

FROM A RAMANUJAN-SELBERG CONTINUED FRACTION TO A JACOBIAN IDENTITY

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ABSTRACT. Jacobi proved an elegant identity involving eight-fold infinite products. In this paper, we give a new proof of this identity. A key ingredient of our proof is an identity satisfied by a Ramanujan-Selberg continued fraction.

1. INTRODUCTION

For $|q| < 1$, we define

$$T(q) := \frac{1}{1} + \frac{q}{1} + \frac{q+q^2}{1} + \frac{q^3}{1} + \frac{q^2+q^4}{1} + \cdots$$

Independently, Ramanujan and Selberg studied this interesting continued fraction. Ramanujan asserted in his notebook [25, p. 290] that

$$(1.1) \quad T(q) = \frac{(-q^2; q^2)_\infty}{(-q; q^2)_\infty}.$$

Here and in the rest of the paper we follow the customary q -product notation: we set (for $|q| < 1$)

$$(c; q)_\infty := \prod_{k=0}^{\infty} (1 - cq^k).$$

It was A. Selberg, unaware of Ramanujan's assertion at the time of writing, who gave the first proof of (1.1) in print [26]. Other proofs were discovered by Ramanathan [24], Andrews, Berndt, Jacobsen and Lamphere [4], and, Zhang [30]. Zhang, building upon his earlier results in [31, 32], gave explicit evaluations of $T(q)$ in terms of class invariants and singular moduli in [33]. See also the paper by Baruah and Saikia [7] for a penetrating study of $T(q)$ (and related continued fractions).

It may not be obvious how $T(q)$ is related to the following amazing identity discovered by Jacobi:

Theorem 1.1 (Jacobi). *For $|q| < 1$,*

$$(1.2) \quad (q; q^2)_\infty^8 + 16q(-q^2; q^2)_\infty^8 = (-q; q^2)_\infty^8,$$

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or, in standard notation,

$$\prod_{n=1}^{\infty} (1 - q^{2n-1})^8 + 16q \prod_{n=1}^{\infty} (1 + q^{2n})^8 = \prod_{n=1}^{\infty} (1 + q^{2n-1})^8.$$

Whittaker and Watson note that [29, p. 470] Jacobi was deeply impressed by (1.2). For proofs of (1.2), see the wonderful books by Whittaker and Watson [29] and by the Borweins [11]; see also an elegant paper by Ewell [19].

In this paper, we will give a new proof of (1.2) that is based on the following identity of $T(q)$:

Theorem 1.2. *Let $x(q) = 1/(2T^2(q))$. Then*

$$(1.3) \quad x(q) - \frac{q^{1/2}}{x(q)} = \frac{(q^{1/2}; q^{1/2})_{\infty}^4}{2(q^2; q^2)_{\infty}^4}.$$

In Section 2, we will prove Theorem 1.2. In Section 3, we will prove the Jacobian Identity (1.2). Section 4 is our concluding remarks.

Before we turn to Section 2, we apply Theorem 1.2 to evaluate $T(q)$. In terms of the eta function $\eta(z) = e^{2\pi iz/24} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})$, we can write (1.3) as (with $q = e^{2\pi iz}$)

$$(1.4) \quad x(q) - \frac{q^{1/2}}{x(q)} = \frac{1}{2} q^{1/4} \left(\frac{\eta(z/2)}{\eta(2z)} \right)^4.$$

With the well-known identity, $\eta(-1/z) = \sqrt{z/i} \eta(z)$, we have, at $z = i$ (i.e., $q = e^{-2\pi}$), $\eta(i/2)/\eta(2i) = \sqrt{2}$. This, with (1.4), implies

$$T(e^{-2\pi}) = \sqrt{\frac{e^{\pi/2}}{2(\sqrt{2} + 1)}}.$$

For a comprehensive theory of the evaluations of $T(q)$, consult the wonderful papers by Zhang [33] and by Baruah and Saikia [7].

2. PROOF OF THEOREM 1.2

Our proof is motivated by the recent works [15, 16]. First, we observe that

$$(2.1) \quad T^2(q) = \frac{1}{2} \left(\frac{\sum q^{j^2+j}}{\sum q^{j^2}} \right),$$

where in both sums j runs from $-\infty$ to ∞ . To see this, we note that

$$T^2(q) = \frac{1}{2} \left(\frac{2(q^2; q^2)_{\infty} (-q^2; q^2)_{\infty}^2}{(q^2; q^2)_{\infty} (-q; q^2)_{\infty}^2} \right) = \frac{1}{2} \left(\frac{\sum q^{j^2+j}}{\sum q^{j^2}} \right).$$

The first equation is due to (1.1). The second is from Jacobi's Triple Product Identity (see, e.g., [2, 3, 6, 9])

$$(2.2) \quad \sum_{n=-\infty}^{\infty} (-z)^n Q^{n^2} = \prod_{j=1}^{\infty} (1 - Q^{2j})(1 - zQ^{2j-1})(1 - z^{-1}Q^{2j-1}).$$

This proves (2.1).

For later convenience, we define, for $a = 0$ and 1 ,

$$(2.3) \quad G(a) := \sum_{j=-\infty}^{\infty} q^{j^2+aj}$$

and

$$(2.4) \quad J(z) := \prod_{n=1}^{\infty} (1 - q^{n/2}) (1 + z q^{(2n-1)/4}) (1 + z^{-1} q^{(2n-1)/4}).$$

Note that the dependence of G on q is suppressed in this notation. Also, we can write $x(q) = 1/(2T^2(q))$ (as defined in Theorem 1.2) as

$$(2.5) \quad x(q) = \frac{G(0)}{G(1)}.$$

With the above notation understood, we prove the following lemma, which expresses G in terms of $J(\pm 1)$:

Lemma 2.1.

$$(2.6) \quad G(0) = \frac{1}{2} (J(1) + J(-1)),$$

$$(2.7) \quad q^{1/4} G(1) = \frac{1}{2} (J(1) - J(-1)).$$

Proof. We start with defining, for $a = 0$ and 1 ,

$$(2.8) \quad K_n(a) := \sum_{j=-\infty}^{\infty} q^{j^2+aj} \begin{bmatrix} 2n \\ n + 2j \end{bmatrix}.$$

Here, the q -binomial coefficients (or the Gaussian polynomials) are defined as

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0 & \text{if } B < 0 \text{ or } B > A \text{ or } A < 0, \\ \frac{(1 - q^A)(1 - q^{A-1}) \cdots (1 - q^{A-B+1})}{(1 - q^B)(1 - q^{B-1}) \cdots (1 - q)} & \text{otherwise.} \end{cases}$$

See, e.g., [2, 6]. We will need the following property of the q -binomial numbers (see, e.g., [6]): for fixed m_1 and m_2 , with $R > S$ positive,

$$(2.9) \quad \lim_{N \rightarrow \infty} \begin{bmatrix} RN + m_1 \\ SN + m_2 \end{bmatrix} = \frac{1}{(q; q)_{\infty}}.$$

The key idea of our proof is to express (the limit of) $K_n(a)$ in two different ways.

First, we note that, with $K_{\infty}(a) := \lim_{n \rightarrow \infty} K_n(a)$, we have, by (2.8) and (2.9),

$$(2.10) \quad K_{\infty}(a) = \frac{G(a)}{(q; q)_{\infty}}.$$

Next, we write $K_{\infty}(a)$ in a different way. To proceed, we observe that

$$\begin{aligned} K_n(a) &= \sum_{\sigma \equiv 0 \pmod{2}} q^{\sigma^2/4 + (a/2)\sigma} \begin{bmatrix} 2n \\ n + \sigma \end{bmatrix} \\ &= \frac{1}{2} \sum_{k=0}^1 \left(\sum_{\sigma=-\infty}^{\infty} q^{\sigma^2/4 + (a/2)\sigma} \begin{bmatrix} 2n \\ n + \sigma \end{bmatrix} (-1)^{k\sigma} \right) \\ (2.11) \quad &:= \frac{1}{2} \sum_{k=0}^1 C_n((-1)^k, a). \end{aligned}$$

Note that the second line is due to the fact that

$$\frac{1}{2} \sum_{j=0}^1 (-1)^{j\sigma} = \begin{cases} 1 & \text{if } \sigma \equiv 0 \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C_\infty(z, a) := \lim_{n \rightarrow \infty} C_n(z, a)$; then (2.11) implies

$$(2.12) \quad K_\infty(a) = \frac{1}{2} \sum_{k=0}^1 C_\infty((-1)^k, a).$$

To finish proving the second way of expressing $K_\infty(a)$, we need to write $C_\infty(z, a)$ in terms of $J(z)$. We claim that

$$(2.13) \quad C_\infty(z, 0) = \frac{J(z)}{(q; q)_\infty},$$

$$(2.14) \quad C_\infty(z, 1) = \frac{1}{z q^{1/4}} \left(\frac{J(z)}{(q; q)_\infty} \right).$$

To prove (2.14), we observe that

$$\begin{aligned} C_\infty(z, 1) &= \frac{1}{(q; q)_\infty} \sum_{\sigma=-\infty}^{\infty} z^\sigma q^{\sigma^2/4 + \sigma/2} \\ &= \frac{1}{(q; q)_\infty} \prod_{n=1}^{\infty} (1 - q^{n/2}) \left(1 + zq^{(2n+1)/4}\right) \left(1 + z^{-1}q^{(2n-3)/4}\right) \\ &= \frac{1}{z q^{1/4}} \left(\frac{J(z)}{(q; q)_\infty} \right). \end{aligned}$$

Note that the first line follows from the definition of $C_n(z, a)$ (cf. (2.11)) and the property of the q -binomial numbers (cf. (2.9)). The second line is due to Jacobi's Triple Product Identity (cf. (2.2)). For the last line, we recall the definition of $J(z)$ in (2.4). The proof of (2.13) is similar and will be omitted. Note that (2.12)-(2.14) give us a second way of writing $K_\infty(a)$.

By putting the above together, we prove the lemma. Indeed,

$$\begin{aligned} \frac{G(1)}{(q; q)_\infty} &= \frac{1}{2} \sum_{k=0}^1 C_\infty((-1)^k, 1) \quad \text{by (2.10) and (2.12)} \\ &= \frac{1}{2 q^{1/4} (q; q)_\infty} (J(1) - J(-1)) \quad \text{by (2.14)}. \end{aligned}$$

This proves (2.7). The proof of (2.6) follows the same type of calculation and will be omitted here. \square

Proof of Theorem 1.2. First, we write the two identities in Lemma 2.1 as

$$\begin{aligned} G(0) + q^{1/4} G(1) &= J(1), \\ G(0) - q^{1/4} G(1) &= J(-1). \end{aligned}$$

Next, we multiply them together to get

$$G^2(0) - q^{1/2} G^2(1) = J(1)J(-1).$$

By dividing the last equation by $G(0)G(1)$ we obtain

$$(2.15) \quad x(q) - \frac{q^{1/2}}{x(q)} = \frac{J(1)J(-1)}{G(0)G(1)}$$

(recall that $x = G(0)/G(1)$ by (2.5)). Our last task is to simplify the right-hand side of (2.15). To this end, we observe that

$$\begin{aligned} \frac{J(1)J(-1)}{G(0)G(1)} &= \frac{1}{2} \prod_{n=1}^{\infty} \frac{(1 - q^{n/2})^2(1 + q^{(2n-1)/4})^2(1 - q^{(2n-1)/4})^2}{(1 - q^{2n})^2(1 + q^{2n})^2(1 + q^{2n-1})^2} \\ &= \frac{1}{2} \prod_{n=1}^{\infty} \frac{(1 - q^{n/2})^2}{(1 - q^{2n})^2} \cdot \frac{(1 - q^{(2n-1)/2})^2}{(1 + q^n)^2} \\ &= \frac{1}{2} \prod_{n=1}^{\infty} \frac{(1 - q^{n/2})^4}{(1 - q^{2n})^4}, \end{aligned}$$

which is the right-hand side of (1.3). Note that in the first equality we used Jacobi’s Triple Identity to write $G(a)$ as infinite products. This completes our proof of Theorem 1.2. □

3. A NEW PROOF OF THE JACOBIAN IDENTITY

The first part of our proof mimics a proof for Ramanujan’s “Most Beautiful Identity” (see [3, 9, 18, 22, 23]; see also [12] for a similar identity for Ramanujan’s cubic continued fraction).

By replacing q by q^2 in (1.3), we obtain

$$(3.1) \quad x(q^2) - \frac{q}{x(q^2)} = \frac{(q; q)_{\infty}^4}{2(q^4; q^4)_{\infty}^4}.$$

Replacing q by $-q$ in (3.1) gives

$$(3.2) \quad x(q^2) + \frac{q}{x(q^2)} = \frac{(q^2; q^2)_{\infty}^4 (-q; q^2)_{\infty}^4}{2(q^4; q^4)_{\infty}^4}.$$

Now, square (3.2) and subtract the square of (3.1). Precisely, the left-hand side gives

$$\left(x(q^2) + \frac{q}{x(q^2)}\right)^2 - \left(x(q^2) - \frac{q}{x(q^2)}\right)^2 = 4q.$$

The other side of the same expression gives

$$\frac{(q^2; q^2)_{\infty}^8 (-q; q^2)_{\infty}^8 - (q; q)_{\infty}^8}{4(q^4; q^4)_{\infty}^8}.$$

Equating the last two expressions gives

$$(q; q)_{\infty}^8 + 16q(q^4; q^4)_{\infty}^8 = (q^2; q^2)_{\infty}^8 (-q; q^2)_{\infty}^8.$$

Dividing both sides of the last equation by $(q^2; q^2)_{\infty}^8$ gives (1.2). □

4. CONCLUDING REMARKS

Two remarks are in order. First, we note that Theorem 1.2 could be compared with the following results (i.e., equations (4.1) and (4.2) below).

Let $R(q)$ denote the celebrated Rogers-Ramanujan continued fraction:

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots$$

It is known that $R(q)$ satisfies the following remarkable equation:

$$(4.1) \quad \frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)},$$

where $f(-q) := (q; q)_\infty$; cf. [27, 28]. See also the following excellent introductions [2, 5, 8, 10].

Recently, the following was discovered. Denote the Ramanujan cubic continued fraction by

$$v := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \cdots$$

It can be shown that

$$(4.2) \quad \frac{1}{x_v(q)} - q^{1/3} - 2q^{2/3}x_v(q) = \frac{(q^{1/3}; q^{1/3})_\infty (q^{2/3}; q^{2/3})_\infty}{(q^3; q^3)_\infty (q^6; q^6)_\infty},$$

where $x_v(q) = q^{-1/3}v$ (cf. Theorem 2 in [12]; see also its implications in [13, 14]).

Second, we remark that the strategy used in this paper can also be applied to the Ramanujan-Göllnitz-Gordon continued fraction (see p. 229 of Ramanujan's second notebook [25]; see also [1, 20, 21]; and also the paper by Chan and Huang [17], in which the authors provided a comprehensive theory of the Ramanujan-Göllnitz-Gordon continued fraction) and proved certain q -identities. We will address these results on other occasions.

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