SUBANALYTIC BLOW-$C^m$ FUNCTIONS

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(Communicated by Varghese Mathai)

Abstract. We describe rings of subanalytic functions which become continuously differentiable after finitely many local blowings-up with analytic centers.

1. Introduction

An arc-analytic function is a function that is analytic along every analytic arc. Arc-analytic functions were introduced in [13] and have been successfully studied in various papers under the additional hypothesis that they are semialgebraic or subanalytic; see for example [2, 5, 14, 15]. For instance every blow-analytic function in the sense of Kuo [11] is subanalytic and arc-analytic. Also a weak version of the inverse is true [2, 17]: every subanalytic and arc-analytic function becomes analytic after composition with some finite sequences of local blowings-up with smooth analytic centers. In general an arc-analytic function is not analytic. However, if we assume that a function is $C^\infty$ along each $C^\infty$ arc, then this function is actually of the class $C^\omega$; this is due to J. Boman [4].

Let $m > 0$ be an integer. In the present paper we investigate subanalytic functions which become $m$ times continuously differentiable after composition with a finite sequence of local blowings-up. In analogy to Kuo's notation, we call such a function a blow-$C^m$ function. For an introduction of the notion and major properties of blowings-up in the subanalytic setting, see [1]. A general introduction to semialgebraic and subanalytic geometry is provided by [18].

Throughout the paper, every manifold is assumed to be of pure dimension, Hausdorff and equipped with a countable basis for its topology. We consider continuously differentiable versions of the concept of arc-analyticity. There are three versions we will discuss: Let $M$ be a real analytic manifold. A function $f : M \to \mathbb{R}$ is called

- (a) a $C^m_\omega$ function if $f$ is $C^m$-smooth along all analytic arcs,
- (b) a $C^m_{m,\text{sub}}$ function if $f$ is $C^m$-smooth along all subanalytic $C^m$ arcs,
- (c) a $C^m_m$ function if $f$ is $C^m$-smooth along all $C^m$ arcs.

Every $C^m_\omega$ function is $C^m_{m,\text{sub}}$-smooth, and every $C^m_{m,\text{sub}}$ function is $C^m_\omega$-smooth. We will show that these inclusions are proper even in the subanalytic category. In general, a $C^m_\omega$ function is not necessarily continuous; see the example of [3]. But...
subanalytic $C^m$ functions are continuous. This enables us to study them with the help of Parusiński’s Rectilinearization Theorem. As an application of this theorem, we prove the following theorem.

**Theorem 1.1.** Let $M$ be a real analytic manifold, and let $f : M \to \mathbb{R}$ be a subanalytic $C^m$ function. Then $f$ is blow-$C^m$.

A $C^m$-singular point of a function $f$ is a point at which $f$ is not $C^m$-smooth. The centers of a blowing-up are always analytic manifolds whose dimension is bounded by $\dim(M) - 2$. By [19] (see also [2, 12]), the set of $C^m$-singular points of a subanalytic function is again subanalytic. Hence, we obtain the following statement.

**Theorem 1.2.** Let $M$ be a real analytic manifold, and let $f : M \to \mathbb{R}$ be a subanalytic arc-$C^m$ function. Then the set $S$ of $C^m$-singular points of $f$ is subanalytic and satisfies

$$\dim(S) \leq \dim(M) - 2.$$  

In Section 2, we briefly recall Parusiński’s Rectilinearization Theorem and some facts about subanalytic Peano differentiable functions which we need to investigate the examples presented in Section 3. In Section 4 we prove Theorem 1.1

2. Basics

We will use Parusiński’s Rectilinearization Theorem; cf. [17, Theorem 2.7].

**Theorem 2.1** (Parusiński). Let $U$ be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a continuous subanalytic function. Then there exist a locally finite collection $\Psi$ of real analytic morphisms $\phi_\alpha : W_\alpha \to \mathbb{R}^n$ such that

(a) each $W_\alpha$ contains a compact subset $K_\alpha$ such that $\bigcup_\alpha \phi_\alpha(K_\alpha)$ is a neighbourhood of $\text{cl}(U)$;

(b) for each $\alpha$ there exist $\tau_1, \ldots, \tau_n \in \mathbb{N}$, such that $\phi_\alpha = \sigma_\alpha \circ \psi_\alpha$, where $\sigma_\alpha$ is the composition of a finite sequence of local blowings-up with analytic center and

$$\psi_\alpha(x) = (\varepsilon_1 x_1^{\tau_1}, \ldots, \varepsilon_n x_n^{\tau_n})$$

for some $\varepsilon_i = \pm 1$;

(c) for any choice of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1,-1\}^n$ and $\psi_\alpha$ as in Theorem 2.1 (b), the composition $f \circ \sigma_\alpha \circ \psi_\alpha$ is analytic.

We will give examples to distinguish the notions of differentiability along curves. This requires the concept of Peano differentiable functions.

**Definition 2.2.** Let $U \subset \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ is called $m$ times Peano differentiable, in short $f \in \mathcal{P}^m(U, \mathbb{R})$, if for every $u \in U$ there is a polynomial $p$ such that

$$f(x) - f(u) = p(x - u) + o(\|x - u\|^m) \text{ as } x \to u.$$  

By Taylor’s Theorem, every $C^m$ function is $m$ times Peano differentiable. The sets of $C^m$-singular points of $\mathcal{P}^m$ functions have been studied in [9] (see also [7]) for the o-minimal context. Every continuous subanalytic function is locally definable in the o-minimal structure $\mathbb{R}_{an}$ consisting of all globally subanalytic sets; cf. [6] page 506. A subanalytic set $A \subset \mathbb{R}^n$ is called globally subanalytic if $\tau_n(A)$ is subanalytic where

$$\tau_n(x) = \left( \frac{x_1}{\sqrt{1 + x_1^2}}, \ldots, \frac{x_n}{\sqrt{1 + x_n^2}} \right);$$
see for example [6] page 506]. The theorems in [9] are stated for o-minimal expansions of real closed fields. However, for the subanalytic category the result of our interest (cf. [9, Theorem 1.1]) reads as follows:

**Theorem 2.3.** Let \( U \subset \mathbb{R}^n \) be open, and let \( f : U \to \mathbb{R} \) be a subanalytic \( \mathcal{P}^m \) function. Then the set \( S \) of \( \mathcal{C}^m \)-singular points is subanalytic and

\[
\dim(S) \leq n - 2.
\]

In particular unary subanalytic \( \mathcal{P}^m \) functions are \( \mathcal{C}^m \)-smooth. Note that every subanalytic \( \mathcal{P}^m \) function is actually \( \mathcal{C}^m_{\omega} \). Hence from Theorem [11] follows:

**Corollary 2.4.** Every subanalytic \( \mathcal{P}^m \) function is blow-\( \mathcal{C}^m \).

3. Examples

Next we discuss the announced examples.

**Example 3.1.** Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a semialgebraic \( \mathcal{C}^m \) function that vanishes outside of \((0, 2)\) and for which \( \varphi(1) = 1 \). Let

\[
A := \{(x, y) \in \mathbb{R}^2 : x > 0, \ x^{m+1/2} < y < 3x^{m+1/2}\}.
\]

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be the function

\[
f(x, y) = \begin{cases} x^{m/2+1/8}\varphi \left( \frac{y}{x^{m+1/2}} - 1 \right), & \text{if } (x, y) \in A, \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( f \) is a semialgebraic \( \mathcal{C}^m_{\omega} \) function that is not \( \mathcal{C}^m_{m, \text{sub}} \)-smooth.

**Proof.** First we prove that \( f \) is \( \mathcal{C}^m_{\omega} \)-smooth.

Outside of \((0, 0)\) the function \( f \) is \( \mathcal{C}^m \)-smooth. It remains to study the origin. Let

\[
\phi = (\phi_1, \phi_2) : (-1, 1) \to \mathbb{R}^2
\]

be an analytic curve with \( \phi(0) = (0, 0) \).

Assume that \( \phi'(0) = 0 \), and that \( \phi_1(t) > 0 \) for \( t > 0 \) small enough. Then

\[
\phi_1(t) = O \left( t^2 \right) \text{ as } t \to 0
\]

so that

\[
f \circ \phi(t) \text{ is } O \left( t^{m+1/4} \right) \text{ as } t \to 0.
\]

Hence \( f \circ \phi \) is \( m \) times Peano differentiable at \( t = 0 \). Note that \( f \circ \phi(t) \) restricted to \((-1/2, 1/2)\) is definable. Thus \( f \circ \phi \) is \( \mathcal{C}^m \)-smooth in some pointed neighbourhood of \( 0 \). By Theorem 2.3 the function \( f \circ \phi \) is \( \mathcal{C}^m \)-smooth.

If \( \phi'(0) \neq 0 \), then we claim that \( f \circ \phi \) is locally zero at \( 0 \). Again we may assume that \( \phi_1(t) > 0 \) for \( t > 0 \) sufficiently small. If \( \phi_2'(0) > 0 \), then the germ of \( \phi \) at \( 0^+ \) lies above \( A \), and if \( \phi_2'(0) \leq 0 \), then the germ lies below \( A \). In both cases, the function \( f \circ \phi(t) = 0 \) for \( t \) sufficiently close to \( 0 \).

Hence \( f \) is a \( \mathcal{C}^m_{\omega} \) function.

To see that \( f \) is not a \( \mathcal{C}^m_{m, \text{sub}} \) function we show that \( f \) is not \( \mathcal{C}^m \)-smooth along the semialgebraic \( \mathcal{C}^m \) curve \( \phi : (-1, 1) \to \mathbb{R}^2 \) given by

\[
\phi(t) := \begin{cases} (t, 2t^{m+1/2}), & \text{if } t > 0, \\ (t, 0), & \text{if } t \leq 0. \end{cases}
\]
The composition \( f \circ \phi(t) = 0 \) for \( t \leq 0 \). But for \( t > 0 \),

\[
f \circ \phi(t) = t^{m/2+1/8},
\]

which cannot be extended to 0 as a \( C^m \) function. \( \square \)

**Remark 3.1.** By the previous example we see that the class of subanalytic \( C^m \)
functions is not closed under compositions. The classes of subanalytic \( C^m \) and \( C^{m,\text{sub}} \)
functions are closed under compositions.

**Example 3.3.** Let the semialgebraic function \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
f(x, y) := \begin{cases} y^{m+1} \varphi \left( \frac{x}{y^{2m^2}} - 2 \right), & y > 0, \\ 0, & y \leq 0, \end{cases}
\]

where \( \varphi : \mathbb{R} \to \mathbb{R} \) is defined by

\[
\varphi(t) := \begin{cases} t (1-t^2)^{m+1}, & t \in (-1, 1), \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( f \) is \( m \) times Peano differentiable. Thus \( f \) is \( C^{m,\text{sub}} \)-smooth. But \( f \) is not a \( C^m \) function.

**Proof.** The function \( f \) is \( C^m \)-smooth outside of the origin. The function \( \varphi \) is bounded, so that

\[
f(x, y) \text{ is } o \left( \| (x, y) \|^{m} \right) \text{ as } (x, y) \to (0, 0).
\]

Hence \( f \) is \( m \) times Peano differentiable, so \( f \) is \( C^{m,\text{sub}} \)-smooth.

Next we present a \( C^m \) curve along which \( f \) is not even \( C^1 \)-smooth. Let \( \phi : (-1, 1) \to \mathbb{R}^2 \) be the curve given by

\[
\phi(t) := \left( 2t^{2m^2} + t^{2m^2+1} \sin \left( t^{-m-1/2} \right), t \right).
\]

It is straightforward to verify that \( \phi \) is a \( C^m \) curve. We note the first derivative of the first component of \( \phi \) for \( t > 0 \):

\[
\phi'_1(t) = 2m^2 t^{2m^2-1} \left( 1 + \sin \left( t^{-m-1/2} \right) \right) - \left( m + \frac{1}{2} \right) t^{2m^2-m-1/2} \cos \left( t^{-m-1/2} \right).
\]

The partial derivative of \( f \) with respect to \( y \) is continuous. Hence it suffices to study \( \partial f / \partial x \). For \( y > 0 \),

\[
\frac{\partial f}{\partial x}(x, y) = y^{-2m^2+m} \varphi' \left( xy^{-2m^2} - 2 \right).
\]

Note that \( \varphi'(0) = 1 \). Hence, combining the equations (3.1) and (3.2) we can write \( (f \circ \phi)'(t) \) for positive \( t \) as follows:

\[
(f \circ \phi)'(t) = \frac{\partial f}{\partial y}(\phi(t)) \varphi'_2(t) + \frac{\partial f}{\partial x}(\phi(t)) \varphi'_1(t)
\]

\[
= \frac{\partial f}{\partial y}(\phi(t)) + \varphi' \left( t \sin \left( t^{-m-1/2} \right) \right) \left( 2m^2 t^{-m-1} \left( 1 + \sin \left( t^{-m-1/2} \right) \right) \right)
\]

\[
+ \varphi' \left( t \sin \left( t^{-m-1/2} \right) \right) \left( - \left( m + \frac{1}{2} \right) t^{-m-1/2} \cos \left( t^{-m-1/2} \right) \right).
\]

The first two summands are bounded, while the third summand is not locally bounded at \( t = 0 \). Thus \( f \circ \phi(t) \) is not continuously differentiable at \( t = 0 \). \( \square \)
4. Proof of the main theorem

We prepare the proof of Theorem 1.1 by the following observation. Let $B_1(0)$ denote the open unit-ball in $\mathbb{R}^n$.

**Lemma 4.1.** Let $f, g : B_1(0) \to \mathbb{R}$ be $C^m$ functions. Assume that the function $f : B_1(0) \to \mathbb{R}$,

$$F(x) := \begin{cases} f(x), & \text{if } x_1 \leq 0, \\ g(x), & \text{if } x_1 > 0, \end{cases}$$

is $C^m$-smooth along every line segment contained in $B_1(0)$. Then $F$ is $C^m$-smooth.

**Proof.** We may assume that $g$ vanishes identically. Then, it remains to prove that for every $\xi \in B_1(0) \cap \{x_1 = 0\}$ and every $\alpha \in \mathbb{N}^n$ with $\alpha_1 + \cdots + \alpha_n \leq m$,

$$D_{\alpha} f(\xi) = 0,$$

because in this case, the Hestenes Lemma (cf. [10], [20]) implies that $F$ is $C^m$-smooth.

We express $D_{\alpha} f(\xi)$ as a linear combination of higher-order directional derivatives. This is possible by [8] Proof of Theorem 1.4. All directional derivatives of $F$ at $\xi$ vanish, as $F = 0$ for $x_1 > 0$. But $f = F = 0$ for $x_1 \leq 0$, so that every directional derivative of $f$ at $\xi$ vanishes. Thus $F$ is $C^m$-smooth. □

**Lemma 4.2.** Any subanalytic $C^m_\omega$ function $f : M \to \mathbb{R}$ is continuous.

**Proof.** Assume that $f$ is not continuous at $a$. Then, by the curve selection (cf. [16], [2], [9] 1.17), there exists an analytic map $\gamma : (-1, 1) \to M$ with $\gamma(0) = a$ and $\gamma(t) \neq a$ for $t \neq 0$ such that $\lim_{t \to 0} f \circ \gamma(t) \neq f(a)$. But $f \circ \gamma(0) = f(a)$ and $f \circ \gamma$ is at least continuous. Thus $f$ must be continuous. □

**Proof of Theorem 1.1.** The problem is local, so that we may assume that $M = U \subset \mathbb{R}^n$ is a neighbourhood of the origin. Since every $C^m_\omega$ function is continuous, we can apply Parusiński’s theorem to $f$ and $U$ and obtain a family $\{\phi_\alpha = \sigma_\alpha \circ \psi_\alpha\}$ which satisfies the conclusion of Theorem 2.1. Let $\sigma = \sigma_\alpha$, and fix the corresponding $r_i \in \mathbb{N}$. Then, for each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in (1, -1]^n$ and $\psi$ as defined in Theorem 2.1 [6], the function $f \circ \sigma \circ \psi$ is analytic. Hence

$$f \circ \sigma \circ \psi(x) = \sum_{\beta} a_\beta \prod_{i=1}^{n} x_i^{\beta_i}.$$ 

On the quadrant $Q_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon_i x_i \geq 0 \text{ for } i = 1, \ldots, n\}$ we have

$$f \circ \sigma(x) = \sum_{\beta} a_\beta \prod_{i=1}^{n} (\varepsilon_i x_i)^{\beta_i/r_i}.$$ 

But $\sigma$ is analytic; hence $f \circ \sigma$ is a $C^m_\omega$ function. Assume that there is a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$ with

$$\frac{\gamma_1}{r_1} + \cdots + \frac{\gamma_n}{r_n} < m$$

such that at least one of the $r_i$ does not divide $\gamma_i$, and $a_\gamma \neq 0$. Then, for generic $c = (c_1, \ldots, c_i, \ldots, c_n)$ with $\varepsilon_j c_j \geq 0$ for $j \neq i$, the function

$$f \circ \sigma(c_1, \ldots, c_{i-1}, \varepsilon_i t, c_{i+1}, \ldots, c_n)$$

...
has the Puiseux expansion with non-zero coefficient at \( t^{\gamma_i/r_i} \). This contradicts the fact that \( f \circ \sigma \) is \( C^m_\omega \)-smooth.

Therefore, the function \( f \circ \sigma \) restricted to \( Q_\varepsilon \) is \( C^m_\omega \)-smooth. By [21], it extends to a \( C^m_\omega \) function \( F_\varepsilon \) defined on some open neighbourhood of \( Q_\varepsilon \). Thus, \( f \circ \sigma \) is the gluing of the \( F_\varepsilon \) restricted to \( Q_\varepsilon \). Recall that \( f \circ \sigma \) is \( C^m_\omega \)-smooth. Lemma 4.1 implies that \( f \circ \sigma \) is \( C^m_\omega \)-smooth in the set

\[
U \setminus \bigcup_{\ell \neq k} \{ x \in \mathbb{R}^n : x_\ell = x_k = 0 \}.
\]

The derivatives of \( f \circ \sigma \) extend continuously to \( U \), so again by [21], the function \( f \circ \sigma \) is \( C^m \)-smooth. \( \square \)

REFERENCES


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