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A REMARK ON POLIGNAC'S CONJECTURE

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Abstract. We make reasonable assumptions which imply respectively:
1) every sufficiently large even integer is the difference of two primes;
2) the set of even positive integers which are the differences of two primes has density 1 in the set of even positive integers.

Introduction

Polignac conjectured that every even positive integer is the difference of two primes (see [1], page 187). This conjecture has never been proved. We shall establish a connection between the set of differences of primes and the number of expressions of even positive integers as differences of primes. Under reasonable assumptions, we shall prove that every sufficiently large even integer is a difference of primes, respectively, the set of even positive integers which are differences of primes has density 1 in the set of even positive integers.

We shall use the following notation. For each $x > 0$, $P[x]$ denotes the largest prime $p \leq x$. For each set E of natural numbers and $x > 0$, $E(x)$ shall denote the set of all $n \in E$ such that $n \leq x$. As usual, $\pi(x)$ denotes the number of primes $p \leq x$. If $x \geq 11$, then it is known that $\pi(x) > \frac{x}{\log x}$ (see [1], page 177).

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It is convenient to say that the positive integer $2n$ is a *Polignac number* if there exist primes $p > q > 2$ such that $2n = p - q$. For every $x > 0$ let $L_x = \{2n > 0 \mid$ there exist primes p, q such that $2n = p - q$ and $x \geq p > q > 2\}$. If $x < x'$, then $L_x \subseteq L_{x'}$. Let $L = \bigcup_{x \geq 1} L_x$, so L is the set of Polignac numbers.

Lemma 1.1. *For every $x \geq 1$ there exists $x' > x$ such that $L_{x'} \supset L_x$. Hence there exist infinitely many Polignac numbers.*

Proof. Let $0 < x < q' < q' + x < p'$ where q', p' are primes; let $x' \geq p'$. Then $x < p' - q' \in L_{x'}$, but if p, q are primes such that $2 < q < p < x$, then $p' - q' \neq p - q$, because $p - q < x$. So $p' - q' \notin L_x$. It follows that L is an infinite set. $\square$

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Let $S = \{(p,q) \mid p,q \text{ are primes}, 2 < q < p\}$. For $x \geq 1$ let $S(x) = \{(p,q) \in S \mid p \leq x\}$. Then

$$\# S(x) = \frac{1}{2}\big[(\pi(x)-1)^2 - (\pi(x)-1)\big] = \binom{\pi(x)-1}{2}.$$

Let $F\colon S \to \{2n \mid n \geq 1\}$ be defined by $F\big((p,q)\big) = p - q$. Thus $F\big(S(x)\big) = L_x$.

If $2n > 0$, let $s(2n) = \#\{(p,q) \in S \mid 2n = p-q\}$. If $x \geq 1$, let $s(2n,x) = \#\{(p,q) \in S(x) \mid 2n = p-q\}$, and let $\sigma(x) = \max_{2n \leq x}\{s(2n,x)\}$.

Lemma 1.2. *1) For $x \geq 1$, $\sigma(x) \geq \dfrac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}$ and $\lim_{x\to\infty} \sigma(x) = \infty$.*

2) $\sup_{2n\geq 1}\{s(2n)\} = \infty$.

Proof. 1) For $x \geq 1$,

$$\# S(x) = \sum_{2n \in F(S(x))} s(2n,x) \leq \# F(S(x)) \cdot \sigma(x) = \# L_x \cdot \sigma(x).$$

The largest $2n \in L_x$ is $P[x]-3$, so $\# L_x \leq \dfrac{P[x]-3}{2}\cdot$ Hence

$$\sigma(x) \geq \frac{\# S(x)}{\# L_x} \geq \frac{\binom{\pi(x)-1}{2}}{\frac{P[x]-3}{2}} = \frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}$$
$$\geq \frac{\left(\frac{x}{\log x} - 1\right)\left(\frac{x}{\log x} - 2\right)}{x}$$

for all $x \geq 11$. Hence $\lim_{x\to\infty} \sigma(x) = \infty$.

2) If $\sup_{2n\geq 1}\{s(2n)\} = r < \infty$ there exists n such that $s(2n) \geq s(2n,x) = \sigma(x) > r$, which is absurd. $\square$

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We make the following reasonable assumption:

(A) There exists $x_0 \geq 1$ such that for all sufficiently large $x > x_0$,

$$\sigma(x) \leq \frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}\left(1 + \frac{x_0}{x}\right).$$

Proposition 2.1. *If* (A) *is assumed to be true, then every sufficiently large integer $2n$ is of the form $2n = p - q$, where p, q are primes.*

Proof. For all sufficiently large $x > x_0$ we have

$$\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}\left(1 + \frac{x_0}{x}\right) \geq \sigma(x) \geq \frac{\# S(x)}{\# L_x}$$
$$\geq \frac{\# S(x)}{\frac{P[x]-3}{2}} = \frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}.$$

Hence

$$\# S(x)\left[\frac{1}{\# L_x}-\frac{1}{\frac{P[x]-3}{2}}\right]\le\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}\times\frac{x_0}{x}\cdot$$

Therefore

$$\frac{P[x]-3}{2}-\# L_x\le\frac{\frac{P[x]-3}{2}\times\# L_x}{\# S(x)}\cdot\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}\times\frac{x_0}{x}$$
$$=\# L_x\times\frac{x_0}{x}\le\frac{P[x]-3}{2}\times\frac{x_0}{x}\le\frac{x_0}{2}\cdot$$

Suppose that $2n_1 < 2n_2 < \cdots < 2n_\ell$ with each $2n_i$ not a Polignac number. For x sufficiently large, each $2n_i < \frac{P[x]-3}{2}$. Since each $2n_i \notin L_x$, then $\ell \le \frac{x_0}{2}\cdot$ We deduce that every sufficiently large even number is a Polignac number. □

Now we make another reasonable assumption:

(B) Given $\varepsilon > 0$, there exists an integer $x_1 > 0$ (depending on ε) such that if $x > x_1$, then

$$\sigma(x)\le\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}+\varepsilon.$$

Lemma 2.2. *If* (A) *is assumed to be true, then* (B) *is also satisfied.*

Proof. For $x_0 \ge 1$ and x sufficiently large, we calculate

$$D_x=\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}\left[\left(1+\frac{x_0}{x}\right)-1\right]=\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}\times\frac{x_0}{x}\cdot$$

It follows from Tschebycheff's theorem that $\pi(x)-\pi\left(\frac{x}{2}+3\right)>1$ when x is sufficiently large; this implies that $P[x]>\frac{x}{2}+3$, so $P[x]-3>\frac{x}{2}\cdot$ Therefore

$$D_x\le\frac{(\pi(x)-1)(\pi(x)-2)}{\frac{x}{2}}\times\frac{x_0}{x}\le\frac{2[\pi(x)]^2\,x_0}{x^2}<\frac{9}{2}\,\frac{x_0}{(\log x)^2}<\varepsilon$$

provided x is sufficiently large.

So if (A) is assumed to be true, then

$$\sigma(x)\le\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}\left(1+\frac{x_0}{x}\right)\le\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}+\varepsilon$$

for all $\varepsilon,\ 0<\varepsilon$ and x sufficiently large. Then (B) is satisfied. □

Proposition 2.3. *If* (B) *is assumed to be true, then the set of Polignac numbers has density* 1 *in the set of even positive integers.*

Proof. In the proof of (2.2) we have seen that if x is sufficiently large, then $P[x]-3>x/2$.

Let $0<\varepsilon<1$, so for all sufficiently large x, by assumption

$$\sigma(x)\le\frac{(\pi(x)-1)(\pi(x)-2)}{P[x]-3}+\varepsilon\le\frac{(\pi(x)-1)(\pi(x)-2)}{\frac{x}{2}}+\varepsilon.$$

For the number $L(x)$ of Polignac numbers less than or equal to x we have:

$$\begin{aligned}
\frac{L(x)}{\frac{x}{2}} \geq \frac{L_x}{\frac{x}{2}} \geq \frac{1}{\frac{x}{2}} \times \frac{\# S(x)}{\sigma(x)} &\\
\geq \frac{(\pi(x)-1)(\pi(x)-2)}{\frac{x}{2}\left[\frac{\pi(x)-1)(\pi(x)-2)}{\frac{x}{2}} + \varepsilon\right]} &\\
= \frac{\pi(x)-1)(\pi(x)-2)}{(\pi(x)-1)(\pi(x)-2) + \varepsilon\,\frac{x}{2}} &\\
= \frac{1}{1 + \dfrac{\varepsilon x}{2(\pi(x)-1)(\pi(x)-2)}} \geq \frac{1}{1+\varepsilon^2} \geq 1 - \varepsilon, &
\end{aligned}$$

because $1 > (1-\varepsilon)(1+\varepsilon^2)$. Since the above inequality holds for x sufficiently large, this concludes the proof of the proposition. $\square$

References

[1] Ribenboim, P., *The Little Book of Bigger Primes*. Second edition, Springer-Verlag, New York, 2004. MR2028675 (2004i:11003)

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