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RESONANCE OF BASIS-CONJUGATING AUTOMORPHISM GROUPS

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ABSTRACT. We determine the structure of the first resonance variety of the cohomology ring of the group of automorphisms of a finitely generated free group which act by conjugation on a given basis.

1. RESONANCE OF $P\Sigma_n$

Let F_n be the free group generated by $x_1, \dots, x_n$. The basis-conjugating automorphism group, or pure symmetric automorphism group, is the group $P\Sigma_n$ of all automorphisms of F_n which send each generator x_i to a conjugate of itself. Results of Dahm [4] and Goldsmith [7] imply that this group may also be realized as the "group of loops", the group of motions of a collection of n unknotted, unlinked oriented circles in 3-space, where each circle returns to its original position. McCool [9] found the following presentation for the basis-conjugating automorphism group:

$$P\Sigma_n = \langle \beta_{i,j}, 1 \le i \ne j \le n \mid [\beta_{i,j}, \beta_{k,l}], [\beta_{i,k}, \beta_{j,k}], [\beta_{i,j}, (\beta_{i,k} \cdot \beta_{j,k})] \rangle, \tag{1.1}$$

where $[u, v] = uvu^{-1}v^{-1}$ denotes the commutator, the indices in the relations are distinct, and the generators $\beta_{i,j}$ are the automorphisms of F_n defined by

$$\beta_{i,j}(x_k) = \begin{cases} x_k & \text{if } k \ne j, \\ x_j^{-1} x_i x_j & \text{if } k = i. \end{cases}$$

The purpose of this paper is to determine the structure of the first resonance variety of the cohomology ring of this group.

Let $A = \bigoplus_{k=0}^{\ell} A^k$ be a finite-dimensional, graded, connected algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Since $a \cdot a = 0$ for each $a \in A^1$, multiplication by a defines a cochain complex (A, δ_a):

$$A^0 \xrightarrow{\delta_a} A^1 \xrightarrow{\delta_a} A^2 \xrightarrow{\delta_a} \cdots\cdots \xrightarrow{\delta_a} A^{\ell},$$

where $\delta_a(x) = ax$. The resonance varieties of A are the jumping loci for the cohomology of these complexes: $R_d^j(A) = \{a \in A^1 \mid \dim_{\Bbbk} H^j(A, \delta_a) \ge d\}$. As shown by Falk [6], these algebraic subvarieties of A^1 are isomorphism-type invariants of the algebra A. They have been the subject of considerable recent interest in the context of hyperplane arrangements and related areas; see, for instance, Dimca,

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Papadima, and Suciu [5], Yuzvinsky [14], and the references therein. We will focus on the first resonance variety $R^1(A) = \{a \in A^1 \mid H^1(A, \delta_a) \neq 0\}$.

Since the relations in the presentation (1.1) of $P\Sigma_n$ are all commutators, the first homology group $H_1(P\Sigma_n; \Bbbk)$ is a vector space of dimension $n(n-1)$ with basis $\{[\beta_{p,q}] \mid 1 \leq p \neq q \leq \ell\}$. Let $\{e_{p,q} \mid 1 \leq p \neq q \leq \ell\}$ be the dual basis of $H^1(P\Sigma_n; \Bbbk)$. Denote the first resonance variety of $A = H^*(P\Sigma_n; \Bbbk)$ by $R^1(P\Sigma_n, \Bbbk)$.

Theorem 1.1. *The first resonance variety of the cohomology ring $H^*(P\Sigma_n; \Bbbk)$ of the basis-conjugating automorphism group is*

$$R^1(P\Sigma_n, \Bbbk) = \bigcup_{1 \leq i < j \leq n} C_{i,j} \ \cup \bigcup_{1 \leq i < j < k \leq n} C_{i,j,k},$$

where $C_{i,j} = \operatorname{span}\{e_{i,j}, e_{j,i}\}$ and $C_{i,j,k} = \operatorname{span}\{e_{j,i} - e_{k,i}, e_{i,j} - e_{k,j}, e_{i,k} - e_{j,k}\}$.

This result reveals an interesting relationship between the resonance variety and the Bieri-Neumann-Strebel (BNS) invariant of the basis-conjugating automorphism group. For a finitely generated group G, let $\mathcal{C}$ be the Cayley graph corresponding to a finite generating set. Given an additive character $\chi \colon G \to \mathbb{R}$, let $\mathcal{C}_+(\chi)$ be the full subgraph of $\mathcal{C}$ on the vertex set $\{g \in G \mid \chi(g) \geq 0\}$. Then the (first) BNS invariant of G is the conical subset $\Sigma(G)$ of $\operatorname{Hom}(G, \mathbb{R}) = H^1(G; \mathbb{R})$ defined by

$$\Sigma(G) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \mathcal{C}_+(\chi) \text{ is connected}\}.$$

This invariant of G (which is independent of the choice of generating set) may be used to determine which subgroups above the commutator subgroup $[G, G]$ are finitely generated; see [1].

The BNS invariant of the group $G = P\Sigma_n$ was determined by Orlandi-Korner [11]. Combining her result with the above theorem yields the following:

Theorem 1.2. *The Bieri-Neumann-Strebel invariant of the basis-conjugating automorphism group is given by*

$$\Sigma(P\Sigma_n) = H^1(P\Sigma_n; \mathbb{R}) \setminus R^1(P\Sigma_n, \mathbb{R}).$$

This relationship between the resonance variety and the BNS invariant is known to hold for certain other groups, including right-angled Artin groups; see Meier and VanWyk [10] and Papadima and Suciu [12]. More recently, Papadima and Suciu have shown that the containment $\Sigma(G) \subseteq H^1(G; \mathbb{R}) \setminus R^1(G, \mathbb{R})$ holds for an arbitrary 1-formal group G, where, as above, $R^1(G; \mathbb{R})$ denotes the first resonance variety of $H^*(G; \mathbb{R})$. See [13] for a detailed investigation of the relationship between BNS invariants and (co)homology jumping loci in a number of contexts.

2. Proof of Theorem 1.1

In this section, we recall the structure of the cohomology ring of the basis-conjugating automorphism group and use it to prove Theorem 1.1. The cohomology of $P\Sigma_n$ was computed by Jensen, McCammond, and Meier [8], resolving positively a conjecture of Brownstein and Lee [2].

Theorem 2.1 ([8]). *Let $E_{\mathbb{Z}}$ denote the exterior algebra over $\mathbb{Z}$ generated by degree one elements $e_{p,q}$, $1 \leq p \neq q \leq n$, and let $I_{\mathbb{Z}}$ denote the two-sided ideal in $E_{\mathbb{Z}}$ generated by*

$$\begin{aligned} &\eta_{i,j} = e_{i,j}e_{j,i}, \ 1 \leq i < j \leq n; \\ &\tau_{i,j}^k = (e_{k,i} - e_{j,i})(e_{k,j} - e_{i,j}), \ 1 \leq k \leq n, \ 1 \leq i < j \leq n, \ k \notin \{i, j\}. \end{aligned} \tag{2.1}$$

Then the integral cohomology algebra of the basis-conjugating automorphism group $P\Sigma_n$ is isomorphic to the quotient of $E_{\mathbb{Z}}$ by $I_{\mathbb{Z}}$, $H^*(P\Sigma_n;\mathbb{Z}) \cong E_{\mathbb{Z}}/I_{\mathbb{Z}}$.

Remark 2.2. The above presentation of the cohomology ring $H^*(P\Sigma_n;\mathbb{Z})$ differs slightly from that given in [2, 8] but is easily seen to be equivalent. For instance, the relation in $H^*(P\Sigma_n;\mathbb{Z})$ arising from the generator $\tau_{i,j}^k$ of the ideal I may be obtained by constructing an appropriate linear combination of the relations labeled 2 and 3 in [8, Thm. 6.7].

Let $\Bbbk$ be a field of characteristic zero. From Theorem 2.1 and the Universal Coefficient Theorem, the cohomology algebra $H^*(P\Sigma_n;\Bbbk)$ is isomorphic to $E_{\Bbbk}/I_{\Bbbk}$, where $E_{\Bbbk}$ is the exterior algebra over $\Bbbk$ generated by $e_{p,q}$, $1 \le p \ne q \le n$, and $I_{\Bbbk}$ is the ideal in $E_{\Bbbk}$ generated by the elements $\eta_{i,j}$ and $\tau_{i,j}^k$ above.

Recall from Section 1 that the first resonance variety of $A = H^*(P\Sigma_n;\Bbbk)$ is

$$R^1(P\Sigma_n,\Bbbk) = \{a \in A^1 \mid H^1(A,\delta_a) \ne 0\}.$$

Observe that $A^1 = H^1(P\Sigma_n;\Bbbk)$ is a vector space of dimension $N = n(n-1)$ over $\Bbbk$. Elements of A^1 are of the form $a = \sum_{p\ne q} a_{p,q}e_{p,q}$, where $a_{p,q} \in \Bbbk$. Recall that

$$C_{i,j} = \operatorname{span}\{e_{i,j}, e_{j,i}\} = \{a \in A^1 \mid a_{p,q} = 0 \text{ if } \{p,q\} \ne \{i,j\}\} \tag{2.2}$$

for $1 \le i < j \le n$ and that

$$\begin{aligned} C_{i,j,k} &= \operatorname{span}\{e_{j,i} - e_{k,i}, e_{i,j} - e_{k,j}, e_{i,k} - e_{j,k}\} \\ &= \left\{ a \in A^1 \;\middle|\; \begin{array}{l} a_{j,i} + a_{k,i} = 0,\ a_{i,j} + a_{k,j} = 0,\ a_{i,k} + a_{j,k} = 0, \\ a_{p,q} = 0 \text{ if } \{p,q\} \not\subset \{i,j,k\} \end{array} \right\} \end{aligned} \tag{2.3}$$

for $1 \le i < j < k \le n$. Note that these are linear subspaces of $A^1 \cong \Bbbk^N$ of dimensions 2 and 3 respectively. To prove Theorem 1.1, we must show that $R^1(P\Sigma_n,\Bbbk)$ is equal to the union of these linear subspaces.

Proof of Theorem 1.1. In the case $n = 2$, the group $P\Sigma_n = \langle \beta_{1,2}, \beta_{2,1}\rangle \cong F_2$ is a free group, and the theorem asserts that $R^1(P\Sigma_2,\Bbbk) = C_{1,2} = H^1(P\Sigma_2;\Bbbk)$, which is clear. So assume that $n \ge 3$.

Write $R = R^1(P\Sigma_n,\Bbbk)$ and $C = \bigcup_{1\le i<j\le n} C_{i,j} \cup \bigcup_{1\le i<j<k\le n} C_{i,j,k}$. Observe that $0 \in C$ and $0 \in R$. So it is enough to show that $R \setminus \{0\} = C \setminus \{0\}$.

Write $E = E_{\Bbbk}$ and $I = I_{\Bbbk}$. For $a \in A^1 = E^1$, we have a short exact sequence of chain complexes $0 \longrightarrow (I,\delta_a) \xrightarrow{\iota} (E,\delta_a) \xrightarrow{p} (A,\delta_a) \longrightarrow 0$:

$$\begin{array}{ccccccccc} & & & & I^2 & \xrightarrow{\delta_a} & I^3 & \longrightarrow & \cdots \\ & & & & \downarrow{\scriptstyle \iota^2} & & \downarrow{\scriptstyle \iota^3} & & \\ E^0 & \xrightarrow{\delta_a} & E^1 & \xrightarrow{\delta_a} & E^2 & \xrightarrow{\delta_a} & E^3 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle p^0} & & \downarrow{\scriptstyle p^1} & & \downarrow{\scriptstyle p^2} & & \downarrow{\scriptstyle p^3} & & \\ A^0 & \xrightarrow{\delta_a} & A^1 & \xrightarrow{\delta_a} & A^2 & \xrightarrow{\delta_a} & A^3 & \longrightarrow & \cdots \end{array}$$

where $\iota\colon I \to E$ is the inclusion, $p\colon E \to A$ the projection, and $\delta_a(x) = ax$. Note that since I is generated in degree two, the maps $p^0\colon E^0 \to A^0$ and $p^1\colon E^1 \to A^1$ are identity maps. If $a \ne 0$, then the complex (E,δ_a) is acyclic. Consequently, the corresponding long exact cohomology sequence yields

$$H^1(A,\delta_a) \cong H^2(I,\delta_a) = \ker(\delta_a\colon I^2 \to I^3) = \ker(\iota^3 \circ \delta_a\colon I^2 \to E^3).$$

Thus, $a \in R \setminus \{0\}$ if and only if the map $\psi_a := \iota^3 \circ \delta_a$ fails to inject.

Since the elements $\eta_{i,j}$ and $\tau_{i,j}^k$ recorded in (2.1) generate the ideal I and are of degree two (and are linearly independent in E^2), these elements form a basis for I^2. We record the images of these basis elements under the map ψ_a. For $1 \leq i < j \leq n$,

$$\psi_a(\eta_{i,j}) = \sum_{\{p,q\} \neq \{i,j\}} a_{p,q} e_{p,q} \eta_{i,j} = \sum_{\{p,q\} \neq \{i,j\}} a_{p,q} e_{p,q} e_{i,j} e_{j,i}. \tag{2.4}$$

For $1 \leq k \leq n$, $1 \leq i < j \leq n$, $k \notin \{i,j\}$,

$$\begin{aligned} \psi_a(\tau_{i,j}^k) = (a_{j,i} + a_{k,i}) e_{j,i} e_{k,i} (e_{k,j} - e_{i,j}) - (a_{i,j} + a_{k,j}) e_{i,j} e_{k,j} (e_{k,i} - e_{j,i}) \\ + (a_{i,k} e_{i,k} + a_{j,k} e_{j,k}) \tau_{i,j}^k + \sum_{\{p,q\} \not\subset \{i,j,k\}} a_{p,q} e_{p,q} \tau_{i,j}^k. \end{aligned} \tag{2.5}$$

These calculations immediately yield the containment $C \setminus \{0\} \subseteq R \setminus \{0\}$. If $a \in C_{i,j}$, then $a_{p,q} = 0$ for $\{p,q\} \neq \{i,j\}$. For such an a, we have $\psi_a(\eta_{i,j}) = 0$ by (2.4), so $C_{i,j} \subset R$. If $1 \leq i < j < k \leq n$ and $a \in C_{i,j,k}$, then $a_{j,i} + a_{k,i} = 0$, $a_{i,j} + a_{k,j} = 0$, $a_{i,k} + a_{j,k} = 0$, and $a_{p,q} = 0$ for $\{p,q\} \not\subset \{i,j,k\}$. In this instance, (2.5) may be used to check that $a_{j,i}\tau_{i,j}^k - a_{i,k}\tau_{j,k}^i$ and $a_{i,j}\tau_{i,j}^k - a_{i,k}\tau_{i,k}^j$ are elements of $\ker(\psi_a)$. If $a \in C_{i,j,k}$ is nonzero, at least one of $a_{i,j}, a_{i,k}, a_{j,i}$ must be nonzero. Consequently, ψ_a has nontrivial kernel, and $C_{i,j,k} \subset R$.

Establishing the reverse containment, $R \setminus \{0\} \subseteq C \setminus \{0\}$, is more involved. We will show that $a \notin C$ implies that $a \notin R$. If $a \notin C$, then $a \neq 0$. So assume without loss that $a_{2,1} \neq 0$. Since $a \notin C_{1,2} \subset C$, we must also have $a_{p,q} \neq 0$ for some $\{p,q\} \neq \{1,2\}$. We will consider several cases depending on the relationship between the sets $\{1,2\}$ and $\{p,q\}$.

Case 1. $\{1,2\} \cap \{p,q\} = \emptyset$

Assume first that $\{1,2\}$ and $\{p,q\}$ are disjoint. Note that $n \geq 4$ in this instance. Permuting indices if need be, we may assume that $a \in H^1(P\Sigma_n; \Bbbk)$ satisfies $a_{2,1} \neq 0$ and $a_{3,4} \neq 0$. We will show that this assumption implies that the map $\psi_a \colon I^2 \to E^3$ injects; hence $a \notin R$. Specifically, we will exhibit a subspace $V \subset E^3$ and a projection $\pi \colon E^3 \twoheadrightarrow V$ so that the composition $\pi \circ \psi_a \colon I^2 \to V$ is an isomorphism.

Let $\mathcal{V}$ be the union of the sets

$$\begin{aligned} &\{e_{1,2}e_{2,1}e_{3,4}\} \cup \{e_{2,1}e_{i,j}e_{j,i} \mid 1 \leq i < j \leq n,\ \{i,j\} \neq \{1,2\}\}, \\ &\{e_{3,4}e_{1,2}e_{k,1},\ e_{3,4}e_{2,1}e_{1,k},\ e_{3,4}e_{1,2}e_{1,k} \mid 3 \leq k \leq n\}, \\ &\{e_{2,1}e_{k,i}e_{k,j},\ e_{2,1}e_{j,i}e_{j,k},\ e_{2,1}e_{i,k}e_{k,j} \mid i \leq 2 < j < k \leq n \text{ or } 3 \leq i < j < k \leq n\}. \end{aligned}$$

There is a bijection between $\mathcal{V}$ and the set of generators of I^2 given by

$$\begin{array}{llll} \eta_{1,2} \leftrightarrow e_{1,2}e_{2,1}e_{3,4}, & \eta_{i,j} \leftrightarrow e_{2,1}e_{i,j}e_{j,i} & (\{i,j\} \neq \{1,2\}), & \\ \tau_{1,2}^k \leftrightarrow e_{3,4}e_{1,2}e_{k,1}, & \tau_{1,k}^2 \leftrightarrow e_{3,4}e_{2,1}e_{1,k}, & \tau_{2,k}^1 \leftrightarrow e_{3,4}e_{1,2}e_{1,k} & (3 \leq k \leq n), \\ \tau_{i,j}^k \leftrightarrow e_{2,1}e_{k,i}e_{k,j}, & \tau_{i,k}^j \leftrightarrow e_{2,1}e_{j,i}e_{j,k}, & \tau_{j,k}^i \leftrightarrow e_{2,1}e_{i,k}e_{k,j} & (i \leq 2 < j < k \leq n). \end{array}$$

In particular, the monomials in $\mathcal{V}$ are distinct. Hence, $\mathcal{V}$ is a linearly independent set in E^3 of cardinality $|\mathcal{V}| = \dim_{\Bbbk} I^2 = \binom{n}{2}(n-1)$. Let $V = \operatorname{span} \mathcal{V} \subset E^3$.

Define $\pi \colon E^3 \twoheadrightarrow V$ on basis elements by

$$\pi(e_{a,b}e_{c,d}e_{p,q}) = \begin{cases} e_{a,b}e_{c,d}e_{p,q} & \text{if } e_{a,b}e_{c,d}e_{p,q} \in \mathcal{V}, \\ 0 & \text{if } e_{a,b}e_{c,d}e_{p,q} \notin \mathcal{V}. \end{cases}$$

Then, a calculation using (2.4) and (2.5) reveals that $\pi \circ \psi_a \colon I^2 \to V$ is an isomorphism. For instance, ordering the bases of I^2 and V appropriately, one can check that the matrix M of $\pi \circ \psi_a$ has determinant $\det \mathsf{M} = y_{2,1}^{m_{2,1}} y_{3,4}^{m_{3,4}} \neq 0$, where $m_{3,4} = 3n - 5$ and $m_{2,1} = \binom{n}{2}(n-1) - m_{3,4}$. Thus, if $a \notin C$ satisfies $a_{2,1} \neq 0$ and $a_{p,q} \neq 0$ for some p, q with $\{1,2\} \cap \{p,q\} = \emptyset$, then $a \notin R$.

Case 2. $\{1,2\} \cap \{p,q\} \neq \emptyset$
Now assume that $a \notin C$, $a_{2,1} \neq 0$, and $a_{r,s} = 0$ for all r, s with $\{1,2\} \cap \{r,s\} = \emptyset$. Since $a \notin C_{1,2} \subset C$, we must have $a_{p,q} \neq 0$ for some p, q with $|\{p,q\} \cap \{1,2\}| = 1$. Permuting indices if necessary, we may assume that $3 \in \{p,q\}$.

In the case $n = 3$, since $a \notin C_{1,2,3} \subset C$, one of the sums $a_{2,1} + a_{3,1}$, $a_{1,2} + a_{3,2}$, $a_{1,3} + a_{2,3}$ must be nonzero; see (2.3). In this instance, ordering bases appropriately, the map $\psi_a \colon I^2 \to E^3$ has matrix

$$\mathsf{M}_3 = \begin{pmatrix}
a_{3,2} & 0 & 0 & 0 & -a_{1,2}-a_{3,2} & 0 \\
a_{3,1} & 0 & 0 & 0 & -a_{2,1}-a_{3,1} & 0 \\
a_{2,3} & 0 & 0 & a_{1,2} & -a_{2,3} & a_{2,1} \\
-a_{1,3} & 0 & 0 & a_{1,2} & a_{1,3} & a_{2,1} \\
0 & a_{3,2} & 0 & -a_{3,2} & a_{1,3} & -a_{3,1} \\
0 & -a_{2,3} & 0 & a_{1,3}+a_{2,3} & 0 & 0 \\
0 & -a_{2,1} & 0 & a_{2,1}+a_{3,1} & 0 & 0 \\
0 & a_{1,2} & 0 & -a_{1,2} & -a_{1,3} & a_{3,1} \\
0 & 0 & -a_{3,1} & a_{3,2} & a_{2,3} & a_{3,1} \\
0 & 0 & a_{2,1} & a_{3,2} & a_{2,3} & -a_{2,1} \\
0 & 0 & a_{1,3} & 0 & 0 & -a_{1,3}-a_{2,3} \\
0 & 0 & a_{1,2} & 0 & 0 & -a_{1,2}-a_{3,2} \\
0 & 0 & 0 & a_{3,2} & -a_{1,3} & -a_{2,1} \\
0 & 0 & 0 & a_{2,1}+a_{3,1} & 0 & 0 \\
0 & 0 & 0 & a_{1,3}+a_{2,3} & 0 & 0 \\
0 & 0 & 0 & a_{1,2} & -a_{2,3} & -a_{3,1} \\
0 & 0 & 0 & 0 & a_{1,2}+a_{3,2} & 0 \\
0 & 0 & 0 & 0 & a_{2,1}+a_{3,1} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{1,2}+a_{3,2} \\
0 & 0 & 0 & 0 & 0 & a_{1,3}+a_{2,3}
\end{pmatrix}. \tag{2.6}$$

Using the assumptions on $a_{2,1}$, the sums $a_{2,1} + a_{3,1}$, $a_{1,2} + a_{3,2}$, $a_{1,3} + a_{2,3}$, and $a_{p,q} \in \{a_{1,3}, a_{2,3}, a_{3,1}, a_{3,2}\}$, one can check that the matrix M_3 has maximal rank. Hence, if $a \notin C$, then $\psi_a \colon I^2 \to E^3$ injects in the case $n = 3$.

For general n, the assumption that $a \notin C_{1,2,3}$ implies that the set

$$\{a_{2,1} + a_{3,1}, a_{1,2} + a_{3,2}, a_{1,3} + a_{2,3}\} \cup \{a_{r,s} \mid \{r,s\} \not\subset \{1,2,3\}\}$$

contains a nonzero element. Recall that, by Case 1, we may assume that $a_{r,s} = 0$ for all r, s with $\{1,2\} \cap \{r,s\} = \emptyset$. If $a_{r,s} = 0$ for all $\{r,s\} \not\subset \{1,2,3\}$, let W be the subspace of E^3 spanned by the union of the sets

$$\{e_{i_1,j_1} e_{i_2,j_2} e_{i_3,j_3} \mid 1 \leq i_k, j_k \leq 3,\ i_k \neq j_k\},$$
$$\{e_{1,2} e_{2,1} e_{p,q}\} \cup \{e_{2,1} e_{i,j} e_{j,i} \mid 1 \leq i < j \leq n,\ \{i,j\} \neq \{1,2\}\},$$
$$\{e_{2,1} e_{k,i} e_{k,j},\ e_{2,1} e_{j,i} e_{j,k},\ e_{2,1} e_{i,k} e_{k,j} \mid i \leq 2 < j < k \leq n \text{ or } 3 \leq i < j < k \leq n\},$$
$$\left\{\begin{matrix} e_{p,q} e_{k,1} e_{k,2}, & e_{p,q} e_{k,1} e_{1,2}, & e_{p,q} e_{2,1} e_{k,2}, \\ e_{p,q} e_{2,1} e_{2,k}, & e_{p,q} e_{2,1} e_{1,k}, & e_{p,q} e_{k,1} e_{2,k}, \\ e_{p,q} e_{1,2} e_{1,k}, & e_{p,q} e_{1,2} e_{2,k}, & e_{p,q} e_{k,2} e_{1,k} \end{matrix} \;\middle|\; 3 \leq k \leq n \right\}.$$

Define $\pi\colon E^3 \twoheadrightarrow W$ on basis elements as before. Ordering bases appropriately, one can use (2.4) and (2.5) to find a submatrix M of the matrix of $\pi\circ\psi_a\colon I^2 \to W$ of the form

$$\mathsf{M} = \begin{pmatrix} \mathsf{U} & * \\ 0 & \mathsf{M}_3 \end{pmatrix},$$

where M_3 is given by (2.6) and U is upper triangular, with diagonal entries $a_{2,1}\neq 0$ and $a_{p,q}\neq 0$. (The choice of U depends on which $a_{p,q}\in\{a_{1,3},a_{2,3},a_{3,1},a_{3,2}\}$ is nonzero.) Hence, the matrix M has maximal rank. It follows that $\psi_a\colon I^2\to E^3$ injects.

Finally, consider the case where $a\notin C_{1,2,3}\subset C$, $a_{2,1}\neq 0$, $a_{p,q}\neq 0$ for some $a_{p,q}\in\{a_{1,3},a_{2,3},a_{3,1},a_{3,2}\}$, and $a_{r,s}\neq 0$ for some $\{r,s\}\not\subset\{1,2,3\}$. Since we may assume by Case 1 that $a_{r,s}=0$ if $\{1,2\}\cap\{r,s\}=\emptyset$, we have $a_{r,s}\neq 0$ for some r,s with $r\in\{1,2\}$ and $4\le s\le n$. In this instance, let W be the subspace of E^3 spanned by the union of the sets

$$\begin{aligned}
&\{e_{1,2}e_{2,1}e_{p,q}\}\cup\{e_{2,1}e_{i,j}e_{j,i}\mid 1\le i<j\le n,\ \{i,j\}\neq\{1,2\}\},\\
&\{e_{2,1}e_{k,i}e_{k,j},\ e_{2,1}e_{j,i}e_{j,k},\ e_{2,1}e_{i,k}e_{k,j}\mid i\le 2<j<k\le n \text{ or } 3\le i<j<k\le n\},\\
&\{e_{r,s}e_{3,1}e_{3,2},\ e_{r,s}e_{1,3}e_{3,2},\ e_{r,s}e_{2,3}e_{3,1}\},\\
&\{e_{p,q}e_{k,1}e_{k,2},\ e_{p,q}e_{2,k}e_{k,1},\ e_{p,q}e_{1,k}e_{k,2}\mid 4\le k\le n\}.
\end{aligned}$$

Defining $\pi\colon E^3\twoheadrightarrow W$ on basis elements as above, a calculation using (2.4) and (2.5) shows that $\pi\circ\psi_a\colon I^2\to W$ is an isomorphism. For instance, ordering bases appropriately, one can check that the matrix M of $\pi\circ\psi_a$ has determinant $\det\mathsf{M} = a_{2,1}^{m_{2,1}}a_{p,q}^{m_{p,q}}a_{r,s}^3$, where $m_{p,q}=3n-8$ and $m_{2,1}=\binom{n}{2}(n-1)-m_{p,q}$. Hence, ψ_a injects in this final case.

Thus, for any $a\notin C$, the map $\psi_a\colon I^2\to E^3$ injects, and $a\notin R$. This completes the proof of Theorem 1.1. □

Remark 2.3. It follows from Theorem 2.1 that the integral cohomology groups of $P\Sigma_n$ are torsion free, with Betti numbers $b_k(P\Sigma_n)=\operatorname{rank}H^k(P\Sigma_n;\mathbb{Z})$ given by the coefficients of the Poincaré polynomial $\mathfrak{p}(P\Sigma_n,t)=\sum_{k\ge 0}b_k(P\Sigma_n)\cdot t^k=(1+nt)^{n-1}$; see [8, §6]. Thus the cohomology groups cannot distinguish $P\Sigma_n$ from a direct product $F_n\times\cdots\times F_n$ of $n-1$ free groups of rank n. For $n=2$, this is to be expected, since $P\Sigma_2\cong F_2$.

For $n\ge 3$, the groups $P\Sigma_n$ and $F_n\times\cdots\times F_n$ are not isomorphic and are, in fact, distinguished by their cohomology rings. By Theorem 1.1, the irreducible components of $R^1(P\Sigma_n,\Bbbk)$ are two- and three-dimensional. On the other hand, the results of [3] or [12] may be used to show that the irreducible components of the first resonance variety of $H^*(F_n\times\cdots\times F_n;\Bbbk)$ are all n-dimensional.

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