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# THE AVERAGE BEHAVIOR OF FOURIER COEFFICIENTS OF CUSP FORMS OVER SPARSE SEQUENCES

HUIXUE LAO AND AYYADURAI SANKARANARAYANAN

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Abstract. Let $\lambda(n)$ be the $n$th normalized Fourier coefficient of a holomorphic Hecke eigenform $f(z) \in S_k(\Gamma)$. In this paper we are interested in the average behavior of $\lambda^2(n)$ over sparse sequences. By using the properties of symmetric power $L$-functions and their Rankin-Selberg $L$-functions, we are able to establish that for any $\varepsilon > 0$,

$$\sum_{n\leq x} \lambda^2(n^j) = c_{j-1}x + O\left(x^{1-\frac{2}{(j+1)^2+2}+\varepsilon}\right),$$

where $j = 2, 3, 4$.

## 1. Introduction and main results

Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight $k$ for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Suppose that $f(z)$ is an eigenfunction of all Hecke operators belonging to $S_k(\Gamma)$. Then the Hecke eigenform $f(z)$ has the following Fourier expansion at the cusp $\infty$:

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z},$$

where we normalize $f(z)$ such that $a(1) = 1$. Instead of $a(n)$, one often considers the normalized Fourier coefficient

$$\lambda(n) = \frac{a(n)}{n^{\frac{k-1}{2}}}.$$

Then $\lambda(n)$ is real and satisfies the multiplicative property

$$\lambda(m)\lambda(n) = \sum_{d|(m,n)} \lambda\left(\frac{mn}{d^2}\right), \tag{1.1}$$

where $m \geq 1$ and $n \geq 1$ are any integers. The Fourier coefficients of cusp forms are interesting objects. In 1974, P. Deligne [2] proved the Ramanujan-Petersson

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conjecture

$$|\lambda(n)| \le d(n), \tag{1.2}$$

where $d(n)$ is the divisor function. For the sum of normalized Fourier coefficients over natural numbers, Rankin [14] proved that

$$S(x) = \sum_{n \le x} \lambda(n) \ll x^{\frac{1}{3}} (\log x)^{-\delta},$$

where $0 < \delta < 0.06$.

In 2001, Ivić [5] studied the sum of normalized Fourier coefficients over squares, i.e.

$$S_2(x) = \sum_{n \le x} \lambda(n^2).$$

By (1.1), the Rankin-Selberg method, and the zero-free region of the Riemann zeta function, he successfully gave a nontrivial estimate

$$S_2(x) \ll_f x \exp\left(-A(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}\right),$$

where $A$ is a suitable positive constant.

Later Fomenko [3] mentioned that

$$S_2(x) \ll_f x^{\frac{1}{2}} (\log x)^3.$$

Recently Sankaranarayanan [16] showed that

$$S_2(x) \ll x^{\frac{3}{4}} (\log x)^{\frac{19}{2}} \log\log x$$

holds uniformly for any holomorphic cusp form of even integral weight $k$ for the full modular group satisfying $k \ll x^{\frac{1}{3}} (\log x)^{\frac{22}{3}}$. In the same paper, Sankaranarayanan mentioned that it is an open problem to give a nontrivial estimate for the sum of Fourier coefficients over cubes, i.e.

$$S_3(x) = \sum_{n \le x} \lambda(n^3).$$

Recently by using the properties of symmetric power $L$-functions, Lü [12] proved that for any $\varepsilon > 0$,

$$S_3(x) = \sum_{n \le x} \lambda(n^3) \ll_{f,\varepsilon} x^{\frac{3}{4}+\varepsilon}, \qquad S_4(x) = \sum_{n \le x} \lambda(n^4) \ll_{f,\varepsilon} x^{\frac{7}{9}+\varepsilon}.$$

On the other hand, Rankin [13] and Selberg [17] studied the average behavior of $\lambda^2(n)$ over natural numbers and showed that

$$\sum_{n \le x} \lambda^2(n) = cx + O_f(x^{\frac{3}{5}}).$$

Therefore a natural problem is what is the average behavior of $\lambda^2(n)$ over sparse sequences. In this paper we are interested in this problem, namely to study the asymptotic formula of the sum

$$\sum_{n \le x} \lambda^2(n^j),$$

where $j = 2, 3, 4$.

By using the properties of symmetric power $L$-functions and their Rankin-Selberg $L$-functions, which have been established in [4], [8], [9], [10], [11], and [18], we are able to establish the following results.

**Theorem 1.1.** *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight $k$ for the full modular group, and let $\lambda(n)$ denote its $n$th normalized Fourier coefficients. Then for any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \lambda^2(n^2) = c_1 x + O_{f,\varepsilon}(x^{\frac{9}{11}+\varepsilon}).$$

**Theorem 1.2.** *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight $k$ for the full modular group, and let $\lambda(n)$ denote its $n$th normalized Fourier coefficients. Then for any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \lambda^2(n^3) = c_2 x + O_{f,\varepsilon}(x^{\frac{8}{9}+\varepsilon}).$$

**Theorem 1.3.** *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight $k$ for the full modular group, and let $\lambda(n)$ denote its $n$th normalized Fourier coefficients. Then for any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \lambda^2(n^4) = c_3 x + O_{f,\varepsilon}(x^{\frac{25}{27}+\varepsilon}).$$

## 2. Some lemmas

**Lemma 2.1.** *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight $k$ for the full modular group, and let $\lambda(n)$ denote its $n$th normalized Fourier coefficients. For $j = 2, 3, 4$, we introduce*

$$L_j(s) = \sum_{n=1}^{\infty} \frac{\lambda^2(n^j)}{n^s} \tag{2.1}$$

*for $Re(s) > 1$. Let $L(\mathrm{sym}^j f, s)$ be the $j$th symmetric power $L$-function associated to $f$, and let $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ be the Rankin-Selberg $L$-function of* $\mathrm{sym}^j f$ *and* $\mathrm{sym}^j f$.

*Then we have that for* $\mathrm{Re}(s) > 1$,

$$L_j(s) = L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) U_j(s), \tag{2.2}$$

*where $U_j(s)$ converges uniformly and absolutely in the half-plane* $\mathrm{Re}(s) \geq 1/2 + \varepsilon$ *for any $\varepsilon > 0$.*

*Proof.* According to Deligne [2], for any prime number $p$ there are $\alpha(p)$ and $\beta(p)$ such that

$$\lambda(p) = \alpha(p) + \beta(p) \text{ and } |\alpha(p)| = \alpha(p)\beta(p) = 1. \tag{2.3}$$

Then it is easy to show that for $j \geq 1$,

$$\lambda(p^j) = \frac{\alpha(p)^{j+1} - \beta(p)^{j+1}}{\alpha(p) - \beta(p)} = \sum_{m=0}^{j} \alpha(p)^{j-m} \beta(p)^m.$$

In fact for any integer $j \geq 2$, from the theory of Hecke operators we have the following recursive relation:

$$\lambda(p^j) = \lambda(p^{j-1})\lambda(p) - \lambda(p^{j-2}).$$

By induction, we have

$$\lambda(p^j) = \frac{\alpha(p)^j - \beta(p)^j}{\alpha(p) - \beta(p)} \times (\alpha(p) + \beta(p)) - \frac{\alpha(p)^{j-1} - \beta(p)^{j-1}}{\alpha(p) - \beta(p)} = \frac{\alpha(p)^{j+1} - \beta(p)^{j+1}}{\alpha(p) - \beta(p)}.$$

Therefore we have

$$\lambda^2(p^j) = \left( \sum_{m=0}^{j} \alpha(p)^{j-m}\beta(p)^m \right)^2. \tag{2.4}$$

The $j$th symmetric power $L$-function attached to $f \in S_k(\Gamma)$ is defined as

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} (1 - \alpha(p)^{j-m}\beta(p)^m p^{-s})^{-1} \tag{2.5}$$

for $\mathrm{Re}(s) > 1$. The Rankin-Selberg $L$-function associated to $\mathrm{sym}^j f$ and $\mathrm{sym}^j f$ is defined as

$$L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} \prod_{u=0}^{j} (1 - \alpha(p)^{j-m}\beta(p)^m \alpha(p)^{j-u}\beta(p)^u p^{-s})^{-1} \tag{2.6}$$

for $\mathrm{Re}(s) > 1$. The product over primes also gives a Dirichlet series representation for $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$: for $\mathrm{Re}(s) > 1$,

$$L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(n)}{n^s},$$

where $\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(n)$ is a multiplicative function. From (2.3), we have

$$\left|\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(n)\right| \le d_{(j+1)^2}(n), \tag{2.7}$$

where $d_k(n)$ is the $n$th coefficient of the Dirichlet series $\zeta^k(s)$. Then we have that for $\mathrm{Re}(s) > 1$,

$$L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \prod_p \left( 1 + \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots \right). \tag{2.8}$$

By (2.6) and (2.8), we have

$$\begin{aligned} \lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p) &= \sum_{m=0}^{j} \sum_{u=0}^{j} \alpha(p)^{j-m}\beta(p)^m \alpha(p)^{j-u}\beta(p)^u \\ &= \left( \sum_{m=0}^{j} \alpha(p)^{j-m}\beta(p)^m \right)^2. \end{aligned} \tag{2.9}$$

From (2.4) and (2.9), we find that

$$\lambda^2(p^j) = \lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p). \tag{2.10}$$

From (1.2), we learn that

$$L_j(s) = \sum_{n=1}^{\infty} \frac{\lambda^2(n^j)}{n^s} \tag{2.11}$$

is absolutely convergent in the half-plane $\mathrm{Re}(s) > 1$. On noting that $\lambda^2(n^j)$ is a multiplicative function, we have that for $\mathrm{Re}(s) > 1$,

$$L_j(s) = \sum_{n=1}^{\infty} \frac{\lambda^2(n^j)}{n^s} = \prod_p \left(1 + \frac{\lambda^2(p^j)}{p^s} + \frac{\lambda^2(p^{2j})}{p^{2s}} + \cdots + \frac{\lambda^2(p^{kj})}{p^{ks}} + \cdots \right). \tag{2.12}$$

Therefore from (2.8), (2.10) and (2.12), we have that for $\mathrm{Re}(s) > 1$,

$$\begin{aligned} L_j(s) &= L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) \\ &\quad \times \prod_p \left(1 + \frac{\lambda^2(p^{2j}) - \lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p^2)}{p^{2s}} + \cdots \right) \\ &=: L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) U_j(s). \end{aligned}$$

From (1.2) and (2.7), it is obvious that $U_j(s)$ converges uniformly and absolutely in the half-plane $\mathrm{Re}(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. This completes the proof of Lemma 2.1. □

Based on the work of Cogdell and Michel [1], Lau and Wu [11] showed that for $j = 2, 3, 4$, $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ has a meromorphic continuation to the whole complex plane and satisfies a functional equation.

**Lemma 2.2.** *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight $k$. The Rankin-Selberg $L$-function associated to $\mathrm{sym}^j f$ and $\mathrm{sym}^j f$ is defined as (2.6). For $j = 2, 3, 4$, the Archimedean local factor of $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ is*

$$L_\infty(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \Gamma_{\mathbb{R}}(s)^{\delta_{2|j}} \Gamma_{\mathbb{C}}(s)^{[j/2]+\delta_{2\nmid j}} \prod_{v=1}^{j} \Gamma_{\mathbb{C}}(s + v(k-1))^{j-v+1},$$

*where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, $\delta_{2|j} = 1 - \delta_{2\nmid j}$, and*

$$\delta_{2\nmid j} = \begin{cases} 1, & \text{if } 2 \nmid j, \\ 0, & \text{otherwise} . \end{cases}$$

*Then the complete $L$-function*

$$\Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) =: L_\infty(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$$

*is entire except possibly for simple poles at $s = 0, 1$ and satisfies the functional equation*

$$\Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \epsilon_{\mathrm{sym}^j f \times \mathrm{sym}^j f} \Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, 1 - s)$$

*with $|\epsilon_{\mathrm{sym}^j f \times \mathrm{sym}^j f}| = 1$.*

*Proof.* This is Proposition 2.1 in Lau and Wu [11]. □

**Lemma 2.3.** *Let $j = 2, 3, 4$. Then for any $\varepsilon > 0$ and $0 \le \sigma \le 1$, we have*

$$L(\mathrm{sym}^j f \times \mathrm{sym}^j f, \sigma + it) \ll_{f,\varepsilon} (1 + |t|)^{\frac{(j+1)^2}{2}(1-\sigma)+\varepsilon}.$$

*Proof.* From Lemma 2.2, we can follow standard arguments to establish the convexity bound for $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, \sigma + it)$ in the critical strip $\frac{1}{2} \le \sigma \le 1$ (see, for example, Chapter 5 of [7]). □

**Lemma 2.4.** *Let $j = 2, 3, 4$. Then for $T \ge T_0$ (where $T_0$ is sufficiently large), we have the estimate*

$$\int_T^{2T} \left| L\left(\mathrm{sym}^j f \times \mathrm{sym}^j f, \frac{1}{2} + it\right)\right|^2 dt \ll_{f,\varepsilon} T^{\frac{(j+1)^2}{2}+\varepsilon},$$

*where $\varepsilon$ is any positive constant.*

*Proof.* From Lemma 2.2, we observe that the $L$-function $L\left(\mathrm{sym}^j f \times \mathrm{sym}^j f, s\right)$ is of degree $(j+1)^2$ and is being extended as an entire function except possibly with simple poles at $s = 0$ and $s = 1$. It also satisfies a nice functional equation of the Riemann zeta type, and thus we can write the functional equation here as

$$L\left(\mathrm{sym}^j f \times \mathrm{sym}^j f, s\right) = \chi(s) L\left(\mathrm{sym}^j f \times \mathrm{sym}^j f, 1 - s\right),$$

where

$$|\chi(s)| \asymp |t|^{\frac{(j+1)^2}{2}(1-2\sigma)} \quad \text{as} \quad |t| \to \infty$$

in any fixed strip $a \le \sigma \le b$. Now we follow the arguments of (i) of Theorem 4.1 of the paper by Sankaranarayanan [15]. The only necessary changes are that we need the free parameters $Y$ and $Y_1$ therein to be $Y = Y_1 = cT^{\frac{(j+1)^2}{2}}$, where $c$ is a suitable positive constant. This leads to the estimate of this lemma. □

## 3. Proof of Theorems 1.1.–1.3

Recall that for $j = 2, 3, 4$, we define

$$L_j(s) = \sum_{n=1}^{\infty} \frac{\lambda^2(n^j)}{n^s} \tag{3.1}$$

for $\mathrm{Re}(s) > 1$. From Lemma 2.1 and Lemma 2.2, we learn that $L_j(s) = L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) U_j(s)$ can be analytically continued to the half-plane $\mathrm{Re}(s) > 1/2$. In this region, $L_j(s)$ only has a simple pole $s = 1$.

Now we begin to prove our main results. By (3.1) and Perron's formula (see Proposition 5.54 in [7]), we have

$$\sum_{n \le x} \lambda^2(n^j) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L_j(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.2}$$

where $b = 1 + \varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later. Here we have used (1.2).

Next we move the integration to the parallel segment with $\mathrm{Re}(s) = \frac{1}{2} + \varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\le x}\lambda^2(n^j) &= \mathrm{Res}_{s=1}L_j(s)x + \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{\frac{1}{2}+\varepsilon-iT}\right\} L_j(s)\frac{x^s}{s}ds \\ &\quad + O\left(\frac{x^{1+\varepsilon}}{T}\right) \qquad (3.3)\\ &=: c_{j-1}x + I_1 + I_2 + I_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right).\end{aligned}$$

For $I_1$, by Lemma 2.1, we have

$$\begin{aligned} I_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\int_1^T \left|L(\mathrm{sym}^j f\times \mathrm{sym}^j f, 1/2+\varepsilon+it)U_j(1/2+\varepsilon+it)\right| t^{-1}dt \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\int_1^T \left|L(\mathrm{sym}^j f\times \mathrm{sym}^j f, 1/2+\varepsilon+it)\right| t^{-1}dt. \qquad (3.4)\end{aligned}$$

Then by the Cauchy-Schwarz inequality, Gabriel convexity and Lemma 2.4, we have

$$\begin{aligned} I_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\left\{\frac{1}{T_1}\int_{T_1/2}^{T_1}\left|L(\mathrm{sym}^j f\times \mathrm{sym}^j f, 1/2+\varepsilon+it)\right| dt\right\} \\ &\ll x^{\frac{1}{2}+\varepsilon}\log T\max_{T_1\le T}\left\{\frac{1}{T_1}\left(\int_{T_1/2}^{T_1}\left|L(\mathrm{sym}^j f\times \mathrm{sym}^j f, 1/2+\varepsilon+it)\right|^2 dt\right)^{\frac{1}{2}}\right. \qquad (3.5)\\ &\qquad\qquad \left.\times\left(\int_{T_1/2}^{T_1} 1dt\right)^{\frac{1}{2}}\right\} + x^{\frac{1}{2}+\varepsilon} \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}T^{\frac{(j+1)^2}{4}-\frac{1}{2}+\varepsilon} \ll x^{\frac{1}{2}+\varepsilon}T^{\frac{(j+1)^2}{4}-\frac{1}{2}+\varepsilon}.\end{aligned}$$

For the integrals over the horizontal segments, we use Lemma 2.3 to get

$$\begin{aligned} I_2+I_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^\sigma\left|L(\mathrm{sym}^j f\times \mathrm{sym}^j f, \sigma+iT)\right| T^{-1}d\sigma \\ &\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le b} x^\sigma T^{\frac{(j+1)^2}{2}(1-\sigma)+\varepsilon}T^{-1} = \max_{\frac{1}{2}+\varepsilon\le\sigma\le b}\left(\frac{x}{T^{\frac{(j+1)^2}{2}}}\right)^\sigma T^{\frac{(j+1)^2}{2}-1+\varepsilon} \qquad (3.6)\\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{(j+1)^2}{4}-1+\varepsilon}.\end{aligned}$$

From (3.3), (3.5) and (3.6), we have

$$\sum_{n\le x}\lambda^2(n^j) = c_{j-1}x + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(x^{\frac{1}{2}+\varepsilon}T^{\frac{(j+1)^2}{4}-\frac{1}{2}+\varepsilon}\right). \qquad (3.7)$$

On taking $T = x^{\frac{2}{(j+1)^2+2}}$ in (3.7), we have

$$\sum_{n\le x}\lambda^2(n^j) = c_{j-1}x + O\left(x^{1-\frac{2}{(j+1)^2+2}+\varepsilon}\right).$$

By taking $j = 2, 3, 4$ respectively, we have

$$\sum_{n\le x} \lambda^2(n^2) = c_1 x + O(x^{\frac{9}{11}+\varepsilon}),$$

$$\sum_{n\le x} \lambda^2(n^3) = c_2 x + O(x^{\frac{8}{9}+\varepsilon}),$$

and

$$\sum_{n\le x} \lambda^2(n^4) = c_3 x + O(x^{\frac{25}{27}+\varepsilon}).$$

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Department of Mathematics, Shandong Normal University, Jinan Shandong, 250014, People's Republic of China
*E-mail address*: laohuixue@sina.com

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400 005, India – and – Institute for Number Theory and Probability Theory, University of Ulm, D-89069, Ulm, Germany
*E-mail address*: sank@math.tifr.res.in
*E-mail address*: ayyadurai.sankaranarayanan@uni-ulm.de