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ON SUMS INVOLVING COEFFICIENTS OF AUTOMORPHIC L-FUNCTIONS

GUANGSHI LÜ

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ABSTRACT. Let $L(s,\pi)$ be the automorphic L-function associated to an automorphic irreducible cuspidal representation π of GL_m over $\mathbb{Q}$, and let $a_\pi(n)$ be the nth coefficient in its Dirichlet series expansion. In this paper we prove that if at every finite place p, π_p is unramified, then for any $\varepsilon > 0$,

$$A_\pi(x) = \sum_{n\le x} a_\pi(n) \ll_{\varepsilon,\pi} \begin{cases} x^{\frac{71}{192}+\varepsilon} & \text{if} \quad m = 2, \\ x^{\frac{m^2-m}{m^2+1}+\varepsilon} & \text{if} \quad m \ge 3. \end{cases}$$

1. Introduction and main results

Let $a(n)$ be an arithmetic function. It is an important problem in number theory to establish the asymptotic formula for the summatory function

$$A(x) = \sum_{n\le x} a(n).$$

The asymptotic behavior of $A(x)$ is often closely linked with the analytic properties of the Dirichlet series

$$\mathcal{A}(s) = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

The Langlands program predicts that the most general L-functions arise from automorphic representations of GL_n over a number field and that such L-functions can be decomposed into products of primitive automorphic L-functions arising from irreducible cuspidal representations of GL_m over $\mathbb{Q}$. Therefore in this paper we focus our attention on primitive automorphic L-functions of GL_m over $\mathbb{Q}$.

To be precise, let us recall some basic facts about primitive automorphic L-functions of GL_m over $\mathbb{Q}$ (see Godement and Jacquet [4], Jacquet and Shalika [8], or Rudnick and Sarnak [11]). Let π be an automorphic irreducible cuspidal

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representation of GL_m over $\mathbb{Q}$ with unitary central character. Then π is a restricted tensor product:

$$\pi = \otimes_p \pi_p.$$

To π one associates an Euler product

$$L(s,\pi) = \prod_p L(s,\pi_p) \tag{1.1}$$

given by a product of local factors. Outside of a finite set of primes, π_p is unramified. To every finite place p where π_p is unramified we associate a semisimple conjugacy class

$$A_\pi(p) = \begin{pmatrix} \alpha_{\pi,p}(1), & & \\ & \ddots & \\ & & \alpha_{\pi,p}(m) \end{pmatrix},$$

and we define the local L-function for the finite place p as

$$L(s,\pi_p) = \det(I - p^{-s}A_\pi(p))^{-1} = \prod_{j=1}^{m} (1 - \alpha_{\pi,p}(j)p^{-s})^{-1}. \tag{1.2}$$

It is possible to write the local factors at ramified primes p in the form of (1.2) with the convention that some of the $\alpha_{\pi,p}(j)$'s may be zero. In fact, the local factors at the ramified primes can best be described by the Langlands parameters of π_p.

The general Ramanujan conjectures for cuspidal automorphic representations π of GL_m over $\mathbb{Q}$ assert that for p unramified, $|\alpha_{\pi,p}(j)| = 1$. For certain π, this conjecture has been proved. But in general it is still open. In this direction, Serre [12] first observed that the analytic properties of the Rankin-Selberg L-function, in conjunction with Landau's lemma, can lead to

$$|\alpha_{\pi,p}(j)| \le p^{1/2 - 1/(m^2+1)}. \tag{1.3}$$

For $m = 2$, this has been refined in [9] to

$$|\alpha_{\pi,p}(j)| \le p^{\frac{7}{64}}. \tag{1.4}$$

The product (1.1) over primes gives a Dirichlet series representation: for $\mathrm{Re}s > 1$,

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s}. \tag{1.5}$$

The aim of this paper is to study the summatory function for the coefficients $a_\pi(n)$ of automorphic L-functions attached to automorphic irreducible cuspidal representations of GL_m over $\mathbb{Q}$, i.e.

$$A_\pi(x) = \sum_{n \le x} a_\pi(n).$$

Our main result is the following.

Theorem 1.1. *Let $L(s,\pi)$ be the automorphic L-function associated to an automorphic irreducible cuspidal representation π of GL_m over $\mathbb{Q}$, and let $a_\pi(n)$ be its nth coefficient in (1.5). If at every finite place p, π_p is unramified, then we have that for any $\varepsilon > 0$,*

$$A_\pi(x) = \sum_{n \le x} a_\pi(n) \ll_{\varepsilon,\pi} \begin{cases} x^{\frac{71}{192}+\varepsilon} & \text{if} \quad m = 2, \\ x^{\frac{m^2-m}{m^2+1}+\varepsilon} & \text{if} \quad m \ge 3, \end{cases}$$

where throughout this paper the notation $\ll_{\varepsilon,\pi}$ means that the implied constant depends on ε and π.

Our Theorem 1.1, for which the Ramanujan-Petersson conjecture is not known to hold, can be compared with the results of Iwaniec and Friedlander [3]: if the Ramanujan-Petersson conjecture is assumed, then the coefficients $a(n)$ of a general L-function of degree m with a functional equation and suitable analytic properties satisfy

$$\sum_{n\leq x} a(n) = \text{ main term } + O_L(x^{\frac{m-1}{m+1}+\varepsilon}).$$

Our result can also be compared with one result of Miller [10], which states that for any $\varepsilon > 0$ and any real number α,

$$\sum_{n\leq x} a(m,n)e(n\alpha) \ll_{\varepsilon,m,\Phi} x^{\frac{3}{4}+\varepsilon},$$

where $a(m,n)$ are the Fourier coefficients of a cusp form Φ for $GL(3,\mathbb{Z})\backslash GL(3,\mathbb{R})$.

As an application of our Theorem 1.1, we shall consider the sum

$$\sum_{n\leq x} t(n^2),$$

where $t(n)$ is the nth normalized Fourier coefficient of a Hecke-Maass cusp form φ corresponding to the eigenvalue $l = \kappa^2 + \frac{1}{4}$ with respect to the full modular group $SL(2,\mathbb{Z})$, which coincides with the eigenvalue of the nth Hecke operator T_n.

Corollary 1.2. *Let $t(n)$ be the nth normalized Fourier coefficient of a Hecke-Maass cusp form φ with respect to the full modular group $SL(2,\mathbb{Z})$. Then for any $\varepsilon > 0$, we have*

$$S(x) = \sum_{n\leq x} t(n^2) \ll_{\varepsilon,\varphi} x^{\frac{3}{5}+\varepsilon},$$

where throughout this paper the notation $\ll_{\varepsilon,\varphi}$ means that the implied constant depends on ε and the Maass cusp form φ.

Our result improves a previous result given by Ivić [6]:

$$S(x) \ll_{\varphi} x \exp\left(-A(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right),$$

where $A > 0$ is a suitable constant.

2. Three lemmas

To prove Theorem 1.1, we need the following three lemmas.

Lemma 2.1. *Let $L(f,s)$ be a Dirichlet series with Euler product of degree $m \geq 1$, which is defined by*

$$L(f,s) = \sum_{n=1}^{\infty} \lambda_f(n)n^{-s} = \prod_{p<\infty}\prod_{j=1}^{m}\left(1 - \frac{\alpha_f(p,j)}{p^s}\right)^{-1},$$

where $\alpha_f(p,j), j = 1,\cdots,m,$ are the local parameters of $L(f,s)$ at prime p. This series and Euler product are absolutely convergent for $\mathrm{Re}s > 1$. *Let the gamma* *factor be given by*

$$L_\infty(f,s)=\prod_{j=1}^{m}\pi^{-\frac{s+\mu_f(j)}{2}}\Gamma\left(\frac{s+\mu_f(j)}{2}\right),$$

where $\mu_f(j), j=1,\cdots,m$, *are the local parameters of* $L(f,s)$ *at* ∞. *We also define the completed* L*-function* $\Lambda(f,s)$ *by*

$$\Lambda(f,s)=q(f)^{\frac{s}{2}}L_\infty(f,s)L(f,s),$$

where $q(f)$ *is the conductor of* $L(f,s)$. *We assume that* $\Lambda(f,s)$ *admits an analytic continuation to the whole complex plane* $\mathbb{C}$ *and is an entire function. Assume that it also satisfies a functional equation*

$$\Lambda(f,s)=\epsilon_f\Lambda(\tilde{f},1-s)$$

where ϵ_f *is the root number with* $|\epsilon_f|=1$ *and* $\tilde{f}$ *is the dual of* f *such that* $\lambda_{\tilde{f}}(n)=\overline{\lambda_f(n)}$, $\mu_{\tilde{f}}(j)=\overline{\mu_f(j)}$, *and* $q(\tilde{f})=q(f)$.

Then for every $\eta\geq 0$ *we have*

$$\sum_{n\leq x}\lambda_f(n)\ll_f x^{\frac{1}{2}-\frac{1}{2m}+(\frac{m}{2}-\frac{1}{2})\eta}+\sum_{x<n\leq x+x^{1-\frac{1}{m}-\eta}}|\lambda_f(n)|.$$

Proof. This is a special case of Theorem 4.1 in Chandrasekharan and Narasimhan [2] with

$$\delta=1,\quad A=\frac{m}{2},\quad \beta=1,\quad u=\frac{1}{2}-\frac{1}{2m}\quad\text{and}\quad q=-\infty.$$

We reformulate it in the language used in Chapter 5 of Iwaniec and Kowalski [7].

Lemma 2.2. *With the same notation as in Lemma* 2.1, *we assume that the Dirichlet series* $L(f,s)$ *with Euler product of degree* $m\geq 1$ *has non-negative coefficients, i.e.* $\lambda_f(n)\geq 0$, *and converges for* $\mathrm{Re}s$ *sufficiently large. Suppose further that* $L(f,s)$ *has a meromorphic continuation to* $\mathbb{C}$ *with, at most, poles of finite order at* $s=0,1$. *Assume also that* $L(f,s)$ *is of finite order and satisfies a functional equation*

$$\Lambda(f,s)=\epsilon_f\Lambda(f,1-s).$$

Then we have that for any $\varepsilon>0$,

$$\sum_{n\leq x}\lambda_f(n)=P(\log x)x+O_{\varepsilon,f}\left(x^{\frac{m-1}{m+1}+\varepsilon}\right),$$

where P *is a polynomial depending only on* L, *whose degree equals the order of the pole of* $L(f,s)$ *at* $s=1$.

Proof. This is a refined version of Landau's lemma; see Barthel and Ramakrishnan [1].

Lemma 2.3. *Let* $b(1)$, $b(2)$, ... *be a sequence of complex numbers. Define the sequence* $a(0)=1$, $a(1)$, $a(2)$, ... *by means of the formal identity*

$$\exp\left(\sum_{k=1}^{\infty}\frac{b(k)}{k}x^k\right)=\sum_{n=0}^{\infty}a(n)x^n.$$

For $j = 1$ or 2, define the sequence $A_j(0) = 1$, $A_j(1)$, $A_j(2)$, ... by means of the formal identity

$$\exp\left(\sum_{k=1}^{\infty} \frac{|b(k)|^j}{k} x^k\right) = \sum_{n=0}^{\infty} A_j(n) x^n.$$

Then $A_j(n) \geq |a(n)|^j$.

Proof. See Lemma 3.1 in Soundararajan [13].

3. Proof of Theorem 1.1

Associated with π, an automorphic representation of GL_m over $\mathbb{Q}$, there is also an Archimedean L-factor defined as

$$L(s, \pi_\infty) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_\pi(j)}{2}} \Gamma\left(\frac{s+\mu_\pi(j)}{2}\right),$$

where $\mu_\pi(j)$, $j = 1, 2, 3, \cdots, m$, are local parameters at ∞. In connection with (1.1), the completed L-function associated to π is defined by

$$\Lambda(s, \pi) = L(s, \pi_\infty) L(s, \pi).$$

This completed L-function has analytic continuation, is entire everywhere (note that in our case $m \geq 2$), and satisfies the functional equation

$$\Lambda(s, \pi) = \epsilon_\pi q_\pi^{\frac{1}{2}-s} \Lambda(1-s, \tilde{\pi}), \tag{3.1}$$

where $\tilde{\pi}$ is the contragredient of π, ϵ_π is a complex number of modulus 1, and q_π is a positive integer called the arithmetic conductor of π. For any place $p \leq \infty$, $\tilde{\pi}_p$ is equivalent to the complex conjugate $\overline{\pi_p}$, and we have

$$\{\alpha_{\tilde{\pi},p}(j)\} = \{\overline{\alpha_{\pi,p}(j)}\}, \qquad \{\mu_{\tilde{\pi}}(j)\} = \{\overline{\mu_\pi(j)}\}.$$

Therefore, from Lemma 2.1 and (3.1), we have

$$A_\pi(x) = \sum_{n \leq x} a_\pi(n) \ll_\pi x^{\frac{1}{2} - \frac{1}{2m} + (\frac{m}{2} - \frac{1}{2})\eta} + \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} |a_\pi(n)|, \tag{3.2}$$

for every $\eta \geq 0$.

For $m = 2$, from (1.4) we have

$$|a_\pi(n)| \leq \tau(n) n^{\frac{7}{64}}, \tag{3.3}$$

where $\tau(n)$ is the divisor function. From (3.2) with $m = 2$, we have

$$A_\pi(x) = \sum_{n \leq x} a_\pi(n) \ll_\pi x^{\frac{1}{4} + \frac{1}{2}\eta} + \sum_{x < n \leq x + x^{\frac{1}{2} - \eta}} |a_\pi(n)|. \tag{3.4}$$

From (3.3), we obtain

$$A_\pi(x) \ll_\pi x^{\frac{1}{4} + \frac{1}{2}\eta} + x^{\frac{39}{64} - \eta + \varepsilon}. \tag{3.5}$$

On taking $\eta = \frac{23}{96}$, we have

$$A_\pi(x) \ll_\pi x^{\frac{71}{192} + \varepsilon}. \tag{3.6}$$

In order to give the result for $m \geq 3$, we recall some basic facts about the Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi})$ associated to π and its contragredient $\tilde{\pi}$. It is defined as a product of local factors:

$$L(s, \pi \times \tilde{\pi}) = \prod_p L(s, \pi_p \times \tilde{\pi}_p). \tag{3.7}$$

For unramified primes p, the local factor is given by

$$L(s, \pi_p \times \tilde{\pi}_p) = \prod_{j=1}^{m} \prod_{k=1}^{m} (1 - \alpha_{\pi,p}(j)\overline{\alpha_{\pi,p}(k)} p^{-s})^{-1}. \tag{3.8}$$

It can be defined similarly at primes p where π_p is ramified. By (1.3), the product $\prod_p L(s, \pi_p \times \tilde{\pi}_p)$ converges absolutely on $\mathrm{Re}s > 2 - \frac{2}{m^2+1}$ (in fact on $\mathrm{Re}s > 1$; see e.g. Jacquet and Shalika [8] or Rudnick and Sarnak [11]). We write this product as a Dirichlet series:

$$L(s, \pi \times \tilde{\pi}) = \prod_p \sum_{k=0}^{\infty} \frac{a_{\pi \times \tilde{\pi}}(p^k)}{p^{ks}} = \sum_{n=1}^{\infty} \frac{a_{\pi \times \tilde{\pi}}(n)}{n^s}. \tag{3.9}$$

The completed Rankin-Selberg L-function is defined by

$$\Lambda(s, \pi \times \tilde{\pi}) = L(s, \pi_\infty \times \tilde{\pi}_\infty) L(s, \pi \times \tilde{\pi})$$

with

$$L(s, \pi_\infty \times \tilde{\pi}_\infty) = \prod_{j=1}^{m^2} \pi^{-\frac{s + \mu_{\pi \times \tilde{\pi}}(j)}{2}} \Gamma\left(\frac{s + \mu_{\pi \times \tilde{\pi}}(j)}{2}\right).$$

When π_∞ is unramified,

$$\{\mu_{\pi \times \tilde{\pi}}(j)\}_{1 \leq j \leq m^2} = \{\mu_\pi(j) + \overline{\mu_\pi(k)}\}_{1 \leq j \leq m, 1 \leq k \leq m}.$$

It is known that $a_{\pi \times \tilde{\pi}}(n) \geq 0$ and $L(s, \pi \times \tilde{\pi})$ has a simple pole at $s = 1$. The completed Rankin-Selberg L-function $\Lambda(s, \pi \times \tilde{\pi})$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation

$$\Lambda(s, \pi \times \tilde{\pi}) = \epsilon_{\pi \times \tilde{\pi}} q_{\pi \times \tilde{\pi}}^{\frac{1}{2} - s} \Lambda(1 - s, \pi \times \tilde{\pi}),$$

where $|\epsilon_{\pi \times \tilde{\pi}}| = 1$ and $q_{\pi \times \tilde{\pi}} > 0$.

Therefore by applying Lemma 2.2 to $L(s, \pi \times \tilde{\pi})$ with degree m^2, we have

$$\sum_{n \leq x} a_{\pi \times \tilde{\pi}}(n) = c_\pi x + O_{\varepsilon,\pi}(x^{\frac{m^2-1}{m^2+1} + \varepsilon}), \tag{3.10}$$

where c_π is a positive constant.

From (3.10), we find that for any $\eta \geq 0$,

$$\sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} a_{\pi \times \tilde{\pi}}(n) \ll_{\varepsilon,\pi} x^{\frac{m^2-1}{m^2+1} + \varepsilon}. \tag{3.11}$$

From (3.7), (3.8) and (3.9), we have that for $\mathrm{Re}s > 2 - \frac{2}{m^2+1}$,

$$\sum_{k=0}^{\infty} \frac{a_{\pi \times \tilde{\pi}}(p^k)}{p^{ks}} = \exp\left(\sum_{v=1}^{\infty} \frac{|\lambda_\pi(p^v)|^2}{v} p^{-vs}\right), \tag{3.12}$$

where

$$\lambda_\pi(p^v) = \sum_{j=1}^{m} \alpha_{\pi,p}(j)^v.$$

From (1.1), (1.2) and (1.5), we have

$$\sum_{k=0}^{\infty} \frac{a_\pi(p^k)}{p^{ks}} = \exp\left(\sum_{v=1}^{\infty} \frac{\lambda_\pi(p^v)}{v} p^{-vs}\right). \tag{3.13}$$

From (3.12), (3.13) and Lemma 2.3 with $j = 2$, we have that for an unramified prime p,

$$|a_\pi(p^k)|^2 \le a_{\pi\times\tilde{\pi}}(p^k),$$

and thus in our case

$$|a_\pi(n)|^2 \le a_{\pi\times\tilde{\pi}}(n).$$

Therefore we have

$$\sum_{x<n\le x+x^{1-\frac{1}{m}-\eta}} |a_\pi(n)|^2 \ll \sum_{x<n\le x+x^{1-\frac{1}{m}-\eta}} a_{\pi\times\tilde{\pi}}(n). \tag{3.14}$$

Now we begin to estimate (3.2). By Cauchy's inequality, we find that the short-interval sum in (3.2) satisfies

$$\sum_{x<n\le x+x^{1-\frac{1}{m}-\eta}} |a_\pi(n)| \le \left(\sum_{x<n\le x+x^{1-\frac{1}{m}-\eta}} |a_\pi(n)|^2\right)^{\frac{1}{2}} \left(\sum_{x<n\le x+x^{1-\frac{1}{m}-\eta}} 1\right)^{\frac{1}{2}}. \tag{3.15}$$

By (3.11) and (3.14), we have

$$\sum_{x<n\le x+x^{1-\frac{1}{m}-\eta}} |a_\pi(n)|^2 \ll_{\varepsilon,\pi} x^{\frac{m^2-1}{m^2+1}+\varepsilon}. \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\sum_{x<n\le x+x^{1-\frac{1}{m}-\eta}} |a_\pi(n)| \ll_{\varepsilon,\pi} x^{\frac{1}{2}-\frac{1}{2m}-\frac{\eta}{2}+\frac{m^2-1}{2m^2+2}+\varepsilon}. \tag{3.17}$$

Inserting (3.17) into (3.2), we have

$$A_\pi(x) = \sum_{n\le x} a_\pi(n) \ll_{\varepsilon,\pi} x^{\frac{1}{2}-\frac{1}{2m}+(\frac{m}{2}-\frac{1}{2})\eta} + x^{\frac{1}{2}-\frac{1}{2m}-\frac{\eta}{2}+\frac{m^2-1}{2m^2+2}+\varepsilon}.$$

On taking $\eta = \frac{m^2-1}{m(m^2+1)}$, we get

$$A_\pi(x) \ll_{\varepsilon,\pi} x^{\frac{m^2-m}{m^2+1}+\varepsilon}.$$

This completes the proof of Theorem 1.1.

4. Proof of Corollary 1.2

To prove Corollary 1.2, we recall some basic facts from the books of Iwaniec and Kowalski [7], and of Goldfeld [5]. Associated to each Hecke-Maass cusp form φ for the full modular group $SL(2,\mathbb{Z})$ there is an L-function $L(\varphi,s)$, which is defined, for $\mathrm{Re}s>1$, by

$$\begin{aligned} L(\varphi,s) &= \sum_{n=1}^{\infty} t(n)n^{-s} = \prod_p (1-t(p)p^{-s}+p^{-2s})^{-1} \\ &= \prod_p \left(1-\frac{\alpha_p}{p^s}\right)^{-1}\left(1-\frac{\alpha_p'}{p^s}\right)^{-1} \end{aligned}$$

with $\alpha_p+\alpha_p'=t(p)$ and $\alpha_p\alpha_p'=1$. The symmetric square L-function $L(\mathrm{Sym}^2\varphi,s)$ is defined, for $\mathrm{Re}s>1$, by

$$\begin{aligned} L(\mathrm{Sym}^2\varphi,s) &= \zeta(2s)\sum_{n=1}^{\infty} t(n^2)n^{-s} \\ &= \prod_p \left(1-\frac{\alpha_p^2}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{{\alpha_p'}^2}{p^s}\right)^{-1}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. Then we have

$$\sum_{n=1}^{\infty} t(n^2)n^{-s} = \frac{L(\mathrm{Sym}^2\varphi,s)}{\zeta(2s)}. \tag{4.1}$$

This gives

$$t(n^2) = \sum_{d^2|n} \mu(d)t^{(2)}\left(\frac{n}{d^2}\right), \tag{4.2}$$

where $t^{(2)}(n)$ is the nth coefficient of the symmetric square L-function $L(\mathrm{Sym}^2\varphi,s)$ with $\mathrm{Re}s>1$.

It follows from the Gelbart-Jacquet lift that $L(\mathrm{Sym}^2\varphi,s)$ is an automorphic L-function of GL_3. Then from Theorem 1.1 with $m=3$, we have

$$\sum_{n\le x} t^{(2)}(n) \ll x^{\frac{3}{5}+\varepsilon}. \tag{4.3}$$

From (4.2) and (4.3), we have

$$S(x) = \sum_{n\le x} t(n^2) \ll x^{\frac{3}{5}+\varepsilon}.$$

This completes the proof of Corollary 1.2.

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Department of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China

E-mail address: gslv@sdu.edu.cn