PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 7, July 2009, Pages 2363–2368
S 0002-9939(09)09796-2
Article electronically published on January 22, 2009

INTEGRAL REPRESENTATION FOR NEUMANN SERIES OF BESSEL FUNCTIONS

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(Communicated by Peter A. Clarkson)

Abstract. A closed integral expression is derived for Neumann series of Bessel functions — a series of Bessel functions of increasing order — over the set of real numbers.

1. Introduction and motivation

The series

$$\mathfrak{N}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n J_{\nu+n}(z), \qquad z \in \mathbb{C}, \tag{1}$$

where ν, α_n are constants and J_μ signifies the Bessel function of the first kind of order μ, is called a *Neumann series* [21, Chapter XVI]. Such series owe their name to the fact that they were first systematically considered (for integer μ) by Carl Gottfried Neumann in his important book [15] in 1867; subsequently, in 1877, Leopold Bernhard Gegenbauer extended such series to $\mu \in \mathbb{R}$ (see [21, p. 522]).

Neumann series of Bessel functions arise in a number of application areas. For example, in connection with random noise, Rice [18, Eqs. (3.10–3.17)] applied Bennett's result,

$$\sum_{n=1}^{\infty} \left(\frac{v}{a}\right)^n J_n(a\mathrm{i}\, v) = \mathrm{e}^{v^2/2} \int_0^v x\mathrm{e}^{-x^2/2} J_0(a\mathrm{i}\, x)\mathrm{d}x. \tag{2}$$

Luke [8, pp. 271–288] proved that

$$1 - \int_0^v \mathrm{e}^{-(u+x)} J_0\big(2\mathrm{i}\sqrt{ux}\big)\mathrm{d}x = \begin{cases} \mathrm{e}^{-(u+v)} \displaystyle\sum_{n=0}^{\infty} \left(\frac{u}{v}\right)^{n/2} J_n\big(2\mathrm{i}\sqrt{uv}\big), & u < v, \\ 1 - \mathrm{e}^{-(u+v)} \displaystyle\sum_{n=1}^{\infty} \left(\frac{v}{u}\right)^{n/2} J_n\big(2\mathrm{i}\sqrt{uv}\big), & u > v; \end{cases}$$

cf. also [16, Eq. (2a)]. In both of these applications $\mathfrak{N}_0$ plays a key role. The function $\mathfrak{N}_0$ also appears as a relevant technical tool in the solution of the infinite dielectric wedge problem by Kontorovich–Lebedev transforms [20, §§4, 5]. It also

Received by the editors May 31, 2007, and, in revised form, September 22, 2008.
2000 *Mathematics Subject Classification.* Primary 33C10, 33C20; Secondary 40A05, 44A20.
Key words and phrases. Bessel function of the first kind $J_\nu(x)$, integral representation of series, Neumann series of Bessel functions.
The first author was supported in part by Research Project No. 112-2352818-2814 of the Ministry of Sciences, Education and Sports of Croatia.

arises in the description of internal gravity waves in a Boussinesq fluid [14], as well as in the study of the propagation properties of diffracted light beams; see, for example, [12, Eqs. (6a,b), (7b), (10a,b)].

Expanding a given function f, say, into a Neumann series of the form

$$\mathfrak{N}_\nu^{\mathsf{w}}(x) = \sum_{n=0}^\infty a_{n\nu} J_{\nu+2n+1}(x), \qquad \nu \geq -1/2,$$

where

$$a_{n\nu} = 2(\nu+2n+1)\int_0^\infty t^{-1} f(t) J_{\nu+2n+1}(t)\mathrm{d}t,$$

Wilkins discussed the question of existence of an integral representation for $\mathfrak{N}_\nu^{\mathsf{w}}(x)$, as well as the conditions under which the Neumann series $\mathfrak{N}_\nu^{\mathsf{w}}(x)$ converges uniformly in x to the 'input' function f [22, §§11–13].

By modifying a result of Watson [21, p. 23, footnote], Maximon represented a simple Neumann series $\mathfrak{N}_\nu$ appearing in the literature in connection with physical problems [11, Eq. (4)] as an indefinite integral expression containing Bessel functions. Meligy expanded into a Neumann series $\mathfrak{N}_{L+1/2}$ of arbitrary argument, containing Bessel functions of order $L + 1/2 + n/2$, where L is the orbital angular momentum quantum number, the wave functions that describe the states of motion of charged particles in a Coulomb field [13, Eqs. (8), (9)]. The inversion probability of a large spin is found *via* modified Neumann series of Bessel functions $J_{(2N+1)(2n-1)\pm 1}$ for integer $N \geq 2$; see, [5, Theorem].

The evaluation of the capacitance matrix of a system of finite-length conductors [2] uses $\mathfrak{N}_p$, with p integer; in [10], free vibrations of a wooden pole were modelled by a coupled system of ordinary differential equations and solved by Neumann series; we note in passing that the analysis of an isotropic medium containing a cylindrical borehole by Love's auxiliary function and the analytical and numerical study of Neumann series of Bessel functions [18] are two further areas in which the unknown coefficients of $\mathfrak{N}_\nu$ are derived and computed from boundary and initial conditions of the problem under consideration.

2. Statement of the main result

In this short note our main goal is to establish a closed integral representation formula for the series $\mathfrak{N}_\nu(z)$. This will be achieved by using the Laplace integral representation of the associated Dirichlet series. Thus, we replace $z \in \mathbb{C}$ with $x \in \mathbb{R}_+$ and assume in what follows that the behaviour of $(\alpha_n)_{n\in\mathbb{N}}$ ensures the convergence of the series (1) over $\mathbb{R}_+$.

Throughout the paper, $[a]$ and $\{a\} = a - [a]$ will denote the integer and fractional part of a real number a, respectively, while χ_S will signify the characteristic function of the set $S \subset \mathbb{R}$.

Consider the real-valued function $x \mapsto a_x = a(x)$ and suppose that $a \in \mathrm{C}^1[k, m]$, $k, m \in \mathbb{Z}$, $k < m$. The classical Euler–Maclaurin summation formula states that

$$\sum_{j=k}^m a_j = \int_k^m a(x)\mathrm{d}x + \frac{1}{2}\big(a_k + a_m\big) + \int_k^m \Big(x - [x] - \frac{1}{2}\Big)a'(x)\mathrm{d}x.$$

On introducing the operator

$$\mathfrak{d}_x := 1 + \{x\}\frac{\mathrm{d}}{\mathrm{d}x},$$

obvious transformations yield the following condensed form of the Euler–Maclaurin formula:

$$\sum_{j=k+1}^{m} a_j = \int_k^m \big(a(x) + \{x\}a'(x)\big)\mathrm{d}x = \int_k^m \mathfrak{d}_x a(x)\,\mathrm{d}x. \tag{3}$$

Theorem. *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$ *and let* $\alpha\big|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$. *Then, for all* x, ν *such that*

$$0 < x < 2\min\left(1, \left(\mathrm{e}\,\overline{\lim_{n\to\infty}}\frac{\sqrt[n]{|\alpha_n|}}{n}\right)^{-1}\right), \qquad \nu > -1/2,$$

we have that

$$\mathfrak{N}_\nu(x) = -\int_1^\infty \frac{\partial}{\partial\omega}\Big(\Gamma(\nu+\omega+1/2)\,J_{\nu+\omega}(x)\Big)\int_0^{[\omega]}\mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)}\Big)\mathrm{d}\eta\,\mathrm{d}\omega. \tag{4}$$

Proof. Consider the integral representation formula [3, 8.411, Eq.(10)]

$$J_\nu(z) = \frac{\big(z/2\big)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_{-1}^{1}\cos(zt)(1-t^2)^{\nu-1/2}\mathrm{d}t, \qquad z\in\mathbb{C},\ \Re\{\nu\} > -1/2. \tag{5}$$

Applying (5) to (1) taking $x > 0$, we get

$$\mathfrak{N}_\nu(x) = \sqrt{\frac{2x}{\pi}}\int_0^1 \cos(xt)\Big(\frac{x(1-t^2)}{2}\Big)^{\nu-1/2}\mathcal{D}_\alpha(t)\,\mathrm{d}t \tag{6}$$

with the Dirichlet series

$$\mathcal{D}_\alpha(t) := \sum_{n=1}^\infty \frac{\alpha_n\big(x(1-t^2)/2\big)^n}{\Gamma(n+\nu+1/2)} = \sum_{n=1}^\infty \frac{\alpha_n\ \exp\big\{-n\ln\frac{2}{x(1-t^2)}\big\}}{\Gamma(n+\nu+1/2)}.$$

Recalling that $\Gamma(s) = \sqrt{2\pi}\,s^{s-1/2}\mathrm{e}^{-s}\big(1+\mathcal{O}(s^{-1})\big)$, $|s|\to\infty$, we see that the Dirichlet series $\mathcal{D}_\alpha(t)$ is absolutely convergent for all $x\in\mathbb{R}_+$ and $t\in(-1,1)$ such that

$$|x|(1-t^2) \le |x| < \frac{2}{\mathrm{e}}\left(\overline{\lim_{n\to\infty}}\frac{\sqrt[n]{|\alpha_n|}}{n}\right)^{-1}.$$

Furthermore, $\mathcal{D}_\alpha(t)$ has a Laplace integral representation when $\ln 2/(x(1-t^2)) > 0$. In this case we can take $x\in(0,2)$ and $t\in(-1,1)$, since the required positivity condition is satisfied when

$$\frac{2}{x(1-t^2)} \ge \frac{2}{x} > 1.$$

Hence, the x–domain becomes

$$0 < x < 2\min\left(1, \left(\mathrm{e}\,\overline{\lim_{n\to\infty}}\frac{\sqrt[n]{|\alpha_n|}}{n}\right)^{-1}\right). \tag{7}$$

Thus, for all such x we deduce that

$$\mathcal{D}_\alpha(t) = \ln\frac{2}{x(1-t^2)}\int_0^\infty \Big(\frac{x(1-t^2)}{2}\Big)^\omega\left(\sum_{j=1}^{[\omega]}\frac{\alpha_j}{\Gamma(j+\nu+1/2)}\right)\mathrm{d}\omega; \tag{8}$$

see, for example, [4, **V**] or [17, §§4, 6]. Now, it remains to sum the so-called *counting function*

$$\mathcal{A}_\alpha(\omega) := \sum_{j=1}^{[w]} \frac{\alpha_j}{\Gamma(j+\nu+1/2)}\,.$$

The Euler–Maclaurin summation formula gives us

$$\mathcal{A}_\alpha(\omega) = \int_0^{[\omega]} \mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)}\Big)\,\mathrm{d}\eta; \tag{9}$$

cf. [17, Lemma 1]. Substituting $\mathcal{A}_\alpha(\omega)$ and $\mathcal{D}_\alpha(t)$ from (9) and (8) into (6), we get

$$\begin{aligned}\mathfrak{N}_\nu(x) = -\sqrt{\frac{x}{2\pi}}\int_0^\infty\int_0^{[\omega]} \mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)}\Big)\\ \times\left(2\int_0^1 \cos(xt)\Big(\frac{x(1-t^2)}{2}\Big)^{\nu+\omega-1/2}\ln\Big(\frac{x(1-t^2)}{2}\Big)\,\mathrm{d}t\right)\mathrm{d}\omega\,\mathrm{d}\eta.\end{aligned} \tag{10}$$

However, the innermost (t–integral) in (10),

$$\mathcal{I}_x(\kappa) := 2\int_0^1 \cos(xt)\Big(\frac{x(1-t^2)}{2}\Big)^{\kappa}\ln\Big(\frac{x(1-t^2)}{2}\Big)\,\mathrm{d}t, \qquad \kappa := \nu+\omega-1/2,$$

can be expressed in terms of the Gamma function and the Bessel function of the first kind by legitimate indefinite integration with respect to κ, as follows. To begin, we define the Fourier cosine transform of a certain function f by

$$\mathcal{F}_c(f;x) := 2\int_0^\infty \cos(xt)\,f(t)\,\mathrm{d}t.$$

Now, we have that

$$\begin{aligned}\int \mathcal{I}_x(\kappa)\,\mathrm{d}\kappa &= 2\Big(\frac{x}{2}\Big)^{\kappa}\int_0^1 \cos(xt)(1-t^2)^\kappa \mathrm{d}t\\ &= \Big(\frac{x}{2}\Big)^{\kappa}\mathcal{F}_c\big((1-t^2)^\kappa\chi_{[0,1)}(t);\,x\big) = \sqrt{\frac{2\pi}{x}}\cdot\Gamma(\kappa+1)\,J_{\kappa+1/2}(x),\end{aligned}$$

where we applied the Fourier cosine transform table [3, 17.34, Eq. (10)]. On observing that $\mathrm{d}\kappa = \mathrm{d}\omega$, we deduce that

$$\mathcal{I}_x(\nu+\omega-1/2) = \sqrt{\frac{2\pi}{x}}\cdot\frac{\partial}{\partial\omega}\Big(\Gamma(\nu+\omega+1/2)\,J_{\nu+\omega}(x)\Big). \tag{11}$$

Substituting (11) into (10) we arrive at the asserted integral expression (4), remarking that the integration domain $\mathbb{R}_+$ changes into $[1,\infty)$ because $[\omega]$ equals zero for all $\omega\in[0,1)$. □

3. Concluding remarks

To conclude, we mention some related integral representation formulæ for Neumann-type series, corresponding to special α's. Bivariate Lommel functions of order

ν are defined by Neumann-type series [21, 16.5, Eqs. (5), (6)] as follows:

$$U_\nu(y,x) := \sum_{m=0}^\infty (-1)^m \Big(\frac{y}{x}\Big)^{\nu+2m} J_{\nu+2m}(x),$$
$$V_\nu(y,x) := \cos\Big(\frac{y}{2}+\frac{x^2}{2y}+\frac{\nu\pi}{2}\Big) + U_{-\nu+2}(y,x), \qquad x,y\in\mathbb{R}.$$

These series converge for unrestricted values of ν.

Now, assuming that $\Re\{\nu\} > 0$, by the formulæ [21, 16.53, Eqs. (1), (2)] we easily deduce that

$$U_{\nu,c}(x) := U_\nu(cx,x) = c^\nu x \int_0^1 t^\nu\, J_{\nu-1}(xt)\cos\big(\tfrac{c}{2}\, x(1-t^2)\big)\,\mathrm{d}t,$$
$$U_{\nu+1,c}(x) = c^\nu x \int_0^1 t^\nu\, J_{\nu-1}(xt)\sin\big(\tfrac{c}{2}\, x(1-t^2)\big)\,\mathrm{d}t.$$

Similarly, by [21, 16.53, Eqs. (11), (12)][1] we also have that

$$V_{\nu,c}(x) := V_\nu(cx,x) = -c^{2-\nu} x \int_1^\infty t^{2-\nu}\, J_{1-\nu}(xt)\cos\big(\tfrac{c}{2}\, x(1-t^2)\big)\,\mathrm{d}t,$$
$$V_{\nu-1,c}(x) = -c^{2-\nu} x \int_1^\infty t^{2-\nu}\, J_{1-\nu}(xt)\sin\big(\tfrac{c}{2}\, x(1-t^2)\big)\,\mathrm{d}t,$$

provided $x, c > 0$, $\Re\{\nu\} > 1/2$.

The integral expressions developed above can be easily adapted to Neumann-type series of the form

$$\sum_{m=0}^\infty \gamma^m\, J_{\nu+2m}(x), \qquad x>0,\ \gamma<0.$$

An interesting open problem, worthy of further study, is the construction of examples with specific coefficients α_n, with known explicit forms of Neumann-type series, that can be derived directly from the representation formula (4).

References

[1] A. Al-Jarrah, K. M. Dempsey, M. L. Glasser, *Generalized series of Bessel functions*, J. Comput. Appl. Math. **143**(2002), 1–8. MR1906499 (2003c:33008)

[2] F. Delfino, R. Procopio, M. Rossi, *Evaluation of capacitance matrix of a finite-length multiconductor transmission line*, IEE Proc.–Sci. Meas. Technol. **151**(2004), 347–353.

[3] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products* (corrected and enlarged edition prepared by A. Jeffrey and D. Zwillinger), sixth ed., Academic Press, San Diego, CA, 2000. MR1773820 (2001c:00002)

[4] J. Karamata, Teorija i praksa Stieltjes–ova integrala, Srpska Akademija Nauka, Posebna izdanja **CLIV**, Matematički institut, Knjiga **I**, Beograd, 1949.

[5] E. A. Karatsuba, P. Moretti, *Inversion time of large spins*, J. Math. Phys. **46(4)**(2005), 042101:1–7. MR2131217 (2005m:81099)

[6] E. C. J. von Lommel, *Die Beugungserscheinungen einer kreisrunden Öffnung und eines kreisrunden Schirmchens theoretisch und experimentell bearbeitet*, Abh. der math. phys. Classe der k. b. Akad. der Wiss. (München) **15**(1884-1886), 229–328.

[7] E. C. J. von Lommel, *Die Beugungserscheinungen geradlinig begrenzter Schirme*, Abh. der math. phys. Classe der k. b. Akad. der Wiss. (München) **15**(1884–1886), 529–664.

[8] Y. L. Luke, *Integrals of Bessel Functions*, McGraw–Hill, New York, 1962. MR0141801 (25:5198)

[1] Watson remarked that all four formulæ that were cited by him [21, 16.53, Eqs. (1), (2), (11), (12)] had been derived by von Lommel (cf. von Lommel's memoirs [6], [7] for further details).

[9] P. A. Martin, *Acoustic waves in slender axisymmetric tubes*, Journal of Sound and Vibration **286**(2005), 55–68.

[10] P. A. Martin, J. R. Berger, *Waves in wood: Free vibrations of a wooden pole*, Journal of the Mechanics and Physics of Solids **49**(2001), 1155–1178.

[11] L. C. Maximon, *On the representation of indefinite integrals containing Bessel functions by simple Neumann series*, Proc. Amer. Math. Soc **7(6)**(1956), 1054–1062. MR0083052 (18:650f)

[12] Z. Mei, D. Zhao, J. Gu, *Comparison of two approximate methods for hard-edged diffracted flat-topped light beams*, Optics Communications **267**(2006), 58–64.

[13] A. S. Meligy, *On Whittaker and Coulomb functions*, J. London Math. Soc. **37**(1962), 141–144. MR0140729 (25:4143)

[14] M. Nadon, L. J. Campbell, *An exact expression for transient forced internal gravity waves in a Boussinesq fluid*, Wave Motion **44**(2007), 340–345. MR2311430 (2008b:76031)

[15] C. G. Neumann, *Theorie der Besselschen Funktionen*, B.G. Teubner Verlag, Leipzig, 1867.

[16] F. Novomestky, *Asymptotic expressions for the unit–step and Dirac delta functions*, SIAM J. Appl. Math. **27(4)**(1974), 521–525. MR0358176 (50:10641)

[17] T. K. Pogány, H. M. Srivastava, Ž. Tomovski, *Some families of Mathieu* **a**-*series and alternating Mathieu* **a**-*series*, Appl. Math. Comput. **173(1)**(2006), 69–108. MR2203374 (2007a:33021)

[18] S. O. Rice, *Mathematical analysis of random noise*. III, Bell System Tech. J. **24**(1945), 46–156. MR0011918 (6:233i)

[19] N. I. Robinson, *An isotropic elastic medium containing a cylindrical borehole with a rigid plug*, Internat. J. Solids Structures **39**(2002), 4889–4904.

[20] M. A. Salem, A. H. Kamel, A. V. Osipov, *Electromagnetic fields in the presence of an infinite dielectric wedge*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **462(2)**(2006), 2503–2522. MR2245179 (2007b:78008)

[21] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1922. MR1349110 (96i:33010)

[22] J. E. Wilkins, Jr., *Neumann series of Bessel functions*, Trans. Amer. Math. Soc. **64**(1948), 359–385. MR0027092 (10:249a)

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