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NOTE ON CONVERSE QUANTUM ERGODICITY

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ABSTRACT. Quantum ergodicity asserts that eigenstates of a system with classical ergodic dynamics must be "equidistributed" in the phase space. In the present note we show that the converse is not true. We provide an example of billiards which are quantum ergodic but not classically ergodic.

1. Converse quantum ergodicity

It is well known that classical ergodicity implies quantum ergodicity. But the converse question also makes sense [MKZ, Z]: *Whether quantum ergodic (QE) systems are necessarily classically ergodic?* The purpose of this note is to give a counterexample. Namely, we show that there exists a certain class of planar QE billiards which are not classically ergodic. To the best of our knowledge this is the first example of such a system.

Let Ω be a bounded domain in $\mathbb{R}^2$ with a piecewise smooth boundary $\partial\Omega = \bigcup_{i=1}^N \partial\Omega_i$ composed of N smooth pieces. We will consider eigenfunctions of the Laplacian Δ in Ω with either Dirichlet or Neumann boundary conditions at each piece $\partial\Omega_i$. Let $\partial\Omega_i$, $i = 1, \ldots, \ell$ be the parts of $\partial\Omega$ with Dirichlet boundary conditions. Then the eigenvalue problem

$$\begin{aligned} &\Delta\varphi_j = \lambda_j^2\varphi_j, \qquad \varphi_j \in L^2(\Omega), \\ &\varphi_j|_{\partial\Omega_k} = 0,\ k = 1,\ldots,\ell, \qquad \partial_n\varphi_j|_{\partial\Omega_k} = 0,\ k = \ell+1,\ldots N, \end{aligned} \tag{1.1}$$

defines the quantum billiard in Ω. We will denote by Δ_Ω the corresponding Laplace operator with the boundary conditions (1.1). Let $\mathbf{Op}(a) \in \Psi^0(\mathbb{R}^2)$ be a pseudo-differential operator of order 0 with the principle symbol a. Quantum ergodicity is concerned with the convergence of the quantum averages of $\mathbf{Op}(a)$ over an orthonormal basis of eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ of Δ_Ω to the classical averages of a over the unit cotangent bundle $S^*\Omega$:

$$\langle \mathbf{Op}(a)\varphi_{j_k}, \varphi_{j_k}\rangle = \int_{S^*\Omega} a\, d\nu_k, \qquad \text{as} \quad j_k \to \infty. \tag{1.2}$$

It is known that any weak-$*$ limit of the sequence ν_k must be a probability measure on $S^*\Omega$ invariant under the billiard flow Φ^t in Ω; see, e.g., [Z]. Δ_Ω is then called

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QE if for almost every (a.e.) sequence of $j_k \to \infty$, ν_k converges to the normalized Liouville measure ν_{Liuv}. Here a.e. means that there exists a sequence of integers of density one such that for any of its subsequences the limiting measure is ν_{Liuv}. If, moreover, every limiting measure in (1.2) is ν_{Liuv} the system is called quantum unique ergodic (QUE).

The quantum ergodicity theorem, first stated by Schnirelman [S1] for smooth Riemannian manifolds and later extended to the billiards with piecewise boundaries [ZZ] asserts that Δ_Ω is QE if Φ^t is classically ergodic. So the converse question seems to be quite natural. Assuming Δ_Ω is QE does it mean that Φ^t must be classically ergodic? The following result provides conditions for classical ergodicity.

Theorem 1.1 ([Z])**.** *Φ^t is ergodic if and only if for every observable $\mathbf{Op}(a) \in \Psi^0(\mathbb{R}^2)$, $\bar{a} = \int a(x)\, d\nu_{\mathrm{Liuv}}$:*

(i) $\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \le \lambda} |\langle \varphi_n, \mathbf{Op}\,(a)\,\varphi_n\rangle - \bar{a}|^2 = 0.$

(ii) $\forall\epsilon, \exists\delta,\ \lim_{\lambda\to\infty} \sup \frac{1}{N(\lambda)} \sum_{\substack{n\neq k,\, \lambda_n,\lambda_k \le \lambda \\ |\lambda_n-\lambda_k|<\delta}} |\langle \varphi_n, \mathbf{Op}\,(a)\,\varphi_k\rangle|^2 < \epsilon.$

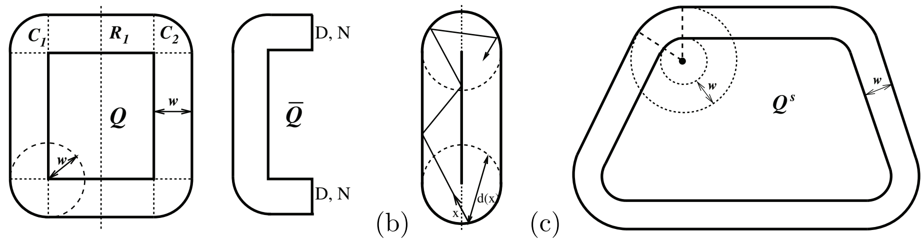
Note that Condition (i) on the diagonal terms is, in fact, equivalent to quantum ergodicity. Thus, an affirmative answer to the converse quantum ergodicity question would mean that Condition (ii) on the non-diagonal terms is satisfied automatically if a system is QE. As we show, this is not the case. Below we give an example of non-ergodic billiards which are QE.

2. Hyperbolic billiards with two ergodic components

Let $Q = \bigcup_i C_i \cup R_i$ be a multiply connected domain as shown in Figs. 1a,b. It is composed of a number of rectangles R_i having the same width w separated by sectors of circles C_i of the radii w. Hence, the exterior ∂Q_{ex} boundary of Q consists of straight lines $\Gamma^{(i)}_{\mathrm{lin}}$ and circle arcs $\Gamma^{(i)}_{\mathrm{arc}}$ connected to each other under the angles π. In addition, we will impose the following conditions. For each circular component $\Gamma^{(i)}_{\mathrm{arc}}$ of ∂Q_{ex} the entire circle containing $\Gamma^{(i)}_{\mathrm{arc}}$ must be inside of ∂Q_{ex}. We will assume that Q has a reflectional symmetry and denote by $\overline{Q}$ a symmetric one half of Q.

Proposition 2.1. *Let Q be as above. Then the classical billiard in Q is hyperbolic with a positive Lyapunov exponent a.e. and contains exactly two ergodic components.*

Proof. First, note that the geometry of Q is such that the billiard ball launched in the clockwise (resp. anti-clockwise) direction will move that way forever. That means the billiard phase space S^*Q can be split into two invariant regions S^*Q_+ and S^*Q_- corresponding to the clockwise and anti-clockwise motions. These two regions are separated from each other by the two-dimensional surface S^*Q_0 composed of all vectors $v \in S^*Q$ whose forward ($t \ge 0$) or backward ($t < 0$) trajectory $\Phi^t \cdot v$ is orthogonal to the billiard boundary at the first bouncing point. Note that non-singular points v of S^*Q_0 (such that $\Phi^t \cdot v$ is defined for all t) form an invariant set under the billiard flow. In spite of the lack of ergodicity, it can be shown by a standard application of the cone field method [W] that the billiard in Q is hyperbolic with a positive Lyapunov exponent a.e. Let us outline the proof. Instead of dealing with the flow Φ^t, it is convenient to consider the corresponding billiard map ϕ acting on a Poincaré section V of S^*Q. More specifically, define the billiard



(a) (b) (c)

FIGURE 1. Examples of uni-directional chaotic billiards Q with piecewise C^0 (a), (b) and C^1 (c) smooth boundaries. The desymmetrized one half of the billiard and the definition of the parameter d are depicted at Figures (a) and (b) respectively.

phase space V as the set of unit vectors attached to the circular parts $\bigcup_i \Gamma^{(i)}_{\rm arc}$ of the boundary and directed inward in Q. Then the action of ϕ on $v \in V$ is defined in the usual way by $\phi \cdot v \equiv \Phi^{t_k+0} \cdot v$, where t_k is the time up to the next bouncing point with $\Gamma^{(i)}_{\rm arc}$'s. (The subset of V where ϕ^n is not defined for all n has measure zero and can be subtracted from V.) The hyperbolicity of Φ^t is implied by the corresponding property of ϕ. To prove the latter we need to find a cone field $\mathcal{C}$ in TV which is preserved under the derivative map $D\phi$. That can be easily achieved through the geometrical parameterization of the cones in terms of the curvatures of infinitesimal beams. Namely, a cone $\mathcal{C}(x) \subset TV_x$ at the point $x \in V$ is defined by an interval $\mathcal{X}(x) = [\chi_1(x), \chi_2(x)]$ for the curvature of the infinitesimal beams with the directions given by x; see [W] for details. An appropriate choice of the cone field $\mathcal{X}(x)$ for the considered family of billiards is $\mathcal{X}(x) = [0, 2d^{-1}(x)]$, where $d(x)$ is the length of the circle chord defined by $x \in V$, as shown in Fig. 1. Then the cone field preservation amounts to $\mathcal{X}(\phi(x)) \subseteq \phi \cdot (\mathcal{X}(x))$ and can be checked by an elementary application of the laws of geometric optics. Furthermore, one can show that this cone field is eventually strictly preserving, i.e., $\mathcal{X}(\phi^n(x)) \subset \phi^n \cdot (\mathcal{X}(x))$ a.e. for some n. By this follows the hyperbolicity of the system. Moreover, the desymmetrized half of the billiard $\overline{Q}$ is a simply connected domain and, in addition to being hyperbolic, contains the unique ergodic component as can be shown by standard arguments for billiards of that type; see [B]. (Note, for instance, that for the billiard in Fig. 1b, $\overline{Q}$ is just a half of the Bunimovich stadium.) This means that the full billiard Q contains exactly two ergodic components corresponding to the clockwise and anti-clockwise motion respectively. □

Remark 2.2. The conditions imposed on the domain Q are overrestrictive and can be considerably softened to include a wider class of billiard shapes. In particular, Proposition 2.1 can also be proven (under some additional geometrical conditions on Q) if circular parts of the domain Q are traded off for strips of constant width with a smooth (bounded) curvature; see Fig. 1c for an example. The resulting domains then have C^1 smooth boundaries (rather than C^0). The proof in that case, however, requires a more elaborate choice of the cone field and we avoid it for the sake of simplicity. Also, the requirement of reflectional symmetry for the billiards seems to be a technical one. It is convenient to deal with such shapes, as one can utilize the results on QE for the desymmetrized half billiard. We believe, however, that the main result (Proposition 3.1) should also hold for hyperbolic billiards of constant width without reflectional symmetries.

3. Proof of QE

Now, consider the quantum billiard in Q, with some boundary conditions of the type (1.1). Let us show that the billiard is QE.

Proposition 3.1. *Let Q be as above. Then the quantum billiard in Q is QE.*

Proof. The basic idea here is to use ergodicity of the half billiard $\overline{Q}$. In conjunction with Δ_Q we will consider quantum billiards in $\overline{Q}$ with the Dirichlet and Neumann boundary conditions at the symmetry line; see Fig. 1a. Let us denote by $\Delta_{\rm D}$, $\Delta_{\rm N}$ the corresponding Laplace operators and by $\varphi_n^{\rm D}$, $\varphi_n^{\rm N}$ their eigenfunctions. By symmetry of Q the spectrum of Δ_Q can be decomposed into the spectra of $\Delta_{\rm D}$, $\Delta_{\rm N}$. Accordingly, each eigenfunction φ_n of Δ_Q restricted to $\overline{Q}$ is an eigenfunction either of $\Delta_{\rm D}$ or of $\Delta_{\rm N}$. Hence for an observable $\mathbf{Op}(a) \in \Psi^0(\mathbb{R}^2)$, the quantum average (1.2) can be separated into the sums over the Dirichlet and Neumann parts of the spectrum:

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n\le\lambda} |\langle\varphi_n, \mathbf{Op}\,(a)\,\varphi_n\rangle - \bar{a}|^2 \tag{3.1}$$
$$= \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \Big(\sum_{\substack{\lambda_n\in\mathrm{Spec}(\Delta_{\rm D})\\ \lambda_n\le\lambda}} |\langle\varphi_n, \mathbf{Op}\,(a)\,\varphi_n\rangle - \bar{a}|^2 + \sum_{\substack{\lambda_n\in\mathrm{Spec}(\Delta_{\rm N})\\ \lambda_n\le\lambda}} |\langle\varphi_n, \mathbf{Op}\,(a)\,\varphi_n\rangle - \bar{a}|^2 \Big).$$

Now, let $\overline{Q}_{\rm L}$, $\overline{Q}_{\rm R}$ be two mirror-symmetric halves of Q (each $\overline{Q}_{\rm L,R}$ is a copy of $\overline{Q}$) and let $a_{\rm L,R}$ denote the restriction of a to $\overline{Q}_{\rm L,R}$. Then for the first sum in (3.1) one has:

$$\sum_{\substack{\lambda_n\in\mathrm{Spec}(\Delta_{\rm D})\\ \lambda_n\le\lambda}} |\langle\varphi_n, \mathbf{Op}\,(a)\,\varphi_n\rangle - \bar{a}|^2 \tag{3.2}$$
$$\le 2 \sum_{\substack{\lambda_n\in\mathrm{Spec}(\Delta_{\rm D})\\ \lambda_n\le\lambda}} |\langle\varphi_n^{\rm D}, \mathbf{Op}\,(a_{\rm L})\,\varphi_n^{\rm D}\rangle - \bar{a}_{\rm L}|^2 + |\langle\varphi_n^{\rm D}, \mathbf{Op}\,(a_{\rm R})\,\varphi_n^{\rm D}\rangle - \bar{a}_{\rm R}|^2 .$$

Since the classical billiard in $\overline{Q}$ is ergodic, it follows immediately by Theorem 1.1 that

$$\lim_{\lambda\to\infty} \frac{1}{N_{\rm D}(\lambda)} \sum_{\substack{\lambda_n\in\mathrm{Spec}(\Delta_{\rm D})\\ \lambda_n\le\lambda}} |\langle\varphi_n^{\rm D}, \mathbf{Op}\,(a_{\rm L,R})\,\varphi_n^{\rm D}\rangle - \bar{a}_{\rm L,R}|^2 = 0, \tag{3.3}$$

where $N_{\rm D}(\lambda)$ is the spectral counting function of $\Delta_{\rm D}$. Together with $\lim_{\lambda\to\infty} \frac{N_{\rm D}(\lambda)}{N(\lambda)} = 1/2$ this implies that the first term in (3.1) must vanish. In complete analogy, the same holds for the second term in (3.1). Therefore the limit (3.1) is, indeed, zero, implying quantum ergodicity for Δ_Q. □

Let us briefly comment on the reasons why QE in the billiards above coexists with the lack of classical ergodicity. As the first condition in Theorem 1.1 amounts to QE, this means that the second condition must fail. In order to clarify the source of this failure it is instructive, first, to consider billiards $Q^{\rm s}$ of constant width with smooth C^∞ (instead of C^1 or C^0) boundaries. These billiards have a seemingly similar phase space structure; however, they are not fully hyperbolic. The "chaotic" components of the dynamics inside clockwise $S^*Q^{\mathrm{s}}_+$ and anticlockwise $S^*Q^{\mathrm{s}}_-$ parts of the phase space are separated here by a three-dimensional dynamical barrier of KAM tori localized in the vicinity of the separation surface $S^*Q^{\mathrm{s}}_0$, [K, G]. As a result, it becomes possible to construct for the Laplace operator $\Delta_{Q^{\mathrm{s}}}$ pairs of quasimodes:

$$\Delta_{Q^{\mathrm{s}}}\varphi^+_j = \tilde{\lambda}^2_j\varphi^+_j + O(\lambda^{-\infty}), \qquad \Delta_{Q^{\mathrm{s}}}\varphi^-_j = \tilde{\lambda}^2_j\varphi^-_j + O(\lambda^{-\infty}), \tag{3.4}$$

whose Wigner transforms are concentrated in $S^*Q^{\mathrm{s}}_+$ and $S^*Q^{\mathrm{s}}_-$ respectively [S2, L, G]. Then by standard arguments (and under the assumption of the lack of systematic spectral degeneracies, see, e.g., [L]), symmetric and antisymmetric combinations of $\varphi^{\pm}_j$:

$$\varphi^s_j = \varphi^+_j + \varphi^-_j + r_1, \qquad \varphi^a_j = \varphi^+_j - \varphi^-_j + r_2, \qquad ||r_{1,2}|| = O(\lambda^{-\infty}) \tag{3.5}$$

must approximate two real quasidegenerate eigenfunctions φ^s_j, φ^a_j of $\Delta_{Q^{\mathrm{s}}}$ with the eigenvalues

$$|\lambda^s_j - \tilde{\lambda}_j| = O(\lambda^{-\infty}), \qquad |\lambda^a_j - \tilde{\lambda}_j| = O(\lambda^{-\infty}).$$

Thus, with the exception of a tiny sequence of eigenfunctions whose Wigner transform is localized at the separation region between $S^*Q^{\mathrm{s}}_+$ and $S^*Q^{\mathrm{s}}_-$, all other eigenfunctions turn out to be quasidegenerate.

For the billiards Q, with C^0 (or C^1) boundaries, one can actually use the same approach. However, here, unlike the billiards with smooth boundaries, the ergodic components in $S^*Q^{\mathrm{s}}_+$ and $S^*Q^{\mathrm{s}}_-$ are separated solely by the two-dimensional surface $S^*Q^{\mathrm{s}}_0$. Due to the diffraction effects at the singular points of the boundary, quasimodes can be constructed only with the discrepancies of finite order in λ. The numerical calculations and an argument in [VPR] suggest that in this case, $||r_{1,2}|| = O(\lambda^0)$ and $\lambda^s_j - \lambda^a_j = O(\lambda^{-1})$. Apparently, the eigenfunctions with such "weak" quasidegeneracies are responsible for the breaking of the second condition in Theorem 1.1. Indeed, for an observable a which is $+1$ inside of S^*Q_+ and -1 inside of S^*Q_- (with a smooth transition in between), Condition (ii) of Theorem 1.1 is violated if the error terms in (3.5) are sufficiently small (e.g., $\|r_{1,2}\| < 1/2$) for a set of eigenfunctions of non-zero counting density.

Finally, let us notice that all billiards considered here allow existence of bouncing ball modes. It is conjectured and widely believed that such systems are not QUE. Recently, in a remarkable paper by A. Hassell [H], it has been proved that "almost all" such billiards are, indeed, not QUE. Therefore, in view of the present result, a refined version of the converse quantum ergodicity question still makes sense: *Let Ω be a bounded domain on a Riemannian manifold such that a quantum billiard problem of the type* (1.1) *in Ω satisfies the QUE property. Does it mean the classical dynamics in Ω is necessarily ergodic?* Note that an affirmative answer to this question would imply in its turn that there are no billiards of constant width which are QUE.

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