VECTOR MEASURES AND THE STRONG OPERATOR TOPOLOGY

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(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. A fundamental result of Nigel Kalton is used to establish a result for operator valued measures which has improved versions of the Vitali-Hahn-Saks Theorem, Phillips's Lemma, the Orlicz-Pettis Theorem and other classical results as straightforward corollaries.

Each of E, F, X, Y, and Z will denote a real ($\mathbf{R}$) Banach space. The space of all bounded linear functions (= operators) from X to Y will be denoted by $L(X, Y)$. Note that $X \cong L(\mathbf{R}, X)$. A generic σ-algebra of sets will be denoted by Σ. If $(A_i)_{i=1}^{\infty}$ is a sequence from Σ, then $\sigma((A_i)_{i=1}^{\infty})$ will denote the σ-algebra of subsets of $\bigcup_{i=1}^{\infty} A_i$ generated by $(A_i)_{i=1}^{\infty}$. If this sequence is pairwise disjoint, then $\sigma((A_i))$ is easily identifiable with $\mathcal{P}$, the power class of $\mathbf{N}$. If (T_i) is a sequence in $L(X, Y)$, recall that (T_i) converges to T in the strong operator topology (*sot*) if and only if $(T_i(x)) \to T(x)$ in Y for all $x \in X$. The reader should consult [3] and [2] for undefined terminology as well as the classical statements of the theorems and corollaries mentioned in this paper. The reader should note that there are non-separable subspaces of ℓ_∞ which do not contain ℓ_∞.

Theorem 1. *Suppose that Σ is a σ-algebra of subsets of Ω, E is separable, and F is a Banach space which does not contain ℓ_∞ but does embed isomorphically into ℓ_∞. If $\mu : \Sigma \to L(E, F)$ is bounded and finitely additive and (A_i) is a pairwise disjoint sequence from Σ, then there is a subsequence (B_i) of (A_i) so that $\mu : \sigma((B_i)) \to (L(E, F), sot)$ is countably additive.*

Proof. Suppose that E, F, Σ, (A_i), and μ are as in the statement of the theorem. As before, let $\mathcal{P}$ be the power class of the positive integers, and identify $\mathcal{P}$ with $\sigma((A_i)_{i=1}^{\infty})$. If $b = (b_i) \in \ell_\infty$, let $T(b) = \int b \, d\mu$. Then $T : \ell_\infty \to L(E, F)$ is a bounded linear operator. For $x \in E$, define $T_x : \ell_\infty \to F$ by $T_x(b) = T(b)(x)$. Since ℓ_∞ does not embed in F, T_x is weakly compact and therefore unconditionally converging; e.g., see [7], [6, p. 270].

Now let $T_n = T(e_n)$, where (e_n) denotes the canonical unit vector basis of $c_0 \subseteq \ell_\infty$. Therefore $\sum T_n$ converges unconditionally in the strong operator topology. In

fact, the map $\Delta : \ell_\infty \to L(E, F)$ defined by

$$\Delta(\gamma)x = \sum_{i=1}^\infty \gamma_i T_i(x) = \sum_{i=1}^\infty \gamma_i T(e_i)x$$

is bounded and linear. Let $L : F \to \ell_\infty$ be an isomorphism, and define operators $U : \ell_\infty \to L(E, \ell_\infty)$ and $V : \ell_\infty \to L(E, \ell_\infty)$ by

$$U(b)(x) = L(T(b)(x))$$

and

$$V(b)(x) = \sum b_i L T_i(x) = L(\sum b_i T_i(x)).$$

Note that $U(e_j)(x) = L(T_j(x)) = V(e_j)(x)$ for each j. Proposition 5 of Kalton [6] applies and yields an infinite set $M \subseteq \mathbf{N}$ so that $U(t) = V(t)$ for all $t = (t_i) \in \ell_\infty(M)$. Therefore $T(s)(x) = \sum s_i T_i(x)$ for $s = (s_i) \in \ell_\infty(M)$. The unconditional convergence of the series $\sum s_i T_i(x)$ and the preceding equality imply that

$$\mu : \sigma((A_i)_{i \in M}) \to (L(E, F), sot)$$

is countably additive.   □

In Corollary 5 below, we show that Theorem 1 leads to an improvement of Phillips's Lemma. Among the many consequences of Phillips's Lemma is a quick proof that c_0 is not complemented in ℓ_∞. The next result shows that this complementation result follows immediately from Theorem 1.

Corollary 2. c_0 *is not complemented in* ℓ_∞.

Proof. Suppose that $P : \ell_\infty \to c_0$ is a projection, and let $\mu : \mathcal{P} \to c_0 \cong L(\mathbf{R}, c_0)$ be the bounded and finitely additive measure so generated. By Theorem 1, there is a subsequence (n_i) of the positive integers so that $\mu : \sigma((n_i)) \to L(\mathbf{R}, c_0)$ is countably additive in the strong operator topology ($=$ the norm topology in this case). Clearly this is false since $||\mu(n_i)|| = 1$ for all i.   □

Now suppose that Σ is a σ-algebra of subsets of Ω, and $\lambda : \Sigma \to \mathbf{R}^+$ is an extended $\mathbf{R}$-valued and countably additive set function. Let X be a Banach space and $\mu_n : \Sigma \to X$ be countably additive so that $\mu_n \ll \lambda$ for all n. Suppose that $(\mu_n(A))$ is norm convergent in X for all $A \in \Sigma$, $\epsilon > 0$, $\lambda(A_n) \to 0$, and $||\mu_n(A_n)|| > 2\epsilon$ for all n. By a standard Cantor diagonalization argument, one may choose a subsequence (A_{n_i}) of (A_n) so that

$$||\mu_{n_i}(A_{n_i} \setminus \bigcup_{j>i} A_{n_j})|| > \epsilon$$

for each i.

Let $\nu_i = \mu_{n_i}$, $B_i = A_{n_i} \setminus \bigcup_{j>i} A_{n_j}$, $U = \{\nu_i(B_j) : i \geq 1, \ j \geq 1\}$, and $Y = [U] = \overline{span}(U)$. Then Y is separable. Thus the space $c(Y) = \{(y_n) : n \in \mathbf{N}\} \subseteq Y^{\mathbf{N}}$ of all convergent sequences in Y is separable when endowed with the usual pointwise operations and the sup norm. Define $\tau : \sigma((B_i)) \to c(Y)$ by

$$\tau(A) = (\nu_i(A)), \ A \in \sigma((B_i)).$$

Note that $c(Y) \cong (L(\mathbf{R}, Y), sot)$. By Theorem 1, we know that there is a subsequence (B_{i_j}) so that $\tau : \sigma((B_{i_j})) \to (L(\mathbf{R}, c(Y)), \ sot)$ is countably additive. However, this is impossible since $||\nu_i(B_i)|| > \epsilon$ for each i. Therefore Theorem 1 produces the following improved versions of the Vitali-Hahn-Saks Theorem [4, p. 24], and [5, p. 158].

Corollary 3. *(i) If $\epsilon > 0$, $\lambda : \Sigma \to \mathbf{R}^+$ is extended $\mathbf{R}$-valued and countably additive, (μ_n) is a sequence in $ca(\sigma, X)$ with $\mu_n \ll \lambda$ for each n, and (A_n) is a pairwise disjoint sequence from Σ so that $(\lambda(A_n)) \to 0$ and $||\mu_n(A_n)|| > \epsilon$ for all n, then no infinite subsequence of (μ_n) can converge setwise on all elements of (A_n).*

(ii) If (μ_n) is a sequence in $ca(\Sigma)$, $\epsilon > 0$, and (A_n) is a pairwise disjoint sequence in Σ so that $|\mu_n(A_n)| > \epsilon$ for each n, then no infinite subsequence of (μ_n) can converge setwise on all elements of the sequence (A_n).

Recall that a finitely additive set function μ defined on an algebra $\mathcal{A}$ is said to be strongly additive provided that $\mu(A_i) \to 0$ whenever (A_i) is a pairwise disjoint sequence from $\mathcal{A}$. It is easy to see that a family of countably additive set functions defined on a σ-algebra Σ is uniformly countably additive if and only if it is uniformly strongly additive. We say that $\mu : \mathcal{A} \to X$ is weakly strongly additive provided that $x^* \circ \mu : \mathcal{A} \to \mathbf{R}$ is strongly additive for all $x^* \in X^*$. Certainly every strongly additive set function is weakly strongly additive. If $c_0 \hookrightarrow X \hookrightarrow L(X^*, \mathbf{R}) \cong X^{**}$, then the following proposition makes it clear that there are weakly strongly additive set functions which are not strongly additive.

Proposition 4. *Every weakly strongly additive set function $\mu : \mathcal{A} \to X$ is strongly additive iff $c_0 \not\hookrightarrow X$.*

Proof. Suppose that $c_0 \hookrightarrow X$ and let $\mathcal{A}$ be the finite–cofinite algebra of subsets of $\mathbf{N}$. Define $\mu : \mathcal{A} \to X$ by

$$\mu(A) = \sum_{n \in A} e_n, \ A \text{ finite}$$

and

$$\mu(A) = \sum_{n \in A^c} e_n, \ A \text{ cofinite}.$$

Suppose that (B_i) is a pairwise disjoint sequence from $\mathcal{A}$. Note that there is an N so that if $n > N$, then B_n is finite. Let $x^* \in X^*$, and choose $\lambda = (\lambda_i) \in \ell_1$ so that $x^*\mu(A) = \langle \lambda, \mu(A) \rangle$ for each A. Since $\sum_{i>n} |\lambda_i| \to 0$ as $n \to \infty$ and at most one member of the sequence (B_i) is cofinite, it follows that $(x^*\mu(B_n)) \xrightarrow{n} 0$. Thus μ is weakly strongly additive and not strongly additive.

Now suppose that $\mathcal{A}$ is an algebra of sets and $\mu : \mathcal{A} \to X$ is weakly strongly additive and not strongly additive. Let (A_i) be a pairwise disjoint sequence in $\mathcal{A}$ and ϵ be a positive number so that $||\mu(A_i)|| > \epsilon$ for each i. If $x^* \in X^*$, then the weak strong additivity guarantees that $\sum |x^*\mu(A_i)| < \infty$. Thus $\sum \mu(A_i)$ is weakly unconditionally convergent and not unconditionally convergent. Classical results of Bessaga and Pelczynski [1] ensure that $c_0 \hookrightarrow X$. $\qquad \square$

Let $c_0(Y)$ be the subspace of $c(Y)$ consisting of null sequences.

Corollary 5 (Phillips's Lemma). *If E and F are separable, $\mu_n : \mathcal{P} \to L(E, F)$ is strongly additive in the sot for each n, and $(\mu_n(A)) \to 0$ in the sot for each A, then*

$$\sum_{k \in A} \mu_n(k) \xrightarrow{n} 0$$

in the sot uniformly for $A \in \mathcal{P}$.

Proof. Suppose not, and select $\epsilon > 0$, (A_i) from $\mathcal{P}$, $x \in E$, and a subsequence (λ_i) of (μ_n) so that $||\sum_{k \in A_i} \lambda_i(k)x|| > 2\epsilon$ for each i. Using the unconditional convergence

of $\sum \lambda_i(k)x$ (strong additivity in the *sot*) and the fact that $(\mu_n(A)x) \to 0$ for all A, obtain a strictly increasing sequence (N_i) of positive integers so that

$$(*) \qquad\qquad ||\sum_{k \in A_{n_i} \cup [N_{i-1}, N_i)} \lambda_{N_i}(k)x|| > \epsilon$$

for each i. Let $B_i = A_{N_i} \cap [N_{i-1}, N_i)$, let $\Sigma = \sigma((B_i))$, and set $\nu(A)(u) = (\lambda_i(A)(u))_i \in c_0(F)$ for $A \in \Sigma$ and $u \in E$. Inequality $(*)$ and an application of Theorem 1 to the set functions ν just defined are incompatible. $\qquad\square$

In the next corollary, $bfa(\mathcal{P}, X)$ is the set of all bounded and finitely additive set functions $\mu : \mathcal{P} \to X$. Equip $bfa(\mathcal{P}, X)$ with the sup norm; i.e., $||\mu|| = sup\{||\mu(A)|| : A \in \mathcal{P}\}$.

Corollary 6. *If X^* is separable and $(\mu_n) \to 0$ weakly in $bfa(\mathcal{P}, X)$, then*

$$\sum_{k=1}^{\infty} |x^*(\mu_n(k))| \overset{n}{\to} 0$$

for all $x^* \in X^*$.

Proof. Since $x^*\mu_n$ is bounded and finitely additive for all $x^* \in X^*$, $x^*\mu_n$ is strongly additive. Further, $(x^*\mu_n(A)) \overset{n}{\to} 0$ for all $x^* \in X^*$ and $A \in \mathcal{P}$. Therefore $(\mu_n(A)) \to 0$ in the (sot) when X is canonically embedded in $X^{**} = L(X^*, \mathbf{R})$. By the preceding corollary, $\sum_{k \in A} x^*\mu_n(k) \overset{n}{\to} 0$ uniformly for $A \in \mathcal{P}$. Thus $\sum_{n=1}^{\infty} |x^*(\mu_n(k))| \overset{n}{\to} 0$. $\qquad\square$

Since the sup norm on $ca(\mathcal{P})$ is equivalent to the total variation norm when ℓ_1 is interpreted in this space, Corollary 6 shows that ℓ_1 has the Schur property.

Corollary 7 (Vitali-Hahn-Saks-Nikodym Theorem)**.** *If X and Y are separable, $\mu_n : \Sigma \to L(X, Y)$ is a bounded and finitely additive set function which is strongly additive in the sot for each n, and $(\mu_n(A)) \to 0$ in the sot for each A, then $sup_n||\mu_n(A_i)x|| \overset{i}{\to} 0$ for each $x \in X$ whenever (A_i) is pairwise disjoint in Σ.*

Proof. Suppose not. Let $\epsilon > 0$, (A_i) be a pairwise disjoint sequence in Σ, and $x \in X$ so that $sup_n||\mu_n(A_i)x|| > \epsilon$ for each i. Let $\Sigma_1 = \sigma((A_i))$, and define $\nu : \Sigma_1 \to L(X, c_0(Y))$ by $\nu(A)(u) = (\mu_n(A)u)_{n=1}^{\infty}$. The preceding inequality and Theorem 1 applied to the vector measure ν provide the contradiction which completes the proof. $\qquad\square$

The proof of the Vitali-Hahn-Saks-Nikodym Theorem in Chapter 1 of [4] shows that there is no loss of generality in insisting that Y be separable in the preceding corollary.

Corollary 8. *Suppose that $\mu : \mathcal{P} \to X$ is bounded and finitely additive, Γ is a subset of X^* which separates the points of X, and μ is countably additive in the weak topology on X defined by Γ. The set function μ is countably additive in the norm topology if and only if $U = \overline{co}\{\mu(A) : A \in \mathcal{P}, \ A \ finite\}$ is Γ-closed.*

Proof. Since Γ separates the points of X and μ is Γ-countably additive, μ is countably additive in the norm topology if μ is strongly additive with respect to the norm on X. Suppose then that U is Γ-closed, and let $W = [U]$. The Γ-countable additivity of μ ensures that $\mu(\mathcal{P}) \subseteq W = L(\mathbf{R}, W)$. By Theorem 1, there does not

exist $\epsilon > 0$ and a disjoint sequence (A_i) in $\mathcal{P}$ so that $||\mu(A_i)|| > \epsilon$ for all i; i.e., μ is strongly additive.

Conversely, suppose that $\mu : \mathcal{P} \to (X, || \bullet ||)$ is countably additive. Thus μ has relatively weakly compact range, and U is weakly compact. Then U is Γ-compact and Γ-closed since Γ separates points. □

Since a norm-closed convex set is weakly closed, the preceding corollary certainly demonstrates that a weakly countably additive vector measure on a σ-algebra is countably additive. Thus this corollary contains the measure-theoretic version of the Orlicz–Pettis Theorem.

Next we note that Theorem 1 also yields a quick proof of the standard statement of the Orlicz–Pettis Theorem: *If $\sum x_n$ is weakly subseries convergent in X, then $\sum x_n$ is unconditionally convergent in X.*

Note first that $\sum_{k=1}^{\infty} |x^*(x_n)| < \infty$ for all $x^* \in X^*$, and let $Y = [\{x_n : n \in \mathbf{N}\}]$. Suppose that $\sum x_n$ is not unconditionally convergent. Let $\epsilon > 0$, $\sigma : \mathbf{N} \to \mathbf{N}$ be a permutation, and (p_i) and (q_i) be intertwining sequences of natural numbers so that $||w_i|| > \epsilon$ for all i, where $w_i = \sum_{n=p_i}^{q_i} x_{\sigma(n)}$. The hypothesis and the weak absolute summability of $\sum x_n$ noted above guarantee that $\nu(S) = weak\text{-}lim \sum_{i \in S} w_i$ defines a bounded and finitely additive set function $\nu : \mathcal{P} \to Y$. Apply Theorem 1 to obtain a contradiction.

Corollary 9. *If K is a relatively weakly compact subset of ca(Σ), then the elements of K are uniformly countably additive.*

Proof. As noted above, it suffices to show that K is uniformly strongly additive. Let (μ_n) be a sequence from K, and, without loss of generality (Eberlein-Smulian Theorem), suppose that (μ_n) converges weakly to a point in $ca(\Sigma)$. Let

$$\nu(A) = (\mu_n(A))_{n=1}^{\infty}$$

for $A \in \Sigma$, and let c denote the space of all convergent sequences of real numbers equipped with the sup norm. Then $\nu : \Sigma \to L(\mathbf{R}, c)$ satisfies the hypotheses of Theorem 1, and we conclude that (μ_n) is uniformly strongly additive. □

At a crucial point in the proof of Theorem 1 above we used the following theorem of Rosenthal [7]: *If ℓ_{∞} does not embed in F, then every operator $T : \ell_{\infty} \to F$ is weakly compact.* A key ingredient in the proof of Rosenthal's theorem is the following lemma.

Rosenthal's Lemma. *If $\mathcal{A}$ is an algebra of sets, (μ_n) is a uniformly bounded sequence of non-negative and real valued measures on $\mathcal{A}$, (A_n) is a pairwise disjoint sequence of sets in $\mathcal{A}$, and $\epsilon > 0$, then there is a subsequence (n_k) of positive integers so that $\sum_{i=1, \ i \neq j}^{N} \mu_{n_j}(A_{n_i}) < \epsilon$ for each N and for each j.*

One may check Chapter 1 of [4] for numerous applications of Rosenthal's Lemma. We close with a particularly quick inductive proof of this lemma. The argument below naturally generates a subsequence (ν_k) of (μ_n) so that the co-finite and terminal subsequences of (ν_k) automatically satisfy the conclusion of Rosenthal's Lemma.

Lemma 10. *Suppose that (μ_n, A_n) is a sequence of pairs of measures and sets so that (A_n) is distinct and pairwise disjoint and (μ_n) is uniformly bounded. If $\epsilon > 0$, then there is an n_0 so that $\{n : \mu_n(A_{n_0}) < \epsilon\}$ is infinite.*

Proof. Suppose that $\|\mu_n\| < M$ for each n, and choose N so that $N\epsilon > M$. Choose distinct positive integers $i_1, i_2, ..., i_N$. If none of these choices satisfy the conclusion, then $S_k = \{n : \mu_n(A_{i_k}) \geq \epsilon\}$ is cofinite for $k = i_1, ..., i_N$. Thus $\bigcap_{k=1}^{N} S_k$ is cofinite. Consequently, $\{n : \mu_n(A_{i_k}) \geq \epsilon, \ k = 1, ..., N\}$ is cofinite. Therefore, for some n, $\mu_n(\bigcup_{k=1}^{N} A_{i_k}) \geq N\epsilon$, and we have a contradiction. $\square$

Use the preceding lemma and choose n_1 so that $S_1 = \{n > n_1 : \mu_n(A_{n_1}) < 1/2\}$ is infinite. Since S_1 is infinite and $\sum_{1=1}^{\infty} \mu_{n_1}(A_i) < \infty$, we may assume that $\sum_{i > n_1} \mu_{n_1}(A_i) < 1/(2^2)$. Now use the lemma again and choose $n_2 \in S_1$ so that $S_2 = \{n \in S_1 : n > n_2, \ \mu_n(A_{n_2} < 1/(2^2)\}$ is infinite. As above, we may assume that

$$\sum_{i > n_2} \mu_{n_2}(A_i) < 1/(2^3).$$

Continue this process inductively to manufacture a subsequence $(\nu_k) = (\mu_{n_k})$. Note that if $\epsilon > 0$ and $1/(2^{k-1}) < \epsilon$, then $(\nu_i)_{i \geq k}$ satisfies the conclusion of Rosenthal's Lemma.

References

[1] C. Bessaga and A. Pelczynski, On bases and unconditional convergence of series in Banach spaces, Studia Math. 17(1958), 151-164. MR0115069 (22:5872)
[2] R. Bilyeu and P. Lewis, Uniform differentiability, uniform absolute continuity, and the Vitali-Hahn-Saks theorem, Rocky Mtn. J. Math. 10(1980), 533-557. MR590217 (82g:46083)
[3] J. Diestel, Sequences and Series in Banach Spaces, Grad. Texts in Math. 92, Springer, New York, 1984. MR737004 (85i:46020)
[4] J. Diestel and J.J. Uhl, Jr., Vector Measures, Amer. Math. Soc., Providence, RI, 1977. MR0453964 (56:12216)
[5] N. Dunford and J. Schwartz, Linear Operators, Part I: General Theory, Wiley, New York, 1958. MR1009162 (90g:47001a)
[6] N. Kalton, Spaces of compact operators, Math. Ann. 208(1974), 267-278. MR0341154 (49:5904)
[7] H. Rosenthal, On relatively disjoint families of measures, with some applications to Banach space theory, Studia Math. 37(1970), 13-36. MR0270122 (42:5015)

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