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LAGRANGIAN BONNET PAIRS IN $\mathbb{C}P^2$

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ABSTRACT. In this paper we introduce Lagrangian Bonnet pairs in the complex projective plane $\mathbb{C}P^2$ and derive a Lawson-Tribuzy type theorem. We also present examples of compact Lagrangian Bonnet pairs with genus one in $\mathbb{C}P^2$.

1. INTRODUCTION

A classical question in the surface theory of $\mathbb{R}^3$ is which data are sufficient to determine a surface up to rigidity motions. According to the Bonnet theorem, a surface is determined by its first and second fundamental forms up to congruence. But this description is too redundant, because the first and second fundamental forms must satisfy the Gauss-Codazzi equations. Bonnet suggested that mean curvature and metric should be sufficient to determine a surface generically [2]. It follows from the local theory (without umbilic points) by Bonnet [2], Cartan [4] and Chern [5] that there are only three exceptions: constant mean curvature surfaces, Bonnet surfaces and Bonnet pairs. Constant mean curvature surfaces have been investigated intensively by various methods. Bonnet surfaces were studied by many mathematicians and recently have been generalized to the homogeneous 3-manifold with a 4-dimensional isometry group [8]. However, much less is known about Bonnet pairs, which are exactly two noncongruent isometric surfaces with the same mean curvature function. The theory of Bonnet pairs in $\mathbb{R}^3$ is closely related to isothermic surfaces in S^3 [9] and can be studied in the framework of the theory of integrable systems [1]. On the other hand, Lawson and Tribuzy [10] showed that for compact oriented surfaces in $\mathbb{R}^3$ with nonconstant mean curvature, there are at most two surfaces with the given metric and mean curvature. Moreover, they proved that there are no Bonnet pairs of genus zero in $\mathbb{R}^3$. Up to now, it is still an open question whether compact Bonnet pairs exist. Refer to a well-written short survey [15] for more details.

So far, the known results on Bonnet pairs are limited to surfaces in $\mathbb{R}^3$ and the 3-dimensional sphere S^3 [15]. By the investigation of Lagrangian surfaces in the complex projective plane $\mathbb{C}P^2$, we introduce a new concept of *Lagrangian Bonnet pairs* in $\mathbb{C}P^2$ in a similar spirit and derive the following Lawson-Tribuzy type theorem:

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Theorem 1.1. *Let M be a compact oriented Lagrangian surface in $\mathbb{C}P^2$. If M is not twistor harmonic, then there exist at most two noncongruent isometric immersions of M in $\mathbb{C}P^2$ with the mean curvature form Φ.*

Similar to the $\mathbb{R}^3$-theory, there is no Lagrangian Bonnet pair of genus zero in $\mathbb{C}P^2$. However, there exist compact Lagrangian Bonnet pairs of genus one in $\mathbb{C}P^2$. This differs from the $\mathbb{R}^3$ case but is the same as the S^3 case [15].

2. Preliminaries

Let $(\mathbb{C}P^2, \omega)$ be the complex projective plane endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4, where ω is the Kähler form. Let $f : M \to \mathbb{C}P^2$ be a Lagrangian immersion of an oriented surface, i.e., $f^*\omega = 0$. The induced metric on M generates a complex structure with respect to which the metric is $g = 2e^u dz d\bar{z}$, where $z = x + iy$ is a local complex coordinate on M and u is a real-valued function defined on M locally. For such a Lagrangian immersion f, there always exists a local horizontal lift $F : \mathrm{U} \to S^5$ such that

$$F_z \cdot \overline{F} = F_{\bar{z}} \cdot \overline{F} = 0. \tag{2.1}$$

where U is an open set of M. In fact, generally, it follows from M being Lagrangian that $dF \cdot \bar{F}$ is a closed one-form for any local lift F. So there exists $\eta \in C^\infty(\mathrm{U})$ such that $d\eta = dF \cdot \bar{F}$. Then $\tilde{F} = e^{-i\eta} F$ is a horizontal lift for f to S^5.

The metric g being conformal gives

$$F_z \cdot \overline{F_{\bar{z}}} = 0, \tag{2.2}$$

$$F_z \cdot \overline{F_z} = F_{\bar{z}} \cdot \overline{F_{\bar{z}}} = e^u. \tag{2.3}$$

Thus the vectors $F_z, F_{\bar{z}}$ as well as F define a Hermitian orthogonal moving frame on the surface, which due to (2.1), (2.2) and (2.3) satisfies the following equations:

$$\sigma_z = \sigma \mathcal{U}, \quad \sigma_{\bar{z}} = \sigma \mathcal{V}, \quad \sigma = (F, F_z, F_{\bar{z}}), \tag{2.4}$$

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & -e^u \\ 1 & u_z + \phi & -\bar{\phi} \\ 0 & e^{-u}\psi & \phi \end{pmatrix}, \mathcal{V} = \begin{pmatrix} 0 & -e^u & 0 \\ 0 & -\bar{\phi} & -e^{-u}\bar{\psi} \\ 1 & \phi & u_{\bar{z}} - \bar{\phi} \end{pmatrix}, \tag{2.5}$$

where

$$\phi = e^{-u} F_{z\bar{z}} \cdot \overline{F_{\bar{z}}}, \quad \psi = F_{zz} \cdot \overline{F_{\bar{z}}}. \tag{2.6}$$

The one-form $\Phi = \phi dz$ and the cubic differential $\Psi = \psi dz^3$ are globally defined on M so as to be independent of the choice of the local lift. We call Φ and Ψ the mean curvature form and the Hopf differential of f, respectively.

The compatibility conditions

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z = [\mathcal{U}, \mathcal{V}]$$

of equations (2.4) and (2.5) have the following form:

$$\phi_{\bar{z}} + \bar{\phi}_z = 0, \tag{2.7}$$

$$u_{z\bar{z}} + e^u + |\phi|^2 - e^{-2u}|\psi|^2 = 0, \tag{2.8}$$

$$e^{-u}\psi_{\bar{z}} = \phi_z - u_z \phi. \tag{2.9}$$

These equations are necessary and sufficient for the existence of the corresponding Lagrangian surface in $\mathbb{C}P^2$. A generic immersed Lagrangian surface $f : M \to \mathbb{C}P^2$

is determined uniquely by its induced metric, mean curvature one-form Φ and Hopf differential Ψ which satisfy equations (2.7), (2.8) and (2.9) (see [17], [11] for more details).

Remark 2.1. Actually this cubic differential Ψ has been introduced by several authors in the study of minimal surfaces in Kähler manifolds, for instance, Eells and Wood [7], Chern and Wolfson [6], Castro and Urbano [3]. Also, instead of the 1-form Φ, Schoen and Wolfson [14], Oh [13] introduced the famous Maslov form σ_H which is the dual of the vector field JH and in fact $\sigma_H = i(\Phi - \bar{\Phi})$, where H is the mean curvature vector field, or Castro and Urbano introduced a global vector field $\mathcal{K} = e^{-u}\bar{\phi}\frac{\partial}{\partial z}$ over M.

Φ has the following geometric interpretations:

Proposition 2.2. *Let $f : M \to \mathbb{C}P^2$ be an oriented immersed surface M. Then f is minimal if and only if $\Phi \equiv 0$. In addition, if f is Lagrangian, then f is Hamiltonian stationary if and only if Φ is holomorphic.*

The twistor space $\mathcal{L}$ of $\mathbb{C}P^2$ can be identified with the flag manifold $SU(3)/SU(1)^3$, which can be endowed with a structure of an Einstein-Kähler manifold. In [3], Castro and Urbano studied twistor harmonic surfaces of $\mathbb{C}P^2$, i.e., surfaces of $\mathbb{C}P^2$ such that their twistor lifts to $\mathcal{L}$ are harmonic maps, and gave a geometric interpretation for Ψ:

Proposition 2.3 ([3])**.** *Let $f : M \to \mathbb{C}P^2$ be a Lagrangian immersion of an oriented surface M. f is twistor harmonic if and only if the Hopf differential Ψ is holomorphic. In particular, f is twistor holomorphic if and only if $\Psi \equiv 0$.*

Proposition 2.4 ([3])**.** *Let $f : M \to \mathbb{C}P^2$ be a twistor harmonic Lagrangian immersion of a compact oriented surface M with genus g. Then*

(a) *if $g \geq 2$, f is minimal;*
(b) *if $g = 0$, f is twistor holomorphic;*
(c) *if $g = 1$, either g is minimal or the function $e^{-3u}\bar{\phi}^3\psi$ is a nonnull constant which has been classified in* [3].

3. Proof of the theorem

We now suppose that we are given three isometric noncongruent Lagrangian immersions $f_k : M \to \mathbb{C}P^2$, $k = 1, 2, 3$, with coinciding mean curvature one-form Φ. As conformal immersions of the same Riemann surface, they are described by the corresponding Hopf differentials Ψ_1, Ψ_2, Ψ_3, the conformal metric $2e^u dz d\bar{z}$ and the mean curvature form Φ. Since the surfaces are noncongruent, the Hopf differentials differ.

It follows from (2.8) and (2.9) that

Proposition 3.1. *Each of the differences $\Psi_{ij} \equiv \Psi_i - \Psi_j$ for $1 \leq i, j \leq 3$ is a holomorphic cubic differential form on M. Moreover,*

$$|\Psi_i| = |\Psi_j|, \quad 1 \leq i, j \leq 3. \tag{3.1}$$

Due to the second statement of Proposition 3.1 the zeros of Ψ_k for $k = 1, 2, 3$ coincide. We call this set the umbilic points set of $f_k, k = 1, 2, 3$. Denote it by

$$\mathcal{U} = \{P \in M \mid \Psi_k(P) = 0\}.$$

Following the standard arguments in [16] and [10], we have

Lemma 3.2. *If the three immersions f_k, $k = 1, 2, 3$, are mutually noncongruent, then*

$$\triangle^0 \log \psi_k = 4|\frac{\partial \log \psi_k}{\partial \bar{z}}|^2 \tag{3.2}$$

for each k, where $\triangle^0 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ is the standard Laplacian in the local coordinate z.

Proof. Due to (3.1), we may write

$$\Psi_k = e^{i\theta_k}\Psi_1, \quad k = 1, 2, 3, \tag{3.3}$$

where θ_k is well defined modulo 2π outside of $\mathcal{U}$. Then $\Psi_1 - \Psi_k = \psi_1(1 - e^{i\theta_k})dz^3$ is a holomorphic cubic differential on M. Denote $h_k = e^{i\theta_k}$. Considering that $\psi_1(1 - h_k)$ is holomorphic, we get

$$0 = \frac{\partial\{\psi_1(1 - h_k)\}}{\partial \bar{z}} = \frac{\partial \psi_1}{\partial \bar{z}}(1 - h_k) - i\psi_1 h_k \frac{\partial \theta_k}{\partial \bar{z}}.$$

Therefore,

$$\frac{\partial \theta_k}{\partial \bar{z}} = -i(\bar{h}_k - 1)\frac{\partial}{\partial \bar{z}} \log \psi_1.$$

Also we have

$$0 = \frac{\partial\{\bar{\psi}_1(1 - \bar{h}_k)\}}{\partial z} = \frac{\partial \bar{\psi}_1}{\partial z}(1 - \bar{h}_k) + i\bar{\psi}_1 \bar{h}_k \frac{\partial \theta_k}{\partial z},$$

so that

$$\frac{\partial \theta_k}{\partial z} = i(h_k - 1)\frac{\partial}{\partial z} \log \bar{\psi}_1. \tag{3.4}$$

It follows from $\frac{\partial^2 \theta_k}{\partial z \partial \bar{z}} = \frac{\partial^2 \theta_k}{\partial \bar{z} \partial z}$ that

$$(\bar{h}_k - 1)\{\frac{\partial^2 \log \psi_1}{\partial z \partial \bar{z}} - |\frac{\partial}{\partial \bar{z}} \log \psi_1|^2\} + (h_k - 1)\{\frac{\partial^2 \log \bar{\psi}_1}{\partial z \partial \bar{z}} - |\frac{\partial}{\partial z} \log \bar{\psi}_1|^2\} = 0.$$

Multiplying the above equation by h_k and changing h_k by h, we obtain

(3.5)

$$\begin{aligned}\{\frac{1}{4}\triangle^0 \log \psi_1 - |\frac{\partial \log \psi_1}{\partial \bar{z}}|^2\} - h\{\frac{1}{4}\triangle^0 \log \psi_1 - |\frac{\partial \log \psi_1}{\partial \bar{z}}|^2 + \frac{1}{4}\triangle^0 \log \bar{\psi} - |\frac{\partial \log \bar{\psi}}{\partial z}|^2\} \\ + h^2\{\frac{1}{4}\triangle^0 \log \psi_1 - |\frac{\partial \log \bar{\psi}_1}{\partial z}|^2\} = 0.\end{aligned}$$

Notice that the left side of equation (3.5) is a complex polynomial P of degree 2 which vanishes at $h_1 \equiv 1, h_2, h_3$. Since the zeros of holomorphic functions $\psi_k - \psi_j, k \neq j$, are isolated, h_1, h_2, h_3 are not equal to each other in a dense set. Therefore the coefficients of this polynomial are identical to zero. In this case, $\triangle^0 \log \psi_1 = 4|\frac{\partial \log \psi_1}{\partial \bar{z}}|^2$. $\square$

From now on we assume that f_1, f_2, f_3 are mutually noncongruent. We will use this fact only for the two Lagrangian immersions f_1 and f_2. Considering $|\psi_1|^2 = |\psi_2|^2$, we may write

$$\psi_2 = \psi_1 e^{i\theta},$$

where θ is well defined outside the zeros of $|\psi_k|^2 = e^{3u}(e^{-u}|\phi|^2 + 1 - K) = e^{3u}(\frac{1}{2}|H|^2 + 1 - K)$ modulo 2π, where K is the Gauss curvature of the induced metric and $|H|$ is the length of the mean curvature vector.

We now consider

$$Q := \frac{\psi_1 - \psi_2}{\psi_1} = 1 - e^{i\theta},$$

which is well defined on $M \backslash \mathcal{U}$. Since $\psi_1 - \psi_2$ is holomorphic, we have

$$\triangle \log Q = \triangle \log |Q| + i \triangle \arg Q \leq 0, \tag{3.6}$$

where $\triangle$ is the Laplace-Beltrami operator on M. From (3.6) we know that

$$\triangle \log |Q| \leq 0, \quad \triangle \arg Q = 0$$

on $M \backslash \mathcal{U}$.

We now observe that since Q is not zero in the connected set $M \backslash \mathcal{U}$, the function θ cannot be zero modulo 2π in this set. Hence we can choose a continuous branch $\theta : M \backslash \mathcal{U} \to (0, 2\pi)$. Then there exists a continuous branch

$$\arg(Q(z)) \in (-\frac{\pi}{2}, \frac{\pi}{2}),$$

for $z \in M \backslash \mathcal{U}$. In particular, $\arg Q$ is a bounded harmonic function on $M \backslash \mathcal{U}$, where $\mathcal{U}$ is a discrete points set. Therefore, by the removable singularities theorem, $\arg Q$ can extend to a smooth harmonic function on M, and hence $\arg Q$ is a constant. It follows that Q is a constant. Consequently, Ψ_1 is holomorphic, and by Proposition 2.3, the surface M is Lagrangian twistor harmonic. This completes the proof.

4. Bonnet pairs

Let f_1, f_2 be a Bonnet pair, i.e., two isometric noncongruent Lagrangian surfaces with coinciding mean curvature form. As conformal immersions of the same Riemann surface

$$f_1 : M \to \mathbb{C}P^2, \quad f_2 : M \to \mathbb{C}P^2,$$

they are described by the corresponding Hopf differentials Ψ_1, Ψ_2, the conformal metric $2e^u dz d\bar{z}$ and the mean curvature form Φ. Since the surfaces are noncongruent, the Hopf differentials differ: $\Psi_1 \neq \Psi_2$.

Proposition 4.1. *Let Ψ_1 and Ψ_2 be the Hopf differentials of a Lagrangian Bonnet pair $f_{1,2} : M \to \mathbb{C}P^2$. Then there exist a holomorphic cubic differential $h = \Psi_1 - \Psi_2$ on M and a smooth real-valued function $\alpha : M \to \mathbb{C}P^2$ such that*

$$\Psi_1 = \frac{1}{2} h(i\alpha + 1), \quad \Psi_2 = \frac{1}{2} h(i\alpha - 1).$$

Proof. Define a smooth cubic differential

$$q = \Psi_1 + \Psi_2.$$

(3.1) implies

$$q\bar{h} + h\bar{q} = 0.$$

Thus $\alpha = -i\frac{q}{h}$ is a real-valued function defined on $M \backslash \mathcal{U}_h$, where

$$\mathcal{U}_h = \{P \in M \mid h(P) = 0\}$$

is the zero set of h. At any $z_0 \in \mathcal{U}_h$ the holomorphic differential h has the form

$$h(z) = (z - z_0)^k h_0(z), \quad h_0(z_0) \neq 0, \quad k \in \mathbb{N}.$$

In a neighborhood of z_0 we have

$$\alpha = -\frac{i}{(z-z_0)^k}\frac{q(z)}{h_0(z)},$$

where q is smooth and h_0 is holomorphic. Real-valuedness of α near z_0 implies

$$q(z) = (z-z_0)^k g_0(z)$$

with g_0 smooth, which implies the smoothness of α at z_0. So α can be smoothly extended to the whole M. □

Corollary 4.2. *Umbilic points of a Lagrangian Bonnet pair are isolated. The umbilic set coincides with the zero set of* h*, i.e.,* $\mathcal{U} = \mathcal{U}_h$.

The number k which is defined above is called the *index* of the umbilic point. We call the zero divisor $D = (h)$ of h the *umbilic divisor* of a Lagrangian Bonnet pair.

In exactly the same way as in the case of Bonnet pairs in $\mathbb{R}^3$ (see [1]), for compact Riemann surfaces, Propositions 3.1, 4.1 imply the following.

Proposition 4.3. (1) *There are no Lagrangian Bonnet pairs of genus zero.*
(2) *Lagrangian Bonnet pairs of genus one have no umbilic points.*
(3) *If Lagrangian Bonnet pairs of genus* $g \geq 1$ *exist, the umbilic divisor* D *is of degree* $6g-6$ *and its class is* $D = 3K$*, where* K *is the canonical divisor.*

The development of Bonnet pairs in $\mathbb{C}P^2$ so far has been completely parallel to the case of Bonnet pairs in $\mathbb{R}^3$. But similar to the case of Bonnet pairs in S^3 [15], Lagrangian Bonnet pairs can be compact in $\mathbb{C}P^2$. Recall we have constructed a class of examples of S^1-equivariant Hamiltonian stationary Lagrangian tori in $\mathbb{C}P^2$ (see [12]) which have the metric $g = 2e^u dz d\bar{z}$, where u only depends on one variable, the mean curvature one-form $\Phi \equiv dz$ and the Hopf differential $\Psi = \psi dz^3$ satisfying that $\psi = -e^u + C$, where C is a constant. It is not hard to see that the case when C is a complex constant in [12] provides us examples of Lagrangian Bonnet pairs of genus one, whose Hopf differentials satisfy that the ψ's are complex conjugate to each other.

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