

LÉVY CONSTANTS OF TRANSCENDENTAL NUMBERS

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ABSTRACT. We prove that every $\gamma \geq \log \frac{1+\sqrt{5}}{2}$ is the Lévy constant of a transcendental number; i.e., there exists a transcendental number α such that $\gamma = \lim_{m \rightarrow \infty} \frac{1}{m} \log q_m(\alpha)$, where $q_m(\alpha)$ denotes the denominator of the m th convergent of α .

1. INTRODUCTION

An irrational number is said to have a Lévy constant $\beta(\alpha)$ if the limit $\lim_{m \rightarrow \infty} \frac{1}{m} \log q_m(\alpha)$ exists and its value is $\beta(\alpha)$. Here $q_m(\alpha)$ denotes the denominator of the m th convergent of the regular continued fraction expansion of α . Classic results of A.Ya. Khintchine [10] and P. Lévy [11] say that almost every real number has a Lévy constant and its value is $\beta(\alpha) = \pi^2/(12 \log 2)$ for almost all α .

Lévy constants satisfy $\beta(\alpha) \geq \log((1 + \sqrt{5})/2)$. This follows from $[0, \underbrace{1, \dots, 1}_m] = F_{m-1}/F_m$ (where F_m denotes the m th Fibonacci number) and therefore

$$q_m(\alpha) \geq F_m \sim \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{m+1} \quad \text{as } m \rightarrow \infty.$$

H. Jager and P. Liardet [9] proved that every quadratic irrationality has a Lévy constant. C. Faivre [5] showed that every possible value is attained; i.e., for all $\gamma \geq \log((1 + \sqrt{5})/2)$ there is an irrational α such that $\beta(\alpha) = \gamma$. E.P. Golubeva [6], [7], [8] studied Lévy constants of quadratic irrationalities and their connections with real quadratic fields and binary quadratic forms. In a recent paper, J. Wu [16] proved that the Lévy constants of quadratic irrationalities are dense in $[\log((1 + \sqrt{5})/2), +\infty)$. This implies trivially that the set $\{\beta(\alpha) \mid \alpha \text{ is algebraic, } \alpha \notin \mathbb{Q}, \beta(\alpha) \text{ exists}\}$ is dense in $[\log((1 + \sqrt{5})/2), +\infty)$. It is the purpose of the present paper to prove the following complementary result:

Theorem 1. (i) *For every $\gamma \geq \log((1 + \sqrt{5})/2)$ there exist non-denumerably many, pairwise not equivalent transcendental α such that $\beta(\alpha) = \gamma$.*

(ii) *For every $\gamma \geq \log((1 + \sqrt{5})/2)$ there exist non-denumerably many, pairwise not equivalent U_2 -numbers α such that $\beta(\alpha) = \gamma$.*

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Remarks. 1) Two numbers α_1, α_2 are called equivalent if their continued fraction expansions have shapes $\alpha_1 = [b_0, b_1, \dots, b_k, a_1, a_2, a_3, \dots]$ and $\alpha_2 = [c_0, c_1, \dots, c_\ell, a_1, a_2, a_3, \dots]$. It is easy to see that equivalent numbers have the same Lévy constant (if one of them has one).

2) Obviously part (ii) of the theorem implies part (i). Nevertheless, we will give separate proofs of both parts as it is instructive to see how certain parameters have to be changed when asking for a U_2 -number instead of a transcendental number.

2. CONTINUED FRACTIONS WITH PRESCRIBED LÉVY CONSTANT

An important technical tool in our proof will be continuants. Let a_1, \dots, a_n be positive integers. The continuant $K_n(a_1, \dots, a_n)$ is defined as the determinant

$$K_n(a_1, \dots, a_n) = \begin{vmatrix} a_1 & 1 & & & \\ -1 & a_2 & 1 & & \\ & -1 & a_3 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & a_n \end{vmatrix}.$$

In addition we set $K_0 := 1$ and $K_{-1} := 0$.

Lemma 2. (i) $K_m(a_1, \dots, a_m) = K_m(a_m, \dots, a_1)$.

(ii) For $0 \leq m \leq n$ we have

$$K_n(a_1, \dots, a_n) = K_m(a_1, \dots, a_m)K_{n-m}(a_{m+1}, \dots, a_n) \\ + K_{m-1}(a_1, \dots, a_{m-1})K_{n-m-1}(a_{m+2}, \dots, a_n).$$

(iii) If $\alpha = [a_0, a_1, a_2, \dots]$, then $q_n(\alpha) = K_n(a_1, \dots, a_n)$.

Proof. Part (i) is trivial. Proofs of parts (ii) and (iii) can be found, e.g., in O. Perron's classic textbook [13]. \square

We will drop the index and write $K(a_1, \dots, a_n)$ from now on.

Lemma 3. $\log K(a_1, \dots, a_n, b_m, \dots, b_1) = \log K(a_1, \dots, a_n) + \log K(b_1, \dots, b_m) + O(1)$ with an absolute implied constant.

Proof. Using Lemma 2 we get

$$\log K(a_1, \dots, a_n, b_m, \dots, b_1) \\ = \log \left(K(a_1, \dots, a_n)K(b_1, \dots, b_m) + K(a_1, \dots, a_{n-1})K(b_1, \dots, b_{m-1}) \right) \\ = \log K(a_1, \dots, a_n) + \log K(b_1, \dots, b_m) \\ + \log \left(1 + \frac{K(a_1, \dots, a_{n-1})K(b_1, \dots, b_{m-1})}{K(a_1, \dots, a_n)K(b_1, \dots, b_m)} \right),$$

which implies the assertion. \square

Lemma 4. If $\alpha = [0, \bar{a}] = [0, a, a, a, \dots]$, then

$$\log q_m(\alpha) = m \log[\bar{a}] + O(1) = m \log \frac{a + \sqrt{a^2 + 4}}{2} + O(1).$$

Proof. Using induction on m one sees that

$$q_m(\alpha) = \frac{1}{\sqrt{a^2 + 4}} \left([\bar{a}]^{m+1} - (-[0, \bar{a}])^{m+1} \right)$$

for all $m \geq -1$, which implies the assertion. □

From now on we will use the shorthand notation

$$w_a = \log[\bar{a}] = \log \frac{a + \sqrt{a^2 + 4}}{2}.$$

Lemma 5. *Let $a < b$ be positive integers and $w_a < \gamma < w_b$. Then there exist non-denumerably many, pairwise not equivalent $\alpha = [0, a_1, a_2, a_3, \dots]$ such that $\beta(\alpha) = \gamma$ and $a_i \in \{a, b\}$ for all $i \geq 1$.*

Proof. Let $x \in (0, 1)$ be such that $\gamma = (1 - x)w_a + xw_b$ and let $(\sigma_k)_{k \geq 1}$ be an increasing sequence of positive integers satisfying $\min\{\sigma_1 x, \sigma_1(1 - x)\} \geq 1$, $\sigma_k \ll k$ as $k \rightarrow \infty$ and $\lim_{k \rightarrow \infty} \sigma_k = +\infty$. These conditions imply $\lim_{k \rightarrow \infty} k/(\sigma_1 + \dots + \sigma_k) = 0$.

Set

$$\alpha = \left[0, \bar{a}^{\lfloor (1-x)\sigma_1 \rfloor}, \bar{b}^{\lfloor x\sigma_1 \rfloor}, \bar{a}^{\lfloor (1-x)\sigma_2 \rfloor}, \bar{b}^{\lfloor x\sigma_2 \rfloor}, \dots \right],$$

where the notation \bar{a}^τ means that the partial quotient a is repeated τ times. Let

$$\sum_{i=1}^k (\lfloor (1-x)\sigma_i \rfloor + \lfloor x\sigma_i \rfloor) < m \leq \sum_{i=1}^{k+1} (\lfloor (1-x)\sigma_i \rfloor + \lfloor x\sigma_i \rfloor).$$

Then

$$(a_1, \dots, a_m) = \left(\bar{a}^{\lfloor (1-x)\sigma_1 \rfloor}, \bar{b}^{\lfloor x\sigma_1 \rfloor}, \dots, \bar{a}^{\lfloor (1-x)\sigma_k \rfloor}, \bar{b}^{\lfloor x\sigma_k \rfloor}, \bar{a}^{\tau_a}, \bar{b}^{\tau_b} \right)$$

for some τ_a, τ_b satisfying either $1 \leq \tau_a \leq \lfloor (1-x)\sigma_{k+1} \rfloor$ and $\tau_b = 0$ or $\tau_a = \lfloor (1-x)\sigma_{k+1} \rfloor$ and $1 \leq \tau_b \leq \lfloor x\sigma_{k+1} \rfloor$. Using Lemmata 3 and 4 we get

$$\begin{aligned} \log q_m(\alpha) &= \log K \left(\bar{a}^{\lfloor (1-x)\sigma_1 \rfloor}, \bar{b}^{\lfloor x\sigma_1 \rfloor}, \dots, \bar{a}^{\lfloor (1-x)\sigma_k \rfloor}, \bar{b}^{\lfloor x\sigma_k \rfloor}, \bar{a}^{\tau_a}, \bar{b}^{\tau_b} \right) \\ &= \sum_{i=1}^k \left(\log K \left(\bar{a}^{\lfloor (1-x)\sigma_i \rfloor} \right) + \log K \left(\bar{b}^{\lfloor x\sigma_i \rfloor} \right) \right) \\ (1) \qquad &+ \log K \left(\bar{a}^{\tau_a} \right) + \log K \left(\bar{b}^{\tau_b} \right) + O(k) \\ &= \sum_{i=1}^k (\lfloor (1-x)\sigma_i \rfloor w_a + \lfloor x\sigma_i \rfloor w_b) + \tau_a w_a + \tau_b w_b + O(k) \\ &= (1-x)(\sigma_1 + \dots + \sigma_k) w_a + x(\sigma_1 + \dots + \sigma_k) w_b + O(k). \end{aligned}$$

Analogously we have $m = \sigma_1 + \dots + \sigma_k + O(k)$, which, together with (1), implies the assertion. □

3. TRANSCENDENCE CRITERIA

The transcendence criteria we use follow ideas that originated with E. Maillet [12] and A. Baker [2], [3]. Recent improvements of their work can be found in papers by J.L. Davison [4] and B. Adamczewski and Y. Bugeaud [1].

Theorem 6 (W.M. Schmidt, [14]; see also [15]). *Let $\alpha \in \mathbb{R}$ be algebraic but neither rational nor a quadratic irrationality and $\delta > 0$. Then there exist only finitely many $\beta \in \mathbb{R}$ which are rational or quadratic irrationalities such that $|\alpha - \beta| < H(\beta)^{-3-\delta}$.*

Remark. Here H denotes the classic absolute height; i.e., if $p(X) = \sum_{i=0}^m a_i X^i \in \mathbb{Z}[X] \setminus \{0\}$ with $\gcd(a_0, \dots, a_m) = 1$, $p(\beta) = 0$ and $\deg p$ is minimal with this property, then $H(\beta) = \max_{0 \leq i \leq m} |a_i|$.

Corollary 7. *Let $\alpha \in \mathbb{R}$ have a quasiperiodic but not periodic continued fraction expansion*

$$\alpha = [0, a_1, \dots, a_{\nu_1-1}, \overline{a_{\nu_1}, \dots, a_{\nu_1+k_1-1}}^{\lambda_1}, \overline{a_{\nu_2}, \dots, a_{\nu_2+k_2-1}}^{\lambda_2}, \dots].$$

Here $\overline{a_{\nu}, \dots, a_{\nu+k}}^{\lambda}$ indicates that the partial quotients $a_{\nu}, \dots, a_{\nu+k}$ should be repeated λ times, i.e., $\nu_n = \nu_1 + \sum_{i=1}^{n-1} \lambda_i k_i$. If α is algebraic, then $\lim_{i \rightarrow \infty} q_{\nu_i+1-1} q_{\nu_i+k_i-1}^{-3-\delta} < +\infty$ for every $\delta > 0$.

Proof. For $i \geq 1$ we define the quadratic irrationality

$$\beta_i = [0, a_1, \dots, a_{\nu_i-1}, \overline{a_{\nu_i}, \dots, a_{\nu_i+k_i-1}}^{\lambda_i}, \dots, \overline{a_{\nu_{i-1}}, \dots, a_{\nu_{i-1}+k_{i-1}-1}}^{\lambda_{i-1}}, \overline{a_{\nu_i}, \dots, a_{\nu_i+k_i-1}}].$$

For $k \leq \nu_{i+1} - 1$ we have $a_k(\alpha) = a_k(\beta_i)$ and we may write p_k/q_k for $p_k(\alpha)/q_k(\alpha) = p_k(\beta_i)/q_k(\beta_i)$. Now $L_i \beta_i^2 + M_i \beta_i + N_i = 0$ with

$$\begin{aligned} L_i &= q_{\nu_i-2} q_{\nu_i+k_i-1} - q_{\nu_i-1} q_{\nu_i+k_i-2} \\ M_i &= q_{\nu_i-1} p_{\nu_i+k_i-2} + p_{\nu_i-1} q_{\nu_i+k_i-2} - p_{\nu_i-2} q_{\nu_i+k_i-1} - q_{\nu_i-2} p_{\nu_i+k_i-1} \\ N_i &= p_{\nu_i-2} p_{\nu_i+k_i-1} - p_{\nu_i-1} p_{\nu_i+k_i-2} \end{aligned}$$

and therefore $H(\beta_i) \leq \max\{|L_i|, |M_i|, |N_i|\} < 2q_{\nu_i+k_i-1}^2$. Theorem 6 implies

$$q_{\nu_{i+1}-1}^{-2} > |\alpha - \beta_i| > C(\alpha, \delta) H(\beta_i)^{-3-\delta} > C(\alpha, \delta) 2^{-3-\delta} q_{\nu_i+k_i-1}^{-6-2\delta}$$

for a certain $C(\alpha, \delta) > 0$. The corollary follows immediately. □

Lemma 8. *Let $a < b$ be positive integers and $w_a < \gamma < w_b$. Then there exist non-denumerably many, pairwise not equivalent transcendental $\alpha = [0, a_1, a_2, a_3, \dots]$ such that $\beta(\alpha) = \gamma$ and $a_i \in \{a, b\}$ for all $i \geq 1$.*

Proof. Let x and $(\sigma_k)_{k \geq 1}$ be as in the proof of Lemma 5. Set

$$\alpha = \left[0, \overline{a^{\lfloor (1-x)\sigma_1 \rfloor}, b^{\lfloor x\sigma_1 \rfloor}}^{\lambda_1}, \overline{a^{\lfloor (1-x)\sigma_2 \rfloor}, b^{\lfloor x\sigma_2 \rfloor}}^{\lambda_2}, \dots \right].$$

Then $\beta(\alpha) = \gamma$ by Lemma 5. Using the notation of Corollary 7 we have

$$(2) \quad k_n = \lfloor (1-x)\sigma_n \rfloor + \lfloor x\sigma_n \rfloor \text{ and } \nu_n - 1 = \sum_{i=1}^{n-1} \lambda_i (\lfloor (1-x)\sigma_i \rfloor + \lfloor x\sigma_i \rfloor) \text{ for all } n \geq 1.$$

Employing that $\overline{a}^{m-1} \leq q_m(\alpha) \leq \overline{b}^m$ for all $m \geq 0$ we get

$$\log q_{\nu_{n+1}-1}(\alpha) \geq \left(-1 + \sum_{i=1}^n \lambda_i (\lfloor (1-x)\sigma_i \rfloor + \lfloor x\sigma_i \rfloor) \right) w_a$$

and

$$\log q_{\nu_n+k_n-1}^{-7/2}(\alpha) \geq -\frac{7}{2} \frac{w_b}{w_a} \left(\sum_{i=1}^{n-1} \lambda_i (\lfloor (1-x)\sigma_i \rfloor + \lfloor x\sigma_i \rfloor) + \lfloor (1-x)\sigma_n \rfloor + \lfloor x\sigma_n \rfloor \right) w_a.$$

Because of Corollary 7, α will be transcendental if the sequence $(\lambda_n)_{n \geq 1}$ is chosen such that

$$\overline{\lim}_{n \rightarrow \infty} \left(\left(\lambda_n - \frac{7}{2} \frac{w_b}{w_a} \right) (\lfloor (1-x)\sigma_n \rfloor + \lfloor x\sigma_n \rfloor) - \left(\frac{7}{2} \frac{w_b}{w_a} - 1 \right) \sum_{i=1}^{n-1} \lambda_i (\lfloor (1-x)\sigma_i \rfloor + \lfloor x\sigma_i \rfloor) \right) = +\infty.$$

As the sequence $(\lambda_n)_{n \geq 1}$ can be chosen in non-denumerably many different ways, the construction yields non-denumerably many, pairwise not equivalent α with $\beta(\alpha) = \gamma$. □

Lemma 9. *Keeping the notation of Corollary 7 we have*

$$0 < |L_n \alpha^2 + M_n \alpha + N_n| < 8q_{\nu_n+k_n-1}^4 q_{\nu_{n+1}-1}^{-2}.$$

Proof. Let $\bar{\beta}_i$ denote the conjugate of β_i . If $|\bar{\beta}_i| \geq 1$, it follows from $L_i \bar{\beta}_i^2 + M_i \bar{\beta}_i + N_i = 0$ that

$$|\bar{\beta}_i|^2 \leq |L_i \bar{\beta}_i^2| = |M_i \bar{\beta}_i + N_i| < 2q_{\nu_i+k_i-1}^2 (|\bar{\beta}_i| + 1) \leq 4q_{\nu_i+k_i-1}^2 |\bar{\beta}_i|$$

and therefore $|\bar{\beta}_i| < 4q_{\nu_i+k_i-1}^2$, which remains true even if $|\bar{\beta}_i| < 1$. This implies

$$|\alpha - \bar{\beta}_i| \leq 1 + |\bar{\beta}_i| < 1 + 4q_{\nu_i+k_i-1}^2 < 8q_{\nu_i+k_i-1}^2$$

and thus

$$\begin{aligned} |L_i \alpha^2 + M_i \alpha + N_i| &= |L_i| \cdot |\alpha - \beta_i| \cdot |\alpha - \bar{\beta}_i| \\ &< q_{\nu_i+k_i-1}^2 \cdot q_{\nu_{i+1}-1}^{-2} \cdot 8q_{\nu_i+k_i-1}^2 = 8q_{\nu_i+k_i-1}^4 q_{\nu_{i+1}-1}^{-2}. \end{aligned}$$

□

Lemma 10. *Let $a < b$ be positive integers and $w_a < \gamma < w_b$. Then there exist non-denumerably many, pairwise not equivalent U_2 -numbers $\alpha = [0, a_1, a_2, a_3, \dots]$ such that $\beta(\alpha) = \gamma$ and $a_i \in \{a, b\}$ for all $i \geq 1$.*

Proof. Let x and $(\sigma_k)_{k \geq 1}$ be as in the proof of Lemma 5 and α as in the proof of Lemma 8. Then $\beta(\alpha) = \gamma$ by Lemma 5 and (2) holds. We have

$$H(L_n X^2 + M_n X + N_n) = \max\{|L_n|, |M_n|, |N_n|\} < 2q_{\nu_n+k_n-1}^2 \leq 2[\bar{b}]^{2(\nu_n+k_n-1)},$$

where H denotes the height of a polynomial. Using Lemma 9 we can estimate

$$\begin{aligned} 0 < |L_n \alpha^2 + M_n \alpha + N_n| &< 8q_{\nu_n+k_n-1}^4 q_{\nu_{n+1}-1}^{-2} \leq 8[\bar{b}]^{4(\nu_n+k_n-1)} [\bar{a}]^{-2(\nu_{n+1}-2)} \\ &= [\bar{b}]^{-(2(\nu_{n+1}-2)w_a - 4(\nu_n+k_n-1)w_b - 3 \log 2)/w_b} = (2[\bar{b}]^{2(\nu_n+k_n-1)})^{-\Psi_n} \end{aligned}$$

with

$$\Psi_n = \frac{2(\nu_{n+1} - 2)w_a - 4(\nu_n + k_n - 1)w_b - 3 \log 2}{2(\nu_n + k_n - 1)w_b + \log 2}.$$

Replacing ν_n and k_n by the expressions given in (2) we see that $\overline{\lim}_{n \rightarrow \infty} \Psi_n = +\infty$ can be achieved by letting the sequence $(\lambda_n)_{n \geq 1}$ grow sufficiently fast, which proves the assertion. □

Remark. As $(\log \frac{1+\sqrt{5}}{2}, +\infty) = \bigcup_{n=2}^{\infty} (w_n, w_{n+2})$, the assertions of Theorem 1 follow from Lemma 8 and Lemma 10 respectively, at least for every $\gamma > \log \frac{1+\sqrt{5}}{2}$. In the remainder we will briefly indicate how one can take care of the remaining value w_1 .

Lemma 11. *If $\alpha = [a_0, a_1, a_2, \dots]$ has a Lévy constant, then*

$$\beta(\alpha) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \log[a_i, a_{i+1}, a_{i+2}, \dots].$$

Proof. Using well-known identities from the theory of regular continued fractions one gets

$$\prod_{i=1}^{m+1} [a_i, a_{i+1}, \dots] = |q_m \alpha - p_m|^{-1} = q_m \cdot [a_{m+1}, a_{m+2}, \dots] + q_{m-1}$$

(where p_m/q_m denotes the m th convergent), which implies

$$\begin{aligned} \sum_{i=1}^{m+1} \log[a_i, a_{i+1}, \dots] &= \log q_m + \log\left([a_{m+1}, a_{m+2}, \dots] + \frac{q_{m-1}}{q_m}\right) \\ &= \log q_m + \log\left([a_{m+1}, a_{m+2}, \dots] + [0, a_m, \dots, a_1]\right) \\ &= \log q_m + \log[a_{m+1}, a_{m+2}, \dots] + O(1). \end{aligned}$$

□

Lemma 12. *Let $\alpha = [a_0, a_1, a_2, \dots]$, $\alpha' = [b_0, b_1, b_2, \dots]$ and let $t(m)$ denote the number of $i \in \{1, \dots, m\}$ with $a_i \neq b_i$. If α has a Lévy constant, $b_i = a_i + O(1)$ as $i \rightarrow \infty$ and $t(m) = o(m)$ as $m \rightarrow \infty$, then $\beta(\alpha') = \beta(\alpha)$.*

Proof. We have

$$\begin{aligned} \sum_{i=1}^m \log[b_i, b_{i+1}, \dots] &= \sum_{i=1}^m \log[a_i, a_{i+1}, \dots] + \sum_{i=1}^m (\log[b_i, b_{i+1}, \dots] - \log[a_i, a_{i+1}, \dots]) \\ &= \sum_{i=1}^m \log[a_i, a_{i+1}, \dots] + O(t(m)) = \sum_{i=1}^m \log[a_i, a_{i+1}, \dots] + o(m), \end{aligned}$$

which together with Lemma 11 yields the assertion. □

Lemma 13. *Let $a < b$ be positive integers and $\alpha = [0, \bar{a}^{\lambda_1}, \bar{b}^{\lambda_2}, \bar{a}^{\lambda_3}, \bar{b}^{\lambda_4}, \dots]$.*

(i) *If*

$$\overline{\lim}_{n \rightarrow \infty} \left(\lambda_1 + \dots + \lambda_n - \frac{7}{2} \frac{w_b}{w_a} (\lambda_1 + \dots + \lambda_{n-1}) \right) = +\infty,$$

then α is transcendental.

(ii) *If $\overline{\lim}_{n \rightarrow \infty} \lambda_n / (\lambda_1 + \dots + \lambda_{n-1}) = +\infty$, then α is a U_2 -number.*

Proof. This can be proved along the same lines as Lemmata 8 and 10. □

Lemma 14. *Let a be a positive integer.*

(i) *There exist non-denumerably many pairwise not equivalent transcendental α such that $\beta(\alpha) = w_a$.*

(ii) *There exist non-denumerably many pairwise not equivalent U_2 -numbers α such that $\beta(\alpha) = w_a$.*

Proof. Set $\alpha = [0, \bar{a}^{\lambda^1}, a+1, \bar{a}^{\lambda^3}, a+1, \bar{a}^{\lambda^5}, \dots]$. By letting the sequence $(\lambda_{2n+1})_{n \geq 0}$ grow fast enough we can guarantee that the assumptions of both Lemma 12 and Lemma 13 are satisfied. \square

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