THE ERDŐS-KAC THEOREM FOR POLYNOMIALS
OF SEVERAL VARIABLES

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ABSTRACT. We prove two versions of the Erdős-Kac type theorem for polynomials of several variables on some varieties arising from translation and affine linear transformation.

1. INTRODUCTION

For a positive integer \( n \), let \( \omega(n) \) be the number of distinct prime divisors of \( n \). The remarkable theorem of Erdős and Kac ([7]) asserts that, for any \( \gamma \in \mathbb{R} \),
\[
\lim_{X \to \infty} \frac{1}{X} \# \left\{ 1 \leq n \leq X : \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq \gamma \right\} = G(\gamma),
\]
where
\[
G(\gamma) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\gamma} e^{-\frac{t^2}{2}} dt
\]
is the Gaussian distribution function.

Erdős and Kac proved this theorem by a probabilistic idea, building upon the work of Hardy and Ramanujan ([10]) and Turán ([21]) on the normal order of \( \omega(n) \). Since then there has been a very rich literature on various aspects of the Erdős-Kac theorem (see, for example, [1, 9, 11, 13, 14, 15, 16, 17, 19, 20]). Interested readers can refer to Granville and Soundararajan’s paper [8] for the most recent account and Elliot’s monograph [6] for a comprehensive treatment of the subject.

In particular, Halberstam in [9] proved that
\[
\lim_{X \to \infty} \frac{1}{X} \# \left\{ n : 1 \leq n \leq X, \frac{\omega(g(n)) - A(n)}{\sqrt{B(n)}} \leq \gamma \right\} = G(\gamma),
\]
where \( g(x) \in \mathbb{Z}[x] \) is an irreducible polynomial,
\[
A(n) = \sum_{p \leq n} \frac{r(p)}{p}, \quad B(n) = \sum_{p \leq n} \frac{r(p)^2}{p},
\]
and \( r(p) \) is the number of solutions of \( g(m) \equiv 0 \) (mod \( p \)), \( 0 \leq m < p \).

In a recent paper [3] Bourgain, Gamburd and Sarnak showed among other things that a large family of polynomials is “factor finite”; that is, the subset at which the polynomial has a bounded number of prime factors is Zariski dense in the orbit obtained by translation and affine linear transformation. By adapting their
proves and applying a criterion of Liu ([15]), in this paper we obtain two versions of the Erdős-Kac type theorem for polynomials of several variables.

To state the first result, we need some notation.

For an additive subgroup $\Lambda \subset \mathbb{Z}^n$ of rank $k$ $(1 \leq k \leq n)$, explicitly given by $\Lambda = \mathbb{Z}_{e_1} \oplus \cdots \oplus \mathbb{Z}_{e_k}$ for $\mathbb{Q}$-linearly independent vectors $e_1, \ldots, e_k \in \mathbb{Z}^n$, we denote by $V = \text{Zcl}(\Lambda)$ the Zariski closure of $\Lambda$ in the affine space $\mathbb{A}^n$ over $\mathbb{Q}$. For any $b \in \mathbb{Z}^n$, denote $\mathcal{O}_b = \Lambda + b$ and for any $L > 0$, denote

$$\mathcal{O}_b(L) = \{ y_1 e_1 + \cdots + y_k e_k + b \in \mathcal{O}_b : |y_i| \leq L, y_i \in \mathbb{Z}, 1 \leq i \leq k \}.$$

**Theorem 1.** Let $\Lambda$ be as above. Suppose each of the polynomials $f_1, \ldots, f_t \in \mathbb{Z}[x_1, \ldots, x_n]$ generates a distinct prime ideal in the coordinate ring $\mathbb{Q}[V]$. Let $f = f_1 \cdots f_t$. Then for any $b \in \mathbb{Z}^n$ and for any $\gamma \in \mathbb{R}$, we have

$$\lim_{L \to \infty} \frac{1}{\# \mathcal{O}_b(L)} \# \left\{ x \in \mathcal{O}_b(L) : \frac{\omega(f(x)) - t \log \log L}{\sqrt{t \log \log L}} \leq \gamma \right\} = G(\gamma).$$

When $k = n = 1$, Theorem 1 coincides with ([11]) in the special case that $g(x) \in \mathbb{Z}[x]$ is absolutely irreducible. As another example we may choose $\Lambda = \mathbb{Z}^2$ and $f_i(x, y) = x^i - y$ for $1 \leq i \leq t$. One sees that this choice of $\Lambda$ and $f_i$’s satisfies all the above conditions.

To state the second result, we use the following notation.

Let $\Lambda \subset \text{GL}(n, \mathbb{Z})$ be a free subgroup generated by the $d$ elements $A_1, \ldots, A_d$. Suppose the Zariski closure $G = \text{Zcl}(\Lambda)$ is isomorphic to $\text{SL}_2$ over $\mathbb{Q}$. Given a matrix $b \in \text{Mat}_{m \times n}(\mathbb{Z})$, $\Lambda$ acts on $b$ by right multiplication. Suppose $\text{Stab}_b(\Lambda)$ is trivial and the $G$ orbit $V = b \cdot G$ is Zariski closed and hence defines a variety over $\mathbb{Q}$. Assume $\dim V > 0$. Denote $\mathcal{O}_b(L) = b \cdot \Lambda$. We turn $\mathcal{O}_b$ into a $2d$-regular tree by joining the vertex $x \in \mathcal{O}_b$ with the vertices $x \cdot A_1, \ldots, x \cdot A_d$. (This is indeed a tree because $\Lambda$ is free on the generators and $\text{Stab}_b(\Lambda)$ is trivial.) For $x, y \in \mathcal{O}_b$, let $v(x, y)$ denote the distance in the tree from $x$ to $y$. For any $L > 0$, we denote

$$\mathcal{O}_b(L) = \{ x \in \mathcal{O}_b : v(x, b) \leq \log L \}.$$

**Theorem 2.** Let $\Lambda, b$ be as above. Suppose each of the polynomials $f_1, \ldots, f_t \in \mathbb{Z}[x_1, \ldots, x_{mn}]$ generates a distinct prime ideal in the coordinate ring $\mathbb{Q}[V]$, and let $f = f_1 \cdots f_t$. Then for any $\gamma \in \mathbb{R}$, we have

$$\lim_{L \to \infty} \frac{1}{\# \mathcal{O}_b(L)} \# \left\{ x \in \mathcal{O}_b(L) : \frac{\omega(f(x)) - t \log \log L}{\sqrt{t \log \log L}} \leq \gamma \right\} = G(\gamma).$$

As an example we may choose $b$ to be the $2 \times 2$ identity matrix, $f_i(x_1, x_2, x_3, x_4) = x_i^4 - x_4$ for each $1 \leq i \leq t$ and the subgroup $\Lambda \subset \text{SL}(2, \mathbb{Z})$ to be generated by two elements:

$$\Lambda = \left\langle \left[ \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right], \left[ \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right] \right\rangle.$$

Since $\Lambda$ is a non-elementary subgroup of $\text{SL}(2, \mathbb{Z})$ and $\Lambda \subset \Gamma(2)$, it is known that $\text{Zcl}(\Lambda) = \text{SL}_2$ and $\Lambda$ is a free group ([2]). One can check that the $f_i$’s generate distinct prime ideals in $\mathbb{Q}[V]$ and $\Lambda$, and the $f_i$’s and $b$ satisfy the conditions of Theorem 2.

This paper is organized as follows. Liu’s criterion is briefly reviewed in Section 2. In Section 3, we use it to prove Theorem 1 by adapting the sieving process of the proof of Theorem 1.6 in [3]. Since the proof of Theorem 2 is similar, it is sketched in Section 4.
2. Preliminaries

We shall need the following criterion obtained by Liu [15]. For completeness and for later applications we reproduce the statement with some adjustments.

Let \( \mathcal{O} \) be an infinite set. For any \( L > 1 \), assign a finite subset \( \mathcal{O}(L) \subseteq \mathcal{O} \) such that \( \#\mathcal{O}(L) \to \infty \) as \( L \to \infty \) and \( \#\mathcal{O}(L^{1/2}) = o(\#\mathcal{O}(L)) \). Let \( f : \mathcal{O} \to \mathbb{Z} \setminus \{0\} \) be a map. Put \( X = X(L) = \#\mathcal{O}(L) \) and write, for each prime \( l \),

\[
\frac{1}{X} \# \{ n \in \mathcal{O}(L) : f(n) \text{ is divisible by } l \} = \lambda_l(X) + e_l(X)
\]
as a sum of the major term \( \lambda_l(X) \) and the error term \( e_l(X) \). For any \( u \) distinct primes \( l_1, l_2, \ldots, l_u \), we write

\[
\frac{1}{X} \# \{ n \in \mathcal{O}(L) : f(n) \text{ is divisible by } l_1 l_2 \cdots l_u \} = \prod_{i=1}^{u} \lambda_{l_i}(X) + e_{l_1 l_2 \cdots l_u}(X).
\]

To ease our notation, the dependence on \( X \) will be dropped when there is no ambiguity.

In order to gain information on the distribution of \( \omega(f(n)) \), some control on \( \lambda_l \) and \( e_l \) is needed. Liu’s criterion uses the conditions below.

Suppose there exist absolute constants \( \beta, c \), where \( 0 < \beta \leq 1 \) and \( c > 0 \), and a function \( Y = Y(X) \leq X^{\beta} \) such that the following hold:

(i) For each \( n \in \mathcal{O}(L) \), the number of distinct prime divisors \( l \) of \( f(n) \) with \( l > X^{\beta} \) is bounded uniformly.

(ii) \( \sum_{Y < \ell \leq X^\beta} \lambda_l = o((\log \log X)^{1/2}) \).

(iii) \( \sum_{Y < \ell \leq X^\beta} |e_l| = o((\log \log X)^{1/2}) \).

(iv) \( \sum_{Y \leq \ell} \lambda_l = c \log \log X + o((\log \log X)^{1/2}) \).

(v) \( \sum_{Y \leq \ell} \lambda_l^2 = o((\log \log X)^{1/2}) \).

The sums in (ii)–(v) are over primes \( l \) in the given range.

(vi) For any \( r \in \mathbb{N} \) and any integer \( u \) with \( 1 \leq u \leq r \), we have

\[
\lim_{X \to \infty} \frac{1}{(\log \log X)^{r/2}} \sum_{l_1 \cdots l_u} |e_{l_1 \cdots l_u}| = 0,
\]

where for each \( u \), the sum \( \sum_{l_1 \cdots l_u} \) extends over all \( u \) distinct primes \( l_1, l_2, \ldots, l_u \) with \( l_i \leq Y \).

**Theorem 3** (Liu [15, Theorem 3]). If \( \mathcal{O} \) and \( f : \mathcal{O} \to \mathbb{Z} \setminus \{0\} \) satisfy all the above conditions, then for \( \gamma \in \mathbb{R} \), we have

\[
\lim_{L \to \infty} \frac{1}{X(L)} \# \left\{ n \in \mathcal{O}(L) : \frac{\omega(f(n)) - c \log \log X(L)}{\sqrt{c \log \log X(L)}} \leq \gamma \right\} = G(\gamma).
\]

While the conditions of Theorem 3 may appear complicated, in our applications, the terms \( \lambda_l \) and \( e_l \) can be easily identified and the conditions easily verified, as we shall see in the proofs of Theorems 1 and 2 below.

3. Proof of Theorem 1

We denote the basis \( \mathbf{x}_i, 1 \leq i \leq k, \) of \( \Lambda \) by \( \mathbf{x}_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n \). Put

\[
A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{kn} \end{pmatrix},
\]

which is a matrix of rank \(k\). For a row vector \(y\), let \(|y|\) be the maximum modulus of its components. Then for \(L\) large, denote

\[
\mathcal{O}_b(L) = \{yA + b : y \in \mathbb{Z}^k, |y| \leq L\}.
\]

We write \(X\) for \(#\mathcal{O}_b(L) = (2[L] + 1)^k\). To apply Theorem [3], one needs to estimate, for each square-free integer \(d\), the sum

\[
\sum_{x \in \mathcal{O}_b(L) \atop f(x) \equiv 0 (\text{mod } d)} 1 = \sum_{y \in \mathbb{Z}^k \atop |y| \leq L} \sum_{x \equiv b \equiv 0 (\text{mod } d) \atop f(xA + b) \equiv 0 (\text{mod } d)} 1.
\]

Suppose \(d \leq L\). The inner sum can be estimated as

\[
\frac{(2[L] + 1)^k}{d^k} + O \left( \frac{(2[L] + 1)^{k-1}}{d^{k-1}} \right) = \frac{X}{d^k} + O \left( \frac{X^{1-\frac{1}{2}}}{d^{k-1}} \right).
\]

Since the affine variety \(V' = V + b\) is absolutely irreducible, and the polynomials \(f_1, \ldots, f_t\) generate distinct prime ideals in the coordinate ring \(\bar{\mathbb{Q}}[V]\), one sees that all the varieties

\[
W_i = V' \cap \{ f_i = 0 \}, \quad i = 1, 2, \ldots, t,
\]

are defined over \(\bar{\mathbb{Q}}\), absolutely irreducible, and of dimension equal to \(\dim V' - 1 = k - 1 \geq 0\). Consider the reduction of the varieties \(V', W_i\) (mod \(p\)). According to Noether’s theorem [18], for \(p\) outside a finite set \(S_1\) of primes, the reductions of \(V'\) and \(W_i, i = 1, \ldots, t\), yield absolutely irreducible affine varieties over \(\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}\). Denote by \(V'(\mathbb{F}_p), V'(\mathbb{Z}/d\mathbb{Z})\), etc., the reduction of the varieties in the corresponding ring. By the Lang-Weil Theorem [12] we have that for \(p \not\in S_1\),

\[
\begin{align*}
\#V'(\mathbb{Z}/p\mathbb{Z}) &= p^k + O \left( p^{k-\frac{1}{2}} \right), \\
\#W_i(\mathbb{Z}/p\mathbb{Z}) &= p^{k-1} + O \left( p^{k-\frac{1}{2}} \right).
\end{align*}
\]

Since the map

\[
\begin{array}{c}
\mathbb{A}^k \\
y
\end{array} \quad \rightarrow \quad \begin{array}{c}
V' \\
yA + b
\end{array}
\]

is a bijection, one obtains

\[
\sum_{y \in (\mathbb{Z}/d\mathbb{Z})^k \atop f(yA + b) \equiv 0 (\text{mod } d)} 1 = \sum_{y \in V'(\mathbb{Z}/d\mathbb{Z}) \atop f(y) \equiv 0 (\text{mod } d)} 1 = \#W(\mathbb{Z}/d\mathbb{Z}),
\]

where

\[
W(\mathbb{Z}/d\mathbb{Z}) = \{y \in V'(\mathbb{Z}/d\mathbb{Z}) : f(y) \equiv 0 \pmod{d}\}.
\]

Let

\[
\lambda_d = \frac{\#W(\mathbb{Z}/d\mathbb{Z})}{d^k}.
\]

By the Chinese Remainder Theorem, \(\lambda_d\) is multiplicative for \(d\) coprime to \(\prod_{p \in S_1} p\). Since

\[
W(\mathbb{Z}/d\mathbb{Z}) = \bigcup_{i=1}^t W_i(\mathbb{Z}/d\mathbb{Z}),
\]
for such square-free \( d \) one has

\[
\sum_{i=1}^{t} \# W_i(\mathbb{Z}/d\mathbb{Z}) \leq \sum_{i=1}^{t} \prod_{p \mid d} \# W_i(\mathbb{Z}/p\mathbb{Z}) = \prod_{p \mid d} \left( p^{k-1} + O(p^{k-3/2}) \right) \ll d^{k-1+\varepsilon}.
\]

Therefore for \( d \leq L \) and \( \gcd \left( d, \prod_{p \in S_1} p \right) = 1 \), we obtain

\[
(3.1) \quad \sum_{\xi \in O_\mathbb{Z}(L)} 1 = X(\lambda_d + e_d), \quad \text{where} \quad e_d \ll d^\varepsilon X^{-\frac{1}{2}}.
\]

It follows from Lemma 3.1 below that the estimate \((3.1)\) still holds if on the left-hand side the points \( x \in O_\mathbb{Z}(L) \) such that \( f(x) = \lambda \) are removed. Thus we may assume that \( f(x) \neq \lambda \) for all \( x \in O_\mathbb{Z}(L) \). Now we return to \( \lambda_d \). For \( d = \lambda \) a prime and \( l \leq S_1 \) we have

\[
W(\mathbb{Z}/l\mathbb{Z}) = \bigcup_{i=1}^{t} W_i(\mathbb{Z}/l\mathbb{Z}).
\]

For fixed \( i \neq j \), the algebraic subset \( W' = W_i(\mathbb{Z}/l\mathbb{Z}) \cap W_j(\mathbb{Z}/l\mathbb{Z}) \) is defined over the finite field \( \mathbb{F}_l = \mathbb{Z}/l\mathbb{Z} \) and has dimension at most \( k - 2 \). Then it follows from Lemma 2.1 of [3] that

\[
\# (W_i(\mathbb{Z}/l\mathbb{Z}) \cap W_j(\mathbb{Z}/l\mathbb{Z})) \ll l^{k-2},
\]

where the implied constant depends on \( f \) and \( V \) only. By the inclusion-exclusion principle,

\[
\sum_{i=1}^{t} \# W_i(\mathbb{Z}/l\mathbb{Z}) - \sum_{1 \leq i < j \leq t} \# (W_i(\mathbb{Z}/l\mathbb{Z}) \cap W_j(\mathbb{Z}/l\mathbb{Z})) \leq \# W(\mathbb{Z}/l\mathbb{Z}) \leq \sum_{i=1}^{t} \# W_i(\mathbb{Z}/l\mathbb{Z}),
\]

from which one obtains

\[
\# W(\mathbb{Z}/l\mathbb{Z}) = tl^{k-1} + O \left( l^{k-\frac{2}{3}} \right).
\]

This implies that

\[
(3.2) \quad \lambda_l = t + O \left( l^{-\frac{2}{3}} \right).
\]

Using \((3.1)\) and \((3.2)\) and choosing

\[
Y = \exp \left( \frac{\log X}{\log \log X} \right), \quad \beta = \frac{1}{2k},
\]

one can verify the conditions (i)–(vi) of Theorem 3 for \( f \) and \( O_\mathbb{Z} \). For example, for (i), noticing that \( f \in \mathbb{Z}[x_1, \ldots, x_n] \) and \( x \in O_\mathbb{Z}(L) \), one has \( f(x) \ll L^{|x|} f \ll X^{\frac{1}{2k} - 1} \). Thus \( \sum_{l \mid f(x)} 1 \ll 1 \); i.e., the number of distinct prime divisors \( l \) of \( f(x) \) with \( l > X^{\beta} \).
is bounded uniformly. For (ii), noticing \( \log \log Y = \log \log X - \log \log \log X \), one has
\[
\sum_{Y < t < X^\beta} \lambda_t \leq \sum_{Y < t < X^\beta} t^{\beta} + O \left( t^{-\frac{1}{2}} \right) \ll t \log \log X^\beta - t \log \log Y + O(1),
\]
which is \( o((\log \log X)^{1/2}) \) as \( X \) goes to infinity. The conditions (iii)--(v) can be verified similarly.

Finally, for (vi), for any fixed \( r \in \mathbb{N} \) and \( 1 \leq u \leq r \),
\[
\sum_{l_1 \leq Y} |e_{l_1, \ldots, l_u}| \leq \epsilon \sum_{l_1 \leq Y} X^{-\frac{1}{r}} (l_1 \cdots l_u)^\epsilon \leq X^{-\frac{1}{r}} Y^{(1+\epsilon)/r} \leq X^{-\frac{1}{r}} (\log X)^{2r},
\]
which is \( o((\log \log X)^{-r/2}) \) as \( X \) goes to infinity.

Since the conditions (i)--(vi) of Theorem 3 are satisfied for \( f \) and \( \mathcal{O}_\mathbb{A} \), the desired conclusion follows from Theorem 3. The proof of Theorem 1 will be completed once we prove Lemma 3.1 below.

**Lemma 3.1.** Let \( W \) be a proper closed subset of \( V' = V + \mathbb{A} \) defined over \( \mathbb{Q} \). Then as \( L \to \infty \) one has\[
\#(\mathcal{O}_\mathbb{A}(L) \cap W) \ll X^{1 - \frac{1}{\text{dim} W}}.
\]

**Proof.** The proof is very similar to that of Proposition 3.2 in [3]. For the sake of completeness we give a detailed proof here.

Since \( V' = V + \mathbb{A} \) is irreducible, \( W \) is defined over \( \mathbb{Q} \) and has dimension at most \( \dim V - 1 = k - 1 \). Let \( W_1, \ldots, W_r \) be the irreducible components of \( W \). Then we have \( W = \bigcup_{j=1}^r W_j \), where the \( W_j \)'s are defined over a finite extension \( K \) of \( \mathbb{Q} \) and \( \dim W_j \leq k - 1 \) for each \( j \). For \( \mathcal{P} \) outside a finite set of prime ideals of the ring of integers \( \mathcal{O}_K \), \( W_j \) is an absolutely irreducible variety over the finite field \( \mathcal{O}_K / \mathcal{P} \) ([13]). Hence by [12] we have
\[
\#W_j(\mathcal{O}_K / \mathcal{P}) \ll N(\mathcal{P})^{\text{dim}(W_j)} \leq N(\mathcal{P})^{k-1}.
\]

Here, as usual, \( N(\mathcal{P}) = \#(\mathcal{O}_K / \mathcal{P}) \). Choose \( p \) so that it splits completely in \( K \) and let \( \mathcal{P}(p) \). Then \( \mathcal{O}_K / \mathcal{P} \cong \mathbb{F}_p \) and we have
\[
(3.3) \quad \#W(\mathbb{Z}/p\mathbb{Z}) \leq \sum_{j=1}^r \#W_j(\mathcal{O}_K / \mathcal{P}) \ll N(\mathcal{P})^{k-1} = p^{k-1}.
\]

Now proceed as before. For \( L \to \infty \) and any large \( p \) as above, we have
\[
\#(\mathcal{O}_\mathbb{A}(L) \cap W) = \sum_{\mathbf{z} \in \mathcal{O}_\mathbb{A}(L)} 1 \leq \sum_{\mathbf{z} \in W(\mathbb{Z}/p\mathbb{Z})} \sum_{y \in \mathbb{Z}^k, |y| \leq L} \sum_{yA+b \equiv \mathbf{z} \pmod{p}} 1.
\]

Similarly the right-hand side can be estimated as
\[
\sum_{\mathbf{z} \in W(\mathbb{Z}/p\mathbb{Z})} \left( \frac{X}{p^r} + O \left( \frac{X^{1-1/k}}{p^{k-1}} \right) \right).
\]

Hence for large \( p \) as in (3.3),
\[
\#(\mathcal{O}_\mathbb{A}(L) \cap W) \ll Xp^{-1} + X^{1-1/k}.
\]
By the Chebotarev density theorem (4) we can choose a \( p \) which splits completely in \( K \) and which satisfies
\[
X^{1/k}/2 \leq p \leq 2X^{1/k}.
\]
With this choice we get the bound claimed in Lemma 4.1. \( \square \)

4. Proof of Theorem 2

It is elementary that the number of points on a \( 2d \)-regular tree whose distance to a given vertex is at most \( \lceil \log L \rceil \) is equal to \( X = \#\mathcal{O}_L(L) = \frac{d(2d-1)^{\lceil \log L \rceil} - 1}{d-1} \). By the assumptions of Theorem 2 \( V \) is an absolutely irreducible affine variety defined over \( \mathbb{Q} \) with \( \dim V > 0 \) and \( f_1, \ldots, f_t \) generate distinct prime ideals in \( \mathbb{Q}[V] \). Hence for \( i = 1, \ldots, t, \) the varieties
\[
W_i = V \cap \{ f_i = 0 \}
\]
are defined over \( \mathbb{Q} \), absolutely irreducible, and of dimension equal to \( \dim V - 1 \). We consider the reduction of the varieties (mod \( p \)). By Noether’s theorem \( \mathcal{E} \) and the Lang-Weil Theorem \( \mathcal{T} \), there is a finite set \( S_1 \) of primes such that if \( p \not\in S_1 \), the varieties \( V(\mathbb{Z}/p\mathbb{Z}) \), \( W_i(\mathbb{Z}/p\mathbb{Z}) \) are absolutely irreducible and
\[
\#V(\mathbb{Z}/p\mathbb{Z}) = p^{\dim V} + O\left(p^{\dim V - \frac{\gamma}{2}}\right),
\]
\[
\#W_i(\mathbb{Z}/p\mathbb{Z}) = p^{\dim V - 1} + O\left(p^{\dim V - \frac{\gamma}{3}}\right).
\]

By using the uniform expansion property of \( \text{SL}_2 \) established in \( \mathcal{E} \) (or assuming a conjecture of Lubotzy for a more general setting), Bourgain, Gamburd and Sarnak proved (Proposition 3.1, \( \mathcal{E} \)) that
\[
\frac{1}{X} \sum_{x \in \mathcal{O}_L(L)} \sum_{\substack{v \mid x, \beta \leq L \\\ f(v) \equiv 0 \text{ (mod } d)}} 1 = \lambda_d + e_d,
\]
for square-free integers \( d \leq X \) coprime to \( \prod_{p \in S_2} p \). Here \( S_2 \) is a finite set of primes containing \( S_1 \) and
\[
\lambda_d = \frac{\# V_0(\mathbb{Z}/d\mathbb{Z})}{\# V(\mathbb{Z}/d\mathbb{Z})}, \quad e_d \ll d^{\dim V - 1 + \epsilon} X^{1 - \frac{\gamma}{2}},
\]
where
\[
V_0(\mathbb{Z}/d\mathbb{Z}) = \{ y \in V(\mathbb{Z}/d\mathbb{Z}) : f(y) \equiv 0 \text{ (mod } d)\},
\]
and the absolute constant \( \gamma < 1 \) is bounded below by some \( \delta > 0 \). Also by Proposition 3.2 in \( \mathcal{E} \), in the sum the terms \( x \in \mathcal{O}_L(L) \) with \( f(x) = 0 \) can also be omitted without altering (4.1). Clearly \( \lambda_d \) is a multiplicative function of \( d \) coprime to \( \prod_{p \in S_2} p \). With similar arguments as in the proof of Theorem 4 for \( d = l \) a prime and \( l \not\in S_2 \) we have
\[
\lambda_l = \frac{l}{l} + O\left(l^{-\frac{1}{2}}\right).
\]
Now using (4.1), (4.2), choosing \( Y = \exp(\log X/\log \log X) \) and \( \beta > 0 \) to be sufficiently small, we can similarly verify that the conditions (i)–(vi) of Theorem 3 for \( f \) and \( \mathcal{O}_L \) hold. This completes the proof of Theorem 2.

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