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COMPACT GRAPHS OVER A SPHERE OF CONSTANT SECOND ORDER MEAN CURVATURE

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Abstract. The aim of this work is to show that a compact smooth star-shaped hypersurface Σ^n in the Euclidean sphere $\mathbb{S}^{n+1}$ whose second function of curvature S_2 is a positive constant must be a geodesic sphere $\mathbb{S}^n(\rho)$. This generalizes a result obtained by Jellett in 1853 for surfaces Σ^2 with constant mean curvature in the Euclidean space $\mathbb{R}^3$ as well as a recent result of the authors for this type of hypersurface in the Euclidean sphere $\mathbb{S}^{n+1}$ with constant mean curvature. In order to prove our theorem we shall present a formula for the operator $L_r(g) = div\,(P_r\nabla g)$ associated with a new support function g defined over a hypersurface M^n in a Riemannian space form M_c^{n+1}.

1. Introduction

This paper continues work of the authors concerning constant mean curvature [5], which was inspired by a paper due to Jellett [11] published in the middle of the nineteenth century in which he proved that a smooth star-shaped constant mean curvature surface $\Sigma^2 \subset \mathbb{R}^3$ is a round sphere. On the other hand, if $\Sigma^n \subset \mathbb{R}^{n+1}$ is an oriented hypersurface and $k_1, \ldots, k_n$ represent the principal curvatures of Σ^n, we may consider similar problems related to the r^{th} elementary symmetric functions S_r given by $S_r = \sum k_{i_1} \ldots k_{i_r}$, $r = 0, 1, \ldots, n$. For instance, Suss [18] proved that a compact convex hypersurface Σ^n in the Euclidean space $\mathbb{R}^{n+1}$ with some S_r constant must be a round sphere. The convexity condition was improved upon by Hsiung [10], who showed that a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ whose classical support function has a well-defined sign with any symmetric function of the principal curvatures constant must also be a round sphere. Later, Ros proved in [14] and [15] that a round sphere is the unique compact embedded hypersurface in Euclidean space if any symmetric function S_r is constant. Subsequently, Montiel and Ros [12] extended this result to any compact embedded hypersurface in the hyperbolic space $\mathbb{H}^{n+1}$ as well as to one contained in an open hemisphere of the Euclidean sphere $\mathbb{S}^{n+1}$. On the other hand, it is well known that a product of spheres produces hypersurfaces in the Euclidean sphere $\mathbb{S}^{n+1}$ with S_r constant for any $r = 1, \ldots, n$. Therefore, for hypersurfaces contained in the Euclidean sphere $\mathbb{S}^{n+1}$ we have a lot of examples with S_r constant which are not round spheres. However, returning

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to Jellett's idea, we may obtain a similar result in the Euclidean sphere $\mathbb{S}^{n+1}$ for compact smooth star-shaped hypersurfaces whose second function of curvature is positive while avoiding the condition that it be contained in an open hemisphere. More precisely, we shall prove the following theorem.

Theorem 1.1. *Let $\Sigma^n \subset \mathbb{S}^{n+1}$ be a compact smooth star-shaped hypersurface with S_2 constant and positive. Then Σ^n is totally umbilical.*

2. Conformal fields

Given a vector field $V \in \chi(\overline{M})$ in a Riemannian manifold $\overline{M}$ and a r-covariant tensor field ω, the Lie derivative of ω with respect to V is defined by

$$(\mathcal{L}_V\omega)(X_1, ..., X_r) = V\left(\omega(X_1, ..., X_r)\right) - \sum_{i=1}^{r} \omega(X_1, ..., [V, X_i], ..., X_r).$$

For instance, if $\omega = \langle\, ,\rangle$ stands for the Riemannian metric of $\overline{M}$, then

$$(\mathcal{L}_V\langle,\rangle)(X, Y) \quad = \quad \langle \nabla_X V, Y\rangle + \langle X, \nabla_Y V\rangle.$$

We say that $V \in \chi(\overline{M})$ is a conformal vector field if there exists $\psi \in \mathcal{D}(\overline{M})$, which is called the conformal factor of V, such that

$$\mathcal{L}_V\langle\, ,\rangle = 2\psi\langle\, ,\rangle.$$

A useful conformal vector field in a space form M_c^{n+1} is the position vector with origin at a fixed point $p_0 \in M_c^{n+1}$. It was introduced for non-Euclidean space forms by Heintze [9] according to the following: Let $d : M_c^{n+1} \to \mathbb{R}$ be the distance function defined by $d(.) = dist(.\ , p_0)$. The position vector field over M_c^{n+1} is given by $V = s(d)\nabla d$ where $s(t)$ is the solution of the differential equation $y'' + cy = 0$ subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

On the other hand Greene and Wu [8] have shown an important property concerning the Hessian form of d which states that

$$s(d)\langle D_X\nabla d, Y\rangle = s'(d)(\langle X, Y\rangle - \langle \nabla d, X\rangle\langle \nabla d, Y\rangle),$$

for any vector fields $X, Y \in \chi(M_c^{n+1})$. Here and elsewhere D stands for the Riemannian connection of M_c^{n+1}. From the last relation we conclude that V is a conformal vector field with conformal factor $\psi = s'(d)$.

3. L_r of a support function

In this section we compute the operator L_r of a new support function defined on a hypersurface M^n in a space form M_c^{n+1}. For generalized Robertson-Walker space a particular case appears in Alias and Colares [1]. Given a conformal vector field V in M_c^{n+1} and an isometric immersion $x : M^n \looparrowright M_c^{n+1}$, the support function is defined on M by

$$g(p) = \langle V, N\rangle(x(p)),$$

where N stands for a unit normal field to M^n. Whenever necessary we shall identify $p \in M^n$ with its image $x(p) \in M_c^{n+1}$.

We now consider on M the second fundamental form A, the r^{th} symmetric function of curvature S_r, the r^{th} Newton tensor P_r and the operator L_r. These Newton tensors are defined inductively according to $P_0 = I$ and, for $1 \le r \le n$, $P_r = S_r I - AP_{r-1}$, whereas the operator $L_r : D(M) \to D(M)$ is given by

$$L_r(f) = tr\left(P_r \circ Hess_f\right).$$

It is important to point out that for hypersurfaces in a space form M_c^{n+1} Rosenberg showed in [16] that each L_r takes a divergence form. More exactly, we have

$$L_r(f) = div\,(P_r \nabla f)\,.$$

Moreover, under the above conditions we list some important properties concerning S_r and P_r that may be found in [13] and [16].

- $tr(P_r) = (n-r)S_r$;
- $tr(AP_r) = (r+1)S_{r+1}$;
- $tr(A^2P_r) = S_1S_{r+1} - (r+2)S_{r+2}$;
- $tr\,(P_r \circ D_X A) = \langle X, \nabla S_{r+1}\rangle$;
- P_r is a self-adjoint operator whose eigenvalues are $\frac{\partial S_{r+1}}{\partial \lambda_i}$, where the λ_i's are the eigenvalues of A.

We point out that for $r = 0$ the operator L_r is the Laplacian Δ. In this case a slight modification of the next theorem was obtained by many authors (see e.g. [2], [4], [5], [7] and [17]), whereas formulae for L_rX and L_rN can be found in Reilly [13] and Rosenberg [16], where X stands for the position vector while N is a unit normal vector field to M.

Theorem 3.1. *Let $x : M^n \looparrowright M_c^{n+1}$ be an oriented isometric immersion with unit normal vector field N. If V is a conformal vector field on M_c^{n+1} and $g = \langle V, N\rangle$ represents the support function on M^n, then*

$$\begin{aligned} L_r(g) &= -(S_1S_{r+1} - (r+2)S_{r+2})g - c(n-r)S_r g \\ &- (n-r)S_rN(\psi) - (r+1)S_{r+1}\psi - \langle V, \nabla S_{r+1}\rangle, \end{aligned}$$

where ψ, which we identify with $\psi \circ x$, is the conformal factor of V.

Proof. Given $p \in x(M^n)$, let $\{e_1(p), ..., e_n(p)\} \subset T_pM$ be an orthonormal basis which diagonalizes A at p and whose associated eigenvalues are $\lambda_1, ..., \lambda_n$, respectively. Denote by $\{e_1, ..., e_n\}$ the geodesic frame that extends the above basis to a neighborhood of p in $x(M^n)$. We may do this on a neighborhood of p in M_c^{n+1} in such a way that $D_Ne_i(p) = 0$. We now denote by $\langle\, ,\rangle$ the Riemannian metric of M_c^{n+1} and use the short notation $\mathcal{L}_{N,N} = (\mathcal{L}_V\langle\, ,\rangle)(N, N)$, $\mathcal{L}_{i,i} = (\mathcal{L}_V\langle\, ,\rangle)(e_i, e_i)$ and $\mathcal{L}_{i,N} = (\mathcal{L}_V\langle\, ,\rangle)(e_i, N)$ for the Lie derivatives. First of all we notice that

$$e_ie_i(g) = \langle D_{e_i}D_{e_i}N, V\rangle + 2\langle D_{e_i}N, D_{e_i}V\rangle + \langle D_{e_i}D_{e_i}V, N\rangle.$$

Because $-D_{e_i}N(p) = A(e_i(p)) = \lambda_ie_i(p)$, we have $\langle D_{e_i}N, D_{e_i}V\rangle(p) = -\frac{\lambda_i}{2}\mathcal{L}_{i,i}(p)$, which implies that

$$e_ie_i(g)(p) = \langle D_{e_i}D_{e_i}N, V\rangle(p) + \langle D_{e_i}D_{e_i}V, N\rangle(p) - \lambda_i\mathcal{L}_{i,i}(p). \tag{3.1}$$

We differentiate $\langle D_{e_i}V, N\rangle + \langle D_NV, e_i\rangle = \mathcal{L}_{i,N}$ in the direction e_i to obtain the following identity:

$$\langle D_{e_i}D_{e_i}V, N\rangle + \langle D_{e_i}V, D_{e_i}N\rangle + \langle D_{e_i}D_NV, e_i\rangle + \langle D_NV, D_{e_i}e_i\rangle = e_i\,(\mathcal{L}_{i,N})\,.$$

Using that $\{e_1, ..., e_n\}$ is a geodesic frame, one gets $(D_{e_i}e_j(p))^\top = 0$, from which we have $D_{e_i}e_i(p) = \lambda_iN(p)$. This yields

$$\langle D_NV, D_{e_i}e_i\rangle(p) = \lambda_i\langle D_NV, N\rangle(p) = \frac{\lambda_i}{2}\mathcal{L}_{N,N}(p).$$

Therefore, we have at p,

$$\langle D_{e_i}D_{e_i}V, N\rangle = e_i\,(\mathcal{L}_{i,N}) + \frac{\lambda_i}{2}\mathcal{L}_{i,i} - \frac{\lambda_i}{2}\mathcal{L}_{N,N} - \langle D_{e_i}D_NV, e_i\rangle. \tag{3.2}$$

On the other hand $[N,e_i](p)=D_Ne_i(p)-D_{e_i}N(p)=-D_{e_i}N(p)=\lambda_ie_i(p)$, since $D_Ne_i(p)=0$. Thus we infer that

$$\langle D_{[N,e_i]}V,e_i\rangle(p)=\lambda_i\langle D_{e_i}V,e_i\rangle(p)=\frac{\lambda_i}{2}\mathcal{L}_{i,i}(p).$$

We now differentiate both sides of $\langle D_{e_i}V,e_i\rangle=\frac{1}{2}\mathcal{L}_{i,i}$ with respect to N to obtain

$$\langle D_ND_{e_i}V,e_i\rangle+\langle D_{e_i}V,D_Ne_i\rangle=\frac{1}{2}N\left(\mathcal{L}_{i,i}\right).$$

Then $\langle D_ND_{e_i}V,e_i\rangle(p)=\frac{1}{2}N\left(\mathcal{L}_{i,i}\right)(p)$. But the definition of curvature yields

$$\langle R(N,e_i)V,e_i\rangle(p)=\langle D_{e_i}D_NV,e_i\rangle(p)-\frac{1}{2}N\left(\mathcal{L}_{i,i}\right)(p)+\frac{\lambda_i}{2}\mathcal{L}_{i,i}(p). \tag{3.3}$$

We now substitute (3.3) in (3.2) and then substitute the result in (3.1) to obtain

$$\begin{aligned} e_ie_i(g)(p) &= -\langle R(N,e_i)V,e_i\rangle(p)+\langle D_{e_i}D_{e_i}N,V\rangle(p)\\ &+ e_i\left(\mathcal{L}_{i,N}\right)(p)-\frac{1}{2}N\left(\mathcal{L}_{i,i}\right)(p)-\frac{\lambda_i}{2}\mathcal{L}_{N,N}(p).\end{aligned}$$

Writing $V=\sum_{j=1}^{n}v_je_j+gN$, where $v_j=\langle V,e_j\rangle$, we have

$$\begin{aligned}\langle D_{e_i}D_{e_i}N,V\rangle &= \langle D_{e_i}D_{e_i}N,\sum_{j=1}^{n}v_je_j+gN\rangle\\ &= \sum_{j=1}^{n}v_j\langle D_{e_i}D_{e_i}N,e_j\rangle+g\langle D_{e_i}D_{e_i}N,N\rangle.\end{aligned}$$

Taking the second covariant derivative of $\langle N,N\rangle$ in the direction e_i, we obtain $\langle D_{e_i}D_{e_i}N,N\rangle(p)=-\lambda_i^2$. We do the same for $\langle N,e_j\rangle$ to obtain

$$\langle D_{e_i}D_{e_i}N,e_j\rangle+2\langle D_{e_i}N,D_{e_i}e_j\rangle+\langle N,D_{e_i}D_{e_i}e_j\rangle=0.$$

Therefore

$$\langle D_{e_i}D_{e_i}N,e_j\rangle(p)=-\langle D_{e_i}D_{e_i}e_j,N\rangle(p). \tag{3.4}$$

Because $\langle D_{e_i}e_j,N\rangle=\langle D_{e_j}e_i,N\rangle$, we differentiate both sides in the direction e_i to find

$$\langle D_{e_i}D_{e_i}e_j,N\rangle+\langle D_{e_i}e_j,D_{e_i}N\rangle=\langle D_{e_i}D_{e_j}e_i,N\rangle+\langle D_{e_j}e_i,D_{e_i}N\rangle.$$

This equality, together with (3.4), allows us to write

$$-\langle D_{e_i}D_{e_i}N,e_j\rangle(p)=\langle D_{e_i}D_{e_i}e_j,N\rangle(p)=\langle D_{e_i}D_{e_j}e_i,N\rangle(p).$$

Again using the curvature tensor as well as $[e_i,e_j](p)=0$ and to $e_j\langle D_{e_i}e_i,N\rangle(p)=\langle D_{e_j}D_{e_i}e_i,N\rangle(p)+\langle D_{e_i}e_i,D_{e_j}N\rangle(p)=\langle D_{e_j}D_{e_i}e_i,N\rangle(p)$, we conclude that

$$\begin{aligned}\langle D_{e_i}D_{e_i}N,e_j\rangle(p) &= \langle R(e_i,e_j)e_i,e_j,N\rangle(p)-\langle D_{e_j}D_{e_i}e_i,N\rangle(p)\\ &= \langle R(e_i,e_j)e_i,N\rangle(p)-e_j\langle D_{e_i}e_i,N\rangle(p).\end{aligned}$$

From this we arrive at

$$\begin{aligned}
\langle D_{e_i} D_{e_i} N, V\rangle(p) &= \sum_{j=1}^{n} v_j \left(\langle R(e_i,e_j)e_i, N\rangle(p) - e_j\langle D_{e_i} e_i, N\rangle(p)\right) - g(p)\lambda_i^2 \\
&= \sum_{j=1}^{n} v_j \langle R(e_i,e_j)e_i, N\rangle(p) - \sum_{j=1}^{n} v_j e_j \langle D_{e_i} e_i, N\rangle(p) - g(p)\lambda_i^2 \\
&= \langle R(e_i, \sum_{j=1}^{n} v_j e_j)e_i, N\rangle(p) - \sum_{j=1}^{n} v_j e_j \langle D_{e_i} e_i, N\rangle(p) - g(p)\lambda_i^2 \\
&= \langle R(e_i, V - gN)e_i, N\rangle(p) - \sum_{j=1}^{n} v_j E_j \langle D_{e_i} e_i, N\rangle(p) - g(p)\lambda_i^2 \\
&= \langle R(e_i, V)e_i, N\rangle(p) - g(p)\langle R(e_i, N)e_i, N\rangle(p) - g(p)\lambda_i^2 \\
&- \sum_{j=1}^{n} v_j e_j \langle D_{e_i} e_i, N\rangle(p),
\end{aligned}$$

which yields

$$\begin{aligned}
\langle D_{e_i} D_{e_i} N, V\rangle(p) - \langle R(e_i, V)e_i, N\rangle(p) &= -g(p)\langle R(e_i, N)e_i, N\rangle(p) - g(p)\lambda_i^2 \\
&- \sum_{j=1}^{n} v_j e_j \langle D_{e_i} e_i, N\rangle(p).
\end{aligned}$$

We now make use of the identity obtained for $e_i e_i(g)$ to conclude that

$$\begin{aligned}
e_i e_i(g)(p) &= e_i\left(\mathcal{L}_{i,N}\right)(p) - \frac{1}{2} N\left(\mathcal{L}_{i,i}\right)(p) - \frac{\lambda_i}{2}\mathcal{L}_{N,N}(p) - g(p)\lambda_i^2 \\
&- g(p)\langle R(e_i, N)e_i, N\rangle(p) - \sum_{l=1}^{n} \langle V, e_l\rangle \left(e_l \langle A(e_i), e_i\rangle\right)(p).
\end{aligned}$$

It is convenient to note that

$$\begin{aligned}
L_r(g)(p) &= tr\left(P_r \circ Hess_g\right)(p) = \sum_{i=1}^{n} \langle P_r Hess_g(e_i), e_i\rangle(p) \\
&= \sum_{i=1}^{n} \langle P_r(D_{e_i}\nabla g), e_i\rangle(p) = \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} \langle D_{e_i}\nabla g, e_i\rangle(p).
\end{aligned}$$

Hence, writing $\nabla g = \sum_{i=1}^{n} g_j e_j$, we find

$$D_{e_i}\nabla g(p) = \sum_{j=1}^{n} (g_{ji} e_j + g_j D_{e_i} e_j)(p) = \sum_{j=1}^{n} g_{ji} e_j(p),$$

which gives us $L_r(g)(p) = \sum_{i=1}^{n} \frac{\partial S_{r+1}}{\partial \lambda_i} g_{ii}(p)$.

Moreover, taking into account that the curvature of a space form M_c^{n+1} is given by $R(X,Y)=c\,(X\wedge Y)$ and that $(\mathcal{L}_V\langle\,,\rangle)(X,Y)=2\psi\langle X,Y\rangle$, we deduce that

$$\begin{aligned} L_r(g)(p) &= -\psi(p)\sum_{i=1}^{n}\lambda_i\frac{\partial S_{r+1}}{\partial\lambda_i}-N(\psi)(p)\sum_{i=1}^{n}\frac{\partial S_{r+1}}{\partial\lambda_i}-cg(p)\sum_{i=1}^{n}\frac{\partial S_{r+1}}{\partial\lambda_i}\\ &- g(p)\sum_{i=1}^{n}\lambda_i^2\frac{\partial S_{r+1}}{\partial\lambda_i}-\sum_{l=1}^{n}\langle V,e_l\rangle\left(e_l\langle A(e_i),e_i\rangle\right)(p)\\ &= -(r+1)S_{r+1}\psi(p)-(n-r)S_rN(\psi)(p)-c(n-r)S_rg(p)\\ &- (S_1S_{r+1}-(r+2)S_{r+2})g(p)-I. \end{aligned}$$

We now notice that

$$\begin{aligned} tr\left(P_rD_XA\right)(p) &= \sum_{i=1}^{n}\langle(P_rD_XA)(e_i),e_i\rangle(p)\\ &= \sum_{i=1}^{n}\langle P_r\left((D_XA)(e_i)\right),e_i\rangle(p)\\ &= \sum_{i=1}^{n}\langle(D_XA)(e_i),P_r(e_i)\rangle(p)\\ &= \sum_{i=1}^{n}\frac{\partial S_{r+1}}{\partial\lambda_i}\langle(D_XA)(e_i)-A(D_Xe_i),e_i\rangle(p). \end{aligned}$$

On the other hand, letting $X=e_l$ in the above equality and using $D_{e_l}e_i(p)=0$, one gets

$$\langle e_l,\nabla S_{r+1}\rangle(p)=tr\left(P_rD_{e_l}A\right)(p)=\sum_{i=1}^{n}\frac{\partial S_{r+1}}{\partial\lambda_i}\langle D_{e_l}A(e_i),e_i\rangle(p).$$

Finally we compute

$$e_l\langle A(e_i),e_i\rangle(p)=\langle D_{e_l}A(e_i),e_i\rangle(p)+\langle A(e_i),D_{e_l}e_i\rangle(p)=\langle D_{e_l}A(e_i),e_i\rangle(p)$$

to deduce that

$$I=\sum_{l=1}^{n}\langle V,e_l\rangle\langle e_l,\nabla S_{r+1}\rangle(p)=\langle V,\nabla S_{r+1}\rangle(p).$$

From the above one gets

$$\begin{aligned} L_r(g) &= -(S_1S_{r+1}-(r+2)S_{r+2})g-c(n-r)S_rg\\ &- (n-r)S_rN(\psi)-(r+1)S_{r+1}\psi-\langle V,\nabla S_{r+1}\rangle, \end{aligned}$$

which concludes the proof of the theorem. □

Now we combine Corollary 2.3 and formula (8.4) of Alias et al. [3] to derive the following lemma.

Lemma 3.2. *Let $x:M^n\looparrowright M_c^{n+1}$ be an oriented isometric immersion with unit normal vector field N. If V is a conformal vector field on M_c^{n+1} with conformal factor ψ and $g=\langle V,N\rangle$ represents the support function on M^n, then*

$$div\left(P_r(V^T)\right)=(n-r)S_r\psi+(r+1)S_{r+1}g.$$

Putting together Theorem 3.1 and Lemma 3.2 we derive the next corollary.

Corollary 3.3. *Let* $x : M^n \looparrowright M_c^{n+1}$ *be an oriented isometric immersion with* S_{r+1} *constant. If* $V = s(d)\nabla d$, $g = \langle V, N\rangle$ *and* $W = (n-r-1)P_r\nabla g + (r+1)P_{r+1}(V^T)$, *then*

$$div(W) = -n(r+2)\binom{n}{r+2}(H_1H_{r+1} - H_{r+2})\, g,$$

where N *is a unit normal vector field to* M^n *and* d *is the distance on* M_c^{n+1} *to a fixed point* p_0.

Proof. First we notice that $V = s(d)\nabla d$ is a conformal vector field with factor $\psi = s'(d)$. Moreover, $N(\psi) = -cg$. On the other hand, since S_{r+1} is constant Theorem 3.1 yields

$$(n-r-1)L_r(g) = -(n-r-1)(r+1)S_{r+1}s'(d) - (n-r-1)\left(S_1S_{r+1} - (r+2)S_{r+2}\right) g.$$

We now use Lemma 3.2 to find

$$(r+1)div\left(P_{r+1}(V^T)\right) = (n-r-1)(r+1)S_{r+1}s'(d) + (r+1)(r+2)S_{r+2}g.$$

Therefore, using $L_r(g) = div\,(P_r\nabla g)$ and $W = (n-r-1)P_r\nabla g + (r+1)P_{r+1}(V^T)$, we infer that

$$div(W) = -(n-r-1)S_1S_{r+1}g + n(r+2)S_{r+2}g.$$

In order to deduce the desired result it is enough to notice that $S_r = \binom{n}{r}H_r$ and $(n-r-1)n\binom{n}{r+1} = n(r+2)\binom{n}{r+2}$. □

4. Compact radial graphs

By a compact smooth radial graph $\Sigma^n \subset \mathbb{R}^{n+1}$ we mean a differentiable graph whose domain is a full Euclidean sphere $\mathbb{S}^n(r)$ of radius $r > 0$. In order to construct such a graph we fix a point $p_0 \in \mathbb{R}^{n+1}$ called the origin, which coincides with the center of the sphere, and for each direction $v \in T_{p_0}\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ we consider a point $p(v) \in \Sigma^n$ that corresponds to the end point of a nontrivial geodesic segment on $\mathbb{R}^{n+1}$ starting from p_0 in the direction of v. We also call this type of graph a smooth star-shaped hypersurface of the Euclidean space $\mathbb{R}^{n+1}$.

A similar construction can be done for a compact smooth radial graph $\Sigma^n \subset \mathbb{S}^{n+1}$. In fact, we fix a point $p_0 \in \mathbb{S}^{n+1}$ and for each direction $v \in T_{p_0}\mathbb{S}^{n+1}$ we consider a point $p(v) \in \Sigma^n$ that corresponds to the end point of a nontrivial geodesic segment on $\mathbb{S}^{n+1}$ starting from p_0 in the direction of v.

Now we consider a compact smooth radial graph $\Sigma^n \subset \mathbb{R}^{n+1}$ whose domain is a Euclidean sphere $\mathbb{S}^n(r)$ of radius $r > 0$. Next, we introduce local coordinates $u = (u_1, ..., u_n)$ and let $X(u)$ and $Y(u)$ be parametrizations of $\mathbb{S}^n(r)$ and Σ^n respectively. If $\rho(u) = |Y(u)| > 0$, then $Y = \rho X$.

Let $f : \Sigma^n \to \mathbb{R}$ be the function defined by $f(Y) = \langle Y, N_Y\rangle$, where N_Y is a unit vector field normal to Σ^n. Letting $\frac{\partial h}{\partial u_i} = h_i$, we have $Y_i = \rho X_i + \rho_i X$, from which we obtain

$$\langle \rho X, \frac{Y_1 \wedge ... \wedge Y_n}{|Y_1 \wedge ... \wedge Y_n|}\rangle = \langle \rho X, \frac{(\rho X_1) \wedge ... \wedge (\rho X_n)}{|Y_1 \wedge ... \wedge Y_n|}\rangle.$$

Hence we deduce that

$$f(Y) = \langle \rho X, \frac{Y_1 \wedge ... \wedge Y_n}{|Y_1 \wedge ... \wedge Y_n|} \rangle = \rho^{n+1} \frac{|X_1 \wedge ... \wedge X_n|}{|Y_1 \wedge ... \wedge Y_n|} \langle X, N_X \rangle$$
$$= -\frac{\rho^{n+1}}{r} \frac{|X_1 \wedge ... \wedge X_n|}{|Y_1 \wedge ... \wedge Y_n|} \langle X, X \rangle = -r\rho^{n+1} \frac{|X_1 \wedge ... \wedge X_n|}{|Y_1 \wedge ... \wedge Y_n|} < 0.$$

Here we are considering $N_X = -\frac{1}{r}X$ as the unit normal vector field to $\mathbb{S}^n(r)$ in such a way that its mean curvature is positive.

Finally, given a compact smooth radial graph $\Sigma^n \subset \mathbb{S}^{n+1}$ as above, we consider the stereographic projection $\pi : \mathbb{S}^{n+1} \backslash \{p_0\} \to \mathbb{R}^{n+1}$ and let $V_{\mathbb{S}^{n+1}}$ be the position vector field with basis point p_0 in $\mathbb{S}^{n+1}$. Hence we have the next lemma.

Lemma 4.1. *Under the above conditions the function $g = \langle V_{\mathbb{S}^{n+1}}, N_Y \rangle$ has a well-defined sign.*

Proof. Let X and Y be parametrizations of $\mathbb{S}^n(r)$ and Σ^n respectively. Then $\pi(X)$ is a parametrization of a sphere on $\mathbb{R}^{n+1}$ while $\pi(Y)$ is a parametrization of a smooth radial graph over $\pi(X)$ on $\mathbb{R}^{n+1}$.

Let $g_1, g_2 : \pi(Y) \to \mathbb{R}$ be continuous functions such that either $g_i > 0$ or $g_i < 0$, $d\pi(V_{\mathbb{S}^{n+1}}) = g_1 V_{\mathbb{R}^{n+1}}$ and $d\pi(N_Y) = g_2 N_{\pi(Y)}$. If e^{ϕ} stands for the conformal factor of π, then we obtain

$$e^{2\phi} \langle V_{\mathbb{S}^{n+1}}, N_Y \rangle = \langle d\pi(V_{\mathbb{S}^{n+1}}), d\pi(N_Y) \rangle = \langle g_1 V_{\mathbb{R}^{n+1}}, g_2 N_{\pi(Y)} \rangle > 0 \text{ (or } < 0),$$

from which we derive the desired result. □

5. Proof of Theorem 1.1

Proof. Letting $r = 1$ in Corollary 3.3 and integrating $divW$ over Σ^n, we obtain

$$\int_{\Sigma} (H_1 H_2 - H_3) g d\Sigma = 0. \tag{5.1}$$

On the other hand, it is well known that for immersions $M^n \looparrowright M_c^{n+1}$,

$$H_r^2 \geq H_{r-1} H_{r+1}, \ \forall \ r \in \{1, ..., n-1\}, \tag{5.2}$$

with equality occurring if and only if M is totally umbilical; see [6] or [1] for more details.

Taking into account that $S_1^2 = |\ A\ |^2 + 2S_2$, we see that $H_1^2 > 0$. Hence we may choose the orientation for Σ^n in such a way that $H_1 > 0$. Since $H_0 = 1$ we obtain $H_1^2 \geq H_2 > 0$. Now letting $r = 2$ on (5.2) we have $-H_1 H_3 \geq -H_2^2$, from which we obtain

$$H_1 (H_1 H_2 - H_3) \geq H_2 \left(H_1^2 - H_2\right) \geq 0. \tag{5.3}$$

Since g has a well-defined sign, we make use of equations (5.1) and (5.3) to arrive at $H_1 H_2 - H_3 = 0$. Thus taking this into account in inequality (5.3), we find that $H_1^2 - H_2 = 0$. Hence we deduce that $x(M^n)$ is totally umbilical, which finishes the proof of the theorem. □

For a compact smooth radial graph $\Sigma^n \subset \mathbb{S}^{n+1}$ of non-null constant mean curvature, we proved in [5] that Σ^n is a round sphere. However, we avoided minimal graphs. In fact, this is easily extended to the minimal case according to the next theorem.

Theorem 5.1. *Let $\Sigma^n \subset \mathbb{S}^{n+1}$ be a compact smooth star-shaped minimal hypersurface. Then Σ^n is totally geodesic.*

Proof. Referring to Lemma 3.2, we have $div\left(P_1(V^T)\right) = 2S_2 g$. This implies that

$$\int_\Sigma S_2 g d\Sigma = 0.$$

Since $2S_2 = -|A|^2 \leq 0$ and g does not change sign, we get $S_1 = S_2 = 0$. Therefore, we deduce that $|A|^2 = 0$, which yields that Σ^n is totally geodesic. □

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