak{J}_p^0 = \{J \in \mathfrak{J}_p : \ a_n a_n^* = I_m, \ b_n = 0 \text{ for all } n \in \mathbb{Z}\}.$$

We now formulate our first theorem.

Theorem 1.1. *i) The following identities hold true:*

$$\mathrm{Iso}_p(J^0) = \mathrm{Uni}_p(J^0) = \mathfrak{J}_p^0. \tag{1.7}$$

ii) Let $J \in \mathrm{Iso}_p(J^0)$ and let each $a_n, n \in \mathbb{Z}$ have only positive eigenvalues. Then $J = J^0$.

Case ii) includes two cases: (1) $a_n = a_n^* > 0$; (2) a_n are lower triangular matrices with real positive entries on the diagonal and J is a so-called $(2m+1)$-band matrix.

The Borg-type uniqueness Theorem 1.1(ii) was proved in [CGR] for the partial case $a_n > 0$, $n \in \mathbb{Z}$. In their proof the authors obtained the Herglotz formula for the Green function of the Jacobi matrix. Then using corresponding asymptotics at high energy, they determined the trace formulas, which were used to prove the Borg-type theorem for periodic Jacobi matrices. We give a simple proof of Theorem 1.1 based on our trace formula (2.1) and the sharp estimate (2.5). The Borg-type result in the scalar case $m = 1$ was first settled by Flaschka [F]. There is an enormous literature on inverse spectral problems for scalar (i.e., $m = 1$) periodic Jacobi matrices (see [BGGK], [K], [KKu1], [KKu2], [vM], book [T] and the references therein), but very little for matrix-valued periodic Jacobi operators (see [CGR], [KKu] and the references therein). Note that the complete solution of the inverse problem for finite matrix-valued Jacobi operators was obtained recently [BCK].

Recall the standard fact about the direct integrals for the operators with periodic coefficients; see [RS]. The operator J is unitarily equivalent to the operator $\widehat{J} = \int_{[0,2\pi)}^{\oplus} K_p(e^{ix}) \frac{dx}{2\pi}$ acting in $\int_{[0,2\pi)}^{\oplus} \mathcal{H} \frac{dx}{2\pi}$, where $\mathcal{H} = \mathbb{C}^{pm}$ and the $pm \times pm$ matrix $K_p(\tau)$ is given by

$$K_p(\tau) = \begin{pmatrix} b_1 & a_1 & 0 & ... & \tau^{-1} a_p \\ a_1^* & b_2 & a_2 & ... & 0 \\ 0 & a_2^* & b_3 & ... & 0 \\ ... & ... & ... & ... & ... \\ \tau a_p^* & 0 & ... & a_{p-1}^* & b_p \end{pmatrix}, \quad \tau \in \mathbb{S}^1 = \{\tau : |\tau| = 1\}; \tag{1.8}$$

e.g., see [KKu]. Let $(\lambda_n(\tau))_{n=1}^{pm}$ be eigenvalues of $K_p(\tau)$. In [KKu] we obtained the spectral data (finite number of eigenvalues of $K_p(\tau)$), which determine uniquely all multipliers $\tau_j(\cdot), j = 1, \dots, 2m$ and the spectrum of J including the multiplicity. In our main Theorem 1.2 we give the finite spectral data which are necessary and sufficient for $J = J^0$. Let $c = \det \prod_{n=1}^{p} a_n$.

Theorem 1.2. *Let $a_n > 0, b_n = b_n^*$ be real $m \times m$ matrices for all $n \in \mathbb{Z}$.*
i) Let $c = 1$. Then $\sum_{n=1}^{mp} \lambda_n^2(\tau) = 2pm$ for some $\tau \in \mathbb{S}^1$ iff $J = J^0$.
ii) Let $c = 1$ and let $\varkappa_1 \in \mathbb{R}$. Then the eigenvalues have the form $\lambda_s(e^{i\varkappa_1}) = 2\cos\frac{\varkappa_1 + 2\pi(s-1)}{p}$ for all $s \in \mathbb{N}_{mp}$ iff $J = J^0$.

iii) Let $\varkappa_1, \varkappa_2 \in \mathbb{R}$ *and* $\cos\varkappa_1 \neq \cos\varkappa_2$. *Then the eigenvalues have the form* $\lambda_s(e^{i\varkappa_1}) = 2\cos\frac{\varkappa_1+2\pi(s-1)}{p}$ *for all* $s \in \mathbb{N}_{mp}$ *and* $\lambda_n(e^{i\varkappa_2}) = 2\cos\frac{\varkappa_2+2\pi n_1}{p}$ *for all* $n \in \mathbb{N}_m$, *for some* $n_1 \in \mathbb{N}_p$ *and* $e^{i2\varkappa_2} \neq 1$ *(if* $e^{i2\varkappa_2} = 1$, *then additional eigenvalues* $\lambda_{n+m}(e^{i\varkappa_2}) = \lambda_1(e^{i\varkappa_2}) \neq \pm 2$ *for all* $n \in \mathbb{N}_m$*) iff* $J = J^0$.

Remark. 1) Note that the condition $e^{i2\varkappa_2} \neq 1$ in iii) is associated with the unperturbed operator J^0, where the endpoints of the spectrum $\sigma(J^0) = [-2, 2]$ have the multiplicity m, as the zeros of the determinant $D_p(z, \pm 1)$, where $D_p(z, \tau) = \det(M_p(z) - \tau I_{2m})$. Each point from $(-2, 2)$ has multiplicity $2m$ as the zero of the determinant $D_p(z, \tau), \tau \neq \pm 1$.
2) Consider the case $m = 2$ and $\varkappa_1 = 0$, $\varkappa_2 = \pi$. Let periodic eigenvalues $\lambda_s(1) = 2\cos\frac{2\pi(s-1)}{p}$ for all $s \in \mathbb{N}_{2p}$ and let anti-periodic eigenvalues $\lambda_s(-1) = 2\cos\frac{\pi(2n+1)}{p} \neq \pm 2$ for all $s \in \mathbb{N}_4$ and for some $n \in \mathbb{N}_p$. Then we deduce that $J = J^0$.

2. Proof of Theorems 1.1 and 1.2

We need the following results. Recall that $c = \det\prod_1^p a_n$.

Lemma 2.1. *i) For any* $\tau \in \mathbb{S}^1$ *the following identities and estimate are fulfilled:*

(2.1)
$$\operatorname{Tr} K_p(\tau) = \sum_{n=1}^{mp} \lambda_n(\tau) = \sum_{n=1}^{p} \operatorname{Tr} b_n, \quad \operatorname{Tr}(K_p(\tau))^2 = \sum_{n=1}^{mp} \lambda_n^2(\tau) = \sum_{n=1}^{p} \operatorname{Tr}(b_n^2 + 2a_n a_n^*),$$

$$\sum_{n=1}^{pm} \lambda_n^2(\tau) \geqslant 2pm|c|^{\frac{2}{pm}}, \tag{2.2}$$

where the identity (2.2) *holds true iff* $a_n a_n^* = |c|^{\frac{1}{pm}} I_m$ *and* $b_n = 0$ *for all* $n \in \mathbb{Z}$.
ii) Each determinant $D_p(z, \tau) = \det(M_p(z) - \tau I_{2m}), (z, \tau) \in \mathbb{C} \times \mathbb{S}^1$ *satisfies*

$$\begin{aligned} D_p(z, \tau) = \prod_{j=1}^{2m} (\tau_j(z) - \tau) &= c^{-1}(-\tau)^m \prod_{n=1}^{mp} (z - \lambda_n(\tau)) \\ &= c^{-1}(-\tau)^m \det(zI_{pm} - K_p(\tau)). \end{aligned} \tag{2.3}$$

Proof. This lemma was proved in [KKu]. For the reader's sake we recall the proof.

i) The identities (2.1) simply follow from (1.8) (also we use $\operatorname{Tr} a_n a_n^* = \operatorname{Tr} a_n^* a_n$). Let $\varepsilon_{j,n}$, $j \in \mathbb{N}_m$ be eigenvalues of the matrix $a_n a_n^*, n \in \mathbb{N}_p$. Then (2.1) gives

$$\begin{aligned} \sum_{n=1}^{pm} \lambda_n^2(\tau) \geqslant \sum_{n=1}^{p} 2\operatorname{Tr} a_n a_n^* &= 2\sum_{n=1}^{pm}\sum_{j=1}^{m} \varepsilon_{j,n} \geqslant 2pm \prod_{j,n} \varepsilon_{j,n}^{\frac{1}{pm}} \\ &= 2pm(\det\prod_1^p a_n a_n^*)^{\frac{1}{pm}} = 2pm|c|^{\frac{2}{pm}}. \end{aligned}$$

Recall that the identity $\sum_{n=1}^p \sum_{j=1}^m \varepsilon_{j,n} = pm\prod_{j,n} \varepsilon_{j,n}^{\frac{1}{pm}}$ holds true iff each $\varepsilon_{j,n} = (\prod_{j,n} \varepsilon_{j,n})^{\frac{1}{pm}} = |c|^{\frac{2}{pm}}$, $(j, n) \in \mathbb{N}_m \times \mathbb{N}_p$, that is $a_n a_n^* = |c|^{\frac{2}{pm}} I_m$ for all n. Also, $\sum_{n=1}^{pm} \lambda_n^2(\tau) = \sum_{n=1}^p 2\operatorname{Tr} a_n a_n^*$ iff $b_n = 0$ for all n (see (2.1)).

ii) We fix $\tau \in \mathbb{S}^1$. Identity (1.2) yields $y_{n+1} = a_n^{-1}((z - b_n)y_n - a_{n-1}^* y_{n-1})$, which gives $\varphi_{p+1} = a_p^{-1} a_{p-1}^{-1} .. a_1^{-1} z^p + O(z^{p-1})$, $\varphi_p = O(z^{p-1})$, $\vartheta_{p+1} = O(z^{p-1})$,

$\vartheta_p = O(z^{p-1})$ as $z \to +\infty$. The substitution of these asymptotics into (1.3) yields $D_p(z,\tau) = c^{-1}(-\tau)^m z^{pm} + O(z^{pm-1})$ as $z \to \infty$, since eigenvalues of $M^{-1}(z)$ and $M^*(z)$, $z \in \mathbb{R}$ coincide.

Let $K_p(\tau)f = \lambda(\tau)f$ for some eigenvalue $\lambda(\tau)$ and some eigenvector $f = (f_n)_1^p \in \mathbb{C}^{pm}$. Then the definition of the matrix M_p gives $M_p(\lambda(\tau))(f_0, f_1)^\top = \tau(f_0, f_1)^\top$, where $f_0 = \tau^{-1} f_p$. Thus τ is an eigenvalue of $M_p(\lambda(\tau))$ and $\lambda(\tau)$ is a zero of $D_p(\cdot,\tau)$.

Firstly, let all eigenvalues $\lambda_n(\tau), n \in \mathbb{N}_{mp}$ of $K_p(\tau)$ be distinct. Then $\lambda_n(\tau), n \in \mathbb{N}_{mp}$ are zeros of $D_p(\cdot,\tau)$, which yields (2.3) for all $z \in \mathbb{C}$, since the orders of the polynomials $\det(K_p(\tau) - zI_{mp})$ and $D_p(z,\tau)$ coincide.

Secondly, consider the general case. Define the Jacobi operator $\mathcal{J}_t = \mathcal{J} + t\,\mathrm{diag}(r_n)_{n\in\mathbb{Z}}, t \in \mathbb{R}$, where $r_n = n, n \in \mathbb{N}_{mp}$ and let $r_{n+pm} = r_n$ for all $n \in \mathbb{Z}$. Then the corresponding matrix $K_p(\tau,t) = K_p(\tau) + t\,\mathrm{diag}(r_n)_{n=1}^{pm}$, and let $D_p(z,\tau,t)$ be the corresponding determinant. Then all eigenvalues $\lambda_n(\tau,t) = tr_n + o(t), n \in \mathbb{N}_{mp}$ of $K_p(\tau,t)$ are distinct as $t \to \infty$ and in this case (2.3) holds true; i.e.,

$$(-1)^{pm}\det(K_p(\tau,t) - zI_{pm}) = \frac{D_p(z,\tau,t)}{c^{-1}(-\tau)^m} = \prod_{n=1}^{pm}(z - \lambda_n(\tau,t)), \tag{2.4}$$

for all $z \in \mathbb{C}$ and all large t. The functions in (2.4) are polynomials in t. Then the identities (2.4) hold true for all $t \in \mathbb{R}$, which yields (2.3), at $t = 0$. □

Lemma 2.2. *For any $r \geqslant 0$, $s \geqslant 2$, the following identity holds true:*

$$\sup_{x\in\mathcal{P}_s(r)} \sum_1^s x_n^2 = 2s\left(\frac{r}{2}\right)^{\frac{2}{s}}, \tag{2.5}$$

$$\mathcal{P}_s(r) = \left\{(x_n)_1^s \in \mathbb{R}^s :\ x_1 \leqslant \cdots \leqslant x_s,\ \sum_{n=1}^s x_n = 0,\ \prod_{n=1}^s |z - x_n| \leqslant r, \forall z \in [x_1, x_s]\right\}.$$

Proof. Let $\|x\|^2 = \sum x_n^2, x = (x_n)_1^s \in \mathbb{R}^s$. The set $\mathcal{P}_s(r)$ is compact. Then $\sup_{x\in\mathcal{P}_s(r)} \|x\|^2 = \|x^0\|^2$ for some $x^0 = (x_n^0)_1^s \in \mathcal{P}_s(r)$. Introduce a polynomial $p_0(z) = \prod_{n=1}^s (z - x_n^0)$. The polynomial p_0' has only real zeros $\widetilde{x}_n, n \in \mathbb{N}_{s-1}$. We will show that each $|p_0(\widetilde{x}_n)| = r, n \in \mathbb{N}_{s-1}$. Assume that there exist $1 \leqslant n_1 < n_2 \leqslant s$ such that

$$x_{n_1-1}^0 < x_{n_1}^0 \leqslant x_{n_2}^0 < x_{n_2+1}^0, \quad \max_{z\in[x_{n_1}, x_{n_2}]} |p_0(z)| < r, \text{ where } x_0^0 = -\infty, x_{s+1}^0 = +\infty. \tag{2.6}$$

Introduce a polynomial $p_\varepsilon(z) = \prod_{n=1}^s (z - x_n^\varepsilon)$, where a vector $x^\varepsilon = (x_n^\varepsilon)_{n=1}^s \in \mathbb{R}^s$ is given by

$$x_n^\varepsilon = x_n^0,\ n \neq n_1,\ n \neq n_2; \quad \text{and} \quad x_{n_1}^\varepsilon = x_{n_1}^0 - \varepsilon,\ x_{n_2}^\varepsilon = x_{n_2}^0 + \varepsilon, \quad \varepsilon > 0. \tag{2.7}$$

Using $x^0 \in \mathcal{P}_s(r)$ and (2.6), we obtain

$$x_1^\varepsilon \leqslant \cdots \leqslant x_s^\varepsilon, \quad \sum_{n=1}^s x_n^\varepsilon = \sum_{n=1}^s x_n^0 = 0, \quad \max_{z\in[x_{n_1}^\varepsilon, x_{n_2}^\varepsilon]} |p_\varepsilon(z)| \leqslant r, \tag{2.8}$$

for sufficiently small $\varepsilon > 0$. We rewrite p_ε in the form

$$p_\varepsilon(z) = p_0(z)g_\varepsilon(z), \quad g_\varepsilon(z) = \frac{(z - x_{n_1}^\varepsilon)(z - x_{n_2}^\varepsilon)}{(z - x_{n_1}^0)(z - x_{n_2}^0)} = \frac{(z - x_{n_1}^0 + \varepsilon)(z - x_{n_2}^0 - \varepsilon)}{(z - x_{n_1}^0)(z - x_{n_2}^0)}. \tag{2.9}$$

Due to $x^0_{n_1} \leqslant x^0_{n_2}$ and $\varepsilon > 0$ we deduce that $|g_\varepsilon(z)| \leqslant 1$, $z \in \mathbb{R} \setminus [x^\varepsilon_{n_1}, x^\varepsilon_{n_2}]$. Then (2.9) yields $|p_\varepsilon(z)| \leqslant |p_0(z)|$, $z \in \mathbb{R} \setminus [x^\varepsilon_{n_1}, x^\varepsilon_{n_2}]$ and thus (2.8) gives $|p_\varepsilon(z)| \leqslant r$, $z \in [x^\varepsilon_1, x^\varepsilon_s]$, and $x^\varepsilon \in \mathcal{P}_s(r)$ for all sufficiently small $\varepsilon > 0$. We can obtain an estimate

$$\|x^\varepsilon\|^2 = \sum_{n=1}^{s} (x^\varepsilon_n)^2 = \sum_{n=1}^{s} (x^0_n)^2 + 4\varepsilon(x^0_{n_2} - x^0_{n_1}) + 2\varepsilon^2 > \|x^0\|^2, \tag{2.10}$$

since $x^0_{n_1} \leqslant x^0_{n_2}$ and $\varepsilon > 0$. But (2.10) and the condition $x^\varepsilon \in \mathcal{P}_s(r)$ contradict the identity $\sup_{x\in\mathcal{P}_s(r)} \|x\|^2 = \|x^0\|^2$. Then the assumption (2.6) is not true and each $|p_0(\widetilde{x}_n)| = r, n \in \mathbb{N}_{s-1}$. Note that only polynomials $r\mathcal{T}_s(\alpha z + \beta)$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}$ (here $\mathcal{T}_s(\cos z) = \cos sz$ are the Tchebychev polynomials) have this property. Then $p_0(z) = r\mathcal{T}_s(\alpha z + \beta)$ for some $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}$. We take α, β such that $p_0(z) = \prod_{n=1}^{s}(z - x^0_n)$, $\sum_{n=1}^{s} x^0_n = 0$ and then $p_0(z) = r\mathcal{T}_s(zr^{-\frac{1}{s}}2^{\frac{1-s}{s}})$, since (see [AS])

$$\mathcal{T}_s(z) = \frac{1}{2}\sum_{k=0}^{[\frac{s}{2}]}(-1)^k \frac{s}{s-k} C^{s-k}_k (2z)^{s-2k} = 2^{s-1}z^s - 2^{s-3}sz^{s-2} + o(z^{s-2}) \quad \text{as } z \to \infty,$$

where $C^n_m = \frac{n!}{m!(n-m)!}$. Then using the Viette formulas we get

$$\begin{aligned}\mathcal{T}_s(z) &= 2^{s-1}\prod_{n=1}^{s}(z - z_n)\\ &= 2^{s-1}\left(z^s - \xi z^{s-1} + \frac{1}{2}\left(\xi^2 - \eta\right)z^{s-2} + o(z^{s-2})\right) \quad \text{as } z \to \infty,\end{aligned}$$

where z_n, $n \in \mathbb{N}_s$ are zeroes of $\mathcal{T}_s$ and $\xi = \sum_1^s z_n$, $\eta = \sum_1^s z_n^2$. Then we get $\xi = 0$ and $\eta = \frac{s}{2}$. This gives

$$\sup_{x\in\mathcal{P}_s(r)} \|x\|^2 = \|x^0\|^2 = r^{\frac{2}{s}} 2^{\frac{2s-2}{s}} \sum_1^s z_n^2 = r^{\frac{2}{s}} 2^{\frac{2s-2}{s}} \eta = 2s\left(\frac{r}{2}\right)^{\frac{2}{s}}. \qquad \square$$

Proof of Theorem 1.1. i.a) The inclusion $\mathrm{Uni}_p(J^0) \subset \mathrm{Iso}_p(J^0)$ is obvious.

i.b) We prove that $\mathfrak{J}^0_p \subset \mathrm{Uni}(J^0)$. Let $J \in \mathfrak{J}^0_p$. Let the unitary operator $U = \mathrm{diag}_{n\in\mathbb{Z}}\, u_n : \ell^2(\mathbb{Z})^m \to \ell^2(\mathbb{Z})^m$, where $u_0 = I_m$; $u_{n+1} = a^*_n u_n$, $n \geqslant 0$; $u_{n-1} = a_{n-1}u_n$, $n \leqslant 0$. Then $U^*JU = J^0$ and $J \in \mathrm{Uni}_p(J^0)$.

i.c) Now we prove $\mathrm{Iso}_p(J^0) \subset \mathfrak{J}^0_p$, which finishes the proof of the theorem. Let $J^1 \in \mathrm{Iso}_p(J^0)$ and let $\alpha, \beta \in \mathbb{R}$ be such that $|c| = |\prod a_p..a_1| = 1$ and $\sum_1^p \mathrm{Tr}\, b_n = 0$ for the new operator $J = \alpha J^1 + \beta$. Then $|\tau_j(z)| = 1$, $(z, j) \in \sigma(J) \times \mathbb{N}_{2m}$ for J, since $J^1 \in \mathrm{Iso}_p(J^0)$. Consider the p-periodic operator J as a pk-periodic operator for some $k \in \mathbb{N}$. Let $\lambda_{n,pk} \equiv \lambda_{n,pk}(1)$ be eigenvalues of $K_{pk}(1)$ (we write two indexes n and pk, since we use eigenvalues for various p (see after (1.8))) and

$$\lambda_- \equiv \lambda_{1,pk} \leqslant \lambda_{2,pk} \leqslant \cdots \leqslant \lambda_+ \equiv \lambda_{pmk,pk}. \tag{2.11}$$

The eigenvalues $\lambda_\pm$ belong to $\sigma(J)$. Then we have that $[\lambda_-, \lambda_+] \subset \sigma(J)$. Using $|\tau_j(z)| = 1$ for all $(z, j) \in [\lambda_-, \lambda_+] \times \mathbb{N}_{2m}$ and (2.3), we obtain

$$\prod_{n=1}^{pmk} |z - \lambda_{n,pk}| = |D_{pk}(z, 1)| = \prod_{j=1}^{2m} |1 - (\tau_j)^k(z)| \leqslant \prod_{j=1}^{2m}(1 + |\tau_j|^k(z)) \leqslant 4^m, \tag{2.12}$$

for all $(z,j) \in [\lambda_-, \lambda_+] \times \mathbb{N}_{2m}$, since $|\det \prod_{n=1}^{pk} a_n| = |c|^k = 1$ and a sequence a_n is p-periodic. A sequence b_n is p-periodic. Then (2.1) yields

$$\sum_{n=1}^{pmk} \lambda_{n,pk} = \sum_{n=1}^{pk} \operatorname{Tr} b_n = k \sum_{n=1}^{p} \operatorname{Tr} b_n = 0. \tag{2.13}$$

The relations (2.11), (2.12), (2.13) and Lemma 2.2 give that

$$(\lambda_{n,pk})_{n=1}^{pmk} \in \mathcal{P}_{pmk}(4^m), \qquad \sum_{n=1}^{pmk} \lambda_{n,pk}^2 \leqslant 2pmkC_k, \qquad C_k = \left(\frac{4^m}{2}\right)^{\frac{2}{pmk}}. \tag{2.14}$$

Then using (2.1) and $a_n = a_{n+p}, b_n = b_{n+p}$ for all $n \in \mathbb{Z}$, we obtain

$$\sum_{n=1}^{pmk} \lambda_{n,pk}^2 = \sum_{n=1}^{pk} \operatorname{Tr}(b_n^2 + 2a_n a_n^*) = k \sum_{n=1}^{p} \operatorname{Tr}(b_n^2 + 2a_n a_n^*) = k \sum_{n=1}^{pm} \lambda_{n,p}^2 \leqslant k(2pm)C_k,$$

which yields $\sum_{n=1}^{p} \lambda_{n,p}^2 \leqslant 2pm$, since we take $k \to +\infty$. Thus Lemma 2.1 implies $a_n a_n^* = I_m$ and $b_n = 0$ for all $n \in \mathbb{Z}$, i.e. $J \in \mathfrak{J}_p^0$. By a), b) we have $\sigma(J) = [-2,2] = [-2\alpha+\beta, 2\alpha+\beta]$, which yields $\alpha = 1$ and $\beta = 0$, i.e. $J^1 = J \in \mathfrak{J}_p^0$.

ii) Using i), we deduce $J \in \mathfrak{J}_p^0$. The matrices a_n, $n \in \mathbb{Z}$ are unitary and they have only positive eigenvalues. Then all eigenvalues of a_n, $n \in \mathbb{Z}$ are equal to 1, i.e. $a_n = I_m$ for all n. Then $J = J^0$. □

Lemma 2.3. *Let $a_n > 0$, b_n, $n \in \mathbb{Z}$ be real matrices.*
i) Let λ be an eigenvalue of $K_p(\tau)$ and have multiplicity m for some $\tau \in \mathbb{S}^1 \backslash \{-1,1\}$. Then the multipliers have the form $\tau_j(\lambda) = \tau, \tau_{j+m}(\lambda) = \tau^{-1}$ for all $j \in \mathbb{N}_m$.
ii) Let λ be an eigenvalue of $K_p(\tau)$ and have multiplicity $2m$ for some $\tau \in \{-1,1\}$. Then each $\tau_j(\lambda) = \tau$, $j \in \mathbb{N}_{2m}$.

Proof. i) The matrix $K_p(\tau)$ is self-adjoint. Let $K_p(\tau) f^k = \lambda f^k$ for some orthogonal eigenvectors $f^k = (f_n^k)_{n=1}^{mp}$, $k \in \mathbb{N}_m$. If $f_0^k = \tau^{-1} f_p^k$, then the definition of the matrix M_p gives $M_p(\lambda)(f_0^k, f_1^k)^\top = \tau(f_0^k, f_1^k)^\top$. Note that $\widetilde{f}^k = (f_0^k, f_1^k)^\top$ define other components of the vector f^k, since $K_p(\tau)$ has special form; see (1.8). Then the vectors $\widetilde{f}^k$, $k \in \mathbb{N}_m$, are linearly independent vectors since f^k are linearly independent vectors. Then τ has multiplicity at least m. The matrix M_p is symplectic. Then $\frac{1}{\tau}$ is an eigenvalue of M_p and $\frac{1}{\tau}$ has multiplicity at least m. We obtain the first statement, since M_p is a $2m \times 2m$ matrix. The proof of ii) is similar. □

Proof of Theorem 1.2. The statement i) follows from Lemma 2.1.
ii) *Sufficiency.* Recall that if $m = 1$ and $J = J^0$, then $\det(M_p - \tau I_2) = \tau(\tau + \frac{1}{\tau} - 2\mathcal{T}_p(\frac{z}{2}))$, where $\mathcal{T}_p(z) = \cos(p \arccos z)$ is the Tchebychev polynomial. Moreover, the zeros of the polynomial $\mathcal{T}_p(z) - \cos \varkappa_1$ are given by $\lambda_s(e^{i\varkappa_1}) = 2\cos\frac{\varkappa_1 + 2\pi(s-1)}{p}$, $s \in \mathbb{N}_p$.

Thus if $J = J^0, m \geqslant 2$, then the corresponding monodromy operators M_p satisfy $\det(M_p - \tau I_{2m}) = \tau^m \prod_{j=1}^m (\tau + \tau^{-1} - 2\mathcal{T}_p(z2))$, which yields at $\tau = e^{i\varkappa_1}$,

$$\det(M_p - e^{i\varkappa_1} I_{2m}) = e^{im\varkappa_1} 2^m (\cos\varkappa_1 - \mathcal{T}_p(z/2))^m. \tag{2.15}$$

This implies $\lambda_s(e^{i\varkappa_1}) = 2\cos\frac{\varkappa_1 + 2\pi(s-1)}{p}$ for all $s \in \mathbb{N}_{mp}$.

Necessity. Let $\lambda_s(e^{i\varkappa_1}) = 2\cos\frac{\varkappa_1 + 2\pi(s-1)}{p}$ for all $s \in \mathbb{N}_{mp}$. Then a direct calculation implies $\sum_{n=1}^{mp} \lambda_n^2(e^{i\varkappa_1}) = 2pm$, and the statement i) gives $J = J^0$.

iii) *The sufficiency* is proved similarly to the case ii).

Necessity. We consider the case $e^{2i\varkappa_2} \neq 1$; the proof of the case $e^{2i\varkappa_2} = 1$ is similar. Let $\lambda_s(e^{i\varkappa_1}) = 2\cos\frac{\varkappa_1+2\pi(s-1)}{p}$ for all $s \in \mathbb{N}_{mp}$. Using (2.3), (2.15), we obtain

$$\begin{aligned}\prod_{j=1}^{2m}(e^{i\varkappa_1} - \tau_j(z)) &= c^{-1}(-1)^m e^{im\varkappa_1}\prod_{n=1}^{mp}(z - \lambda_n(e^{i\varkappa_1})) \\ &= c^{-1}(-2)^m e^{im\varkappa_1}(\mathcal{T}_p(z/2) - \cos\varkappa_1)^m,\end{aligned} \tag{2.16}$$

where $c = \det\prod_{n=1}^{p} a_n$. The eigenvalue $\lambda_1(e^{i\varkappa_2})$ has multiplicity m. Then using Lemma 2.3 and substituting $z = \lambda_1(e^{i\varkappa_2}) = 2\cos\frac{\varkappa_2+2\pi n_1}{p}$ and two multipliers $e^{\pm i\varkappa_2}$ (given by Lemma 2.3) into (2.16), we obtain $(e^{i\varkappa_1} - e^{i\varkappa_2})^m(e^{i\varkappa_1} - e^{-i\varkappa_2})^m = c^{-1}2^m e^{im\varkappa_1}(\cos\varkappa_1 - \cos\varkappa_2)^m$, which yields $c = 1$. Then the statement ii) gives $J = J^0$. □

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