

## LIPSCHITZ $p$ -SUMMING OPERATORS

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(Communicated by Marius Junge)

ABSTRACT. The notion of Lipschitz  $p$ -summing operator is introduced. A nonlinear Pietsch factorization theorem is proved for such operators, and it is shown that a Lipschitz  $p$ -summing operator that is linear is a  $p$ -summing operator in the usual sense.

### 1. INTRODUCTION

In this note we introduce a natural nonlinear version of a  $p$ -summing operator, which we call a Lipschitz  $p$ -summing operator. In section 2 we prove a nonlinear version of the Pietsch factorization theorem, show by example that the strong form of the Pietsch domination theorem is not true for Lipschitz  $p$ -summing operators, and make a few other remarks about these operators. In section 3 we “justify” our nomenclature by proving that for a linear operator, the Lipschitz  $p$ -summing norm is the same as the usual  $p$ -summing norm. Finally, in section 4 we raise some problems which we think are interesting.

### 2. PIETSCH FACTORIZATION

The *Lipschitz  $p$ -summing* ( $1 \leq p < \infty$ ) norm,  $\pi_p^L(T)$ , of a (possibly nonlinear) mapping  $T: X \rightarrow Y$  between metric spaces is the smallest constant  $C$  so that for all  $(x_i), (y_i)$  in  $X$  and all positive reals  $a_i$ ,

$$(2.1) \quad \sum a_i \|Tx_i - Ty_i\|^p \leq C^p \sup_{f \in B_{X^\#}} \sum a_i |f(x_i) - f(y_i)|^p.$$

Here  $B_{X^\#}$  is the unit ball of  $X^\#$ , the Lipschitz dual of  $X$ ; i.e.,  $X^\#$  is the space of all real valued Lipschitz functions under the (semi)-norm  $\text{Lip}(\cdot)$ ; and  $\|x - y\|$  is the distance from  $x$  to  $y$  in  $Y$ . We follow the usual convention of considering  $X$  as a pointed metric space by designating a special point  $0 \in X$  and identifying  $X^\#$  with the Lipschitz functions on  $X$  that are zero at  $0$ . With this convention  $(X^\#, \text{Lip}(\cdot))$  is a Banach space and  $B_{X^\#}$  is a compact Hausdorff space in the topology of pointwise convergence on  $X$ .

Notice that the definition is the same if we restrict to  $a_i = 1$ . Indeed, by approximation it is enough to consider rational  $a_i$  and thus, by clearing denominators, integer  $a_i$ . Then, given  $a_i, x_i$ , and  $y_i$ , consider the new collection of vectors in which

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Received by the editors January 8, 2008.

2000 *Mathematics Subject Classification*. Primary 46B28, 46T99, 47H99, 47L20.

*Key words and phrases*.  $p$ -summing operator, absolutely summing operator.

The second author was supported in part by NSF DMS-0503688.

the pair  $(x_i, y_i)$  is repeated  $a_i$  times. (This observation was made with M. Mendel and G. Schechtman.)

It is clear that  $\pi_p^L$  has the ideal property; i.e.,  $\pi_p^L(ATB) \leq \text{Lip}(A)\pi_p^L(T)\text{Lip}(B)$  whenever the compositions make sense. Also, if  $Y$  is a Banach space, the space of Lipschitz  $p$ -summing maps from any metric space into  $Y$  is a Banach space under the norm  $\pi_p^L$ .

If  $T$  is a linear operator, it is clear that  $\pi_p^L(T) \leq \pi_p(T)$ , where  $\pi_p(\cdot)$  is the usual  $p$ -summing norm [5, p. 31]. In section 3 we prove that the reverse inequality is true.

We begin with a Pietsch factorization theorem for Lipschitz  $p$ -summing operators.

**Theorem 1.** *The following are equivalent for a mapping  $T: X \rightarrow Y$  between metric spaces and  $C \geq 0$ .*

- (1)  $\pi_p^L(T) \leq C$ .
- (2) *There is a probability  $\mu$  on  $B_{X^\#}$  such that*

$$\|Tx - Ty\|^p \leq C^p \int_{B_{X^\#}} |f(x) - f(y)|^p d\mu(f)$$

*(Pietsch domination).*

- (3) *For some (or any) isometric embedding  $J$  of  $Y$  into a 1-injective space  $Z$ , there is a factorization*

$$\begin{array}{ccc} L_\infty(\mu) & \xrightarrow{I_{\infty,p}} & L_p(\mu) \\ A \uparrow & & \downarrow B \\ X & \xrightarrow{T} Y \xrightarrow{J} & Z \end{array}$$

*with  $\mu$  a probability and  $\text{Lip}(A) \cdot \text{Lip}(B) \leq C$  (Pietsch factorization).*

*Proof.* That (2) implies (3) is basically obvious: Let  $A: X \rightarrow L_\infty(\mu)$  be the natural isometric embedding composed with the formal identity from  $C(B_{X^\#})$  into  $L_\infty(\mu)$ . Then (2) says that the Lipschitz norm of  $B$  restricted to  $I_{\infty,p}AX$  is bounded by  $C$ , which is just (3). (We have used implicitly the well known fact that every metric space embeds into  $\ell_\infty(\Gamma)$  for some set  $\Gamma$  and that, by the nonlinear Hahn-Banach theorem,  $\ell_\infty(\Gamma)$  is 1-injective. See Lemma 1.1 in [3].)

For (3) implies (1), use

$$\begin{aligned} \pi_p^L(T) &= \pi_p^L(JT) \leq \text{Lip}(A)\pi_p^L(I_{\infty,p})\text{Lip}(B) \leq \text{Lip}(A)\pi_p(I_{\infty,p})\text{Lip}(B) \\ &= \text{Lip}(A)\text{Lip}(B). \end{aligned}$$

The proof of the main implication, that (1) implies (2), is like the proof of the (linear) Pietsch factorization theorem (see, e.g., [5, p. 44]). Suppose  $\pi_p^L(T) = 1$ . Let  $Q$  be the convex cone in  $C(B_{X^\#})$  consisting of all positive linear combinations of functions of the form  $\|Tx - Ty\| - C^p|f(x) - f(y)|^p$ , as  $x$  and  $y$  range over  $X$ . Condition (1) says that  $Q$  is disjoint from the positive cone  $P = \{F \in C(B_{X^\#}) \mid F(f) > 0 \forall f \in X^\#\}$ , which is an open convex subset of  $C(B_{X^\#})$ . Thus by the separation theorem and the Riesz representation theorem there is a finite signed Baire measure  $\mu$  on  $B_{X^\#}$  and a real number  $c$  so that for all  $G \in Q$  and  $F \in P$ ,  $\int_{X^\#} G d\mu \leq c < \int_{X^\#} F d\mu$ . Since  $0 \in Q$  and all positive constants are in  $P$ , we see that  $c = 0$ , and since  $\int_{X^\#} \cdot d\mu$  is positive on the positive cone  $P$  of  $C(B_{X^\#})$ ,

the signed measure  $\mu$  is a positive measure, which we can assume by rescaling is a probability measure. It is clear that the inequality in (2) is satisfied.  $\square$

It is worth noting that the conditions in Theorem 1 are also equivalent to

- (4) *There is a probability  $\mu$  on  $K$ , the closure in the topology of pointwise convergence on  $X$  of the extreme points of  $B_{X^\#}$ , such that*

$$\|Tx - Ty\|^p \leq C^p \int_K |f(x) - f(y)|^p d\mu(f).$$

The proof that (1) implies (4) is the same as the proof that (1) implies (2) since the supremum on the right side of (2.1), the definition of the Lipschitz  $p$ -summing norm, is the same as

$$\sup_{f \in K} \sum a_i |f(x_i) - f(y_i)|^p.$$

One immediate consequence of Theorem 1 is that  $\pi_p^L(T)$  is a monotonely decreasing function of  $p$ . Another consequence is that there is a version of Grothendieck's theorem (that every linear operator from an  $L_1$  space to a Hilbert space is 1-absolutely summing). In the category of metric spaces with Lipschitz mappings as morphisms, weighted trees play a role analogous to that of  $L_1$  in the linear theory. In particular, every finite weighted tree has the lifting property, which is to say that if  $X$  is a finite weighted tree,  $T: X \rightarrow Y$  is a Lipschitz mapping from  $X$  into a metric space  $Y$ , and  $Q: Z \rightarrow Y$  is a 1-Lipschitz quotient mapping in the sense of [2], [7], then for each  $\varepsilon > 0$  there is a mapping  $S: X \rightarrow Z$  so that  $\text{Lip}(S) \leq \text{Lip}(T) + \varepsilon$  and  $T = QS$ . Letting  $Y$  be a Hilbert space and  $Z$  an  $L_1$  space, we see from Grothendieck's theorem and the ideal property of  $\pi_1^L$  that if every finite subset of  $X$  is a weighted tree (in particular, if  $X$  is a tree or a metric tree; see [7]), then  $\pi_1^L(T) \leq K_G \text{Lip}(T)$ , where  $K_G$  is Grothendieck's constant. Here we use the obvious fact that  $\pi_p^L(T: X \rightarrow Y)$  is the supremum of  $\pi_p^L(T|_K)$  as  $K$  ranges over finite subsets of  $X$ .

The strong form of the Pietsch domination theorem says that if  $X$  is a subspace of  $C(K)$  for some compact Hausdorff space  $K$ , and if  $T$  is a  $p$ -summing linear operator with domain  $X$ , then there is a probability measure  $\mu$  on  $K$  so that for all  $x \in X$ ,  $\|Tx\|^p \leq \pi_p(T)^p \int_K |x(t)|^p d\mu(t)$ . It is easy to see that there is not a nonlinear version of this result. Let  $D_n$  be the discrete metric space with  $n$  points so that the distance between any two distinct points is one. We can embed  $D_n$  into  $C(\{-1, 1\}^n)$  in two essentially different ways. First, if  $D_n = \{x_1, \dots, x_n\}$ , let  $f(x_k) = \frac{1}{2}r_k$ , where  $r_k$  is the projection onto the  $k$ th coordinate. The image of this set under the canonical injection from  $C(\{-1, 1\}^n)$  into  $L_p(\{-1, 1\}^n, \mu)$  with  $\mu$  the uniform probability on  $\{-1, 1\}^n$  is a discrete set with the  $p$ -th power of all distances one-half. This shows that the identity on  $D_n$  has Lipschitz  $p$ -summing norm at most two. Secondly, let  $g(k)$ ,  $1 \leq k \leq n$ , be disjointly supported unit vectors in  $C(\{-1, 1\}^n)$ . Then for any probability measure  $\nu$  on  $\{-1, 1\}^n$ , the injection from  $C(\{-1, 1\}^n)$  into  $L_p(\{-1, 1\}^n, \nu)$  shrinks the distance between some pair of the  $g(k)$ 's to at most  $(2/n)^{1/p}$ .

Incidentally,  $\pi_p^L(I_{D_n})$  tends to  $2^{\frac{1}{p}}$  as  $n \rightarrow \infty$  and can be computed exactly. To see this, note that the extreme points,  $K_n$ , of  $B_{D_n^\#}$  are of the form  $\pm\chi_A$  with  $A$  a nonempty subset of  $D_n \sim \{0\}$ . This can be calculated directly or deduced from Theorem 1 in [6]. We calculate  $\pi_p^L(I_{D_n})$  in the (easier) case that  $n$  is even. Define

a probability  $\mu$  on  $K_n$  by letting  $\mu$  be the uniform measure on  $J_{n/2} := \{\chi_A : |A| = n/2, A \subset D_n \sim \{0\}\}$  (so that  $\mu(e) = 0$  for elements  $e$  of  $K_n \sim J_{n/2}$ ). Then for each pair of distinct points  $x$  and  $y$  in  $D_n$ ,  $\int_{K_n} |f(x) - f(y)|^p d\mu(f) = \frac{n}{2(n-1)}$ , so that  $\pi_p^L(I_{D_n}) \leq (2 - \frac{2}{n})^{\frac{1}{p}}$ . To see that  $\mu$  is a Pietsch measure for  $I_{D_n}$ , let  $\nu$  be any Pietsch probability for  $I_{D_n}$  on  $K_n$ . We can clearly assume that  $\nu$  is supported on the positive elements in  $K_n$ . By averaging  $\nu$  against the permutations of  $D_n$  which fix 0, which is a group of isometries on  $D_n$ , we get another Pietsch probability for  $I_{D_n}$  (which we continue to denote by  $\nu$ ) so that if we condition  $\nu$  on  $J_k := \{\chi_A : |A| = k, A \subset D_n \sim \{0\}\}$ ,  $1 \leq k \leq n-1$ , the resulting probability  $\nu_k$  on  $J_k$  is the uniform probability. A trivial calculation shows that for  $x, y$  in  $D_n \sim \{0\}$ ,  $\int_{J_k} |f(x) - f(y)|^p d\nu_k(f) \leq \frac{n}{2(n-1)}$ . This proves that  $\mu$  is a Pietsch measure for  $I_{D_n}$  and hence  $\pi_p^L(I_{D_n}) = (2 - \frac{2}{n})^{\frac{1}{p}}$ .

Our final comment on Lipschitz 1-summing operators is that the concept has appeared in the literature even if the definition is new. In [4], Bourgain proved that every  $n$ -point metric space can be embedded into a Hilbert space with distortion at most  $C \log n$ , where  $C$  is an absolute constant. In fact, he really proved the much stronger result that  $\pi_1^L(I_X) \leq C \log n$  if  $I_X$  is the identity mapping on an  $n$ -point space  $X$  by making use of a special embedding of  $X$  into a space  $C(K_X)$  with  $K_X$  a finite metric space and constructing a probability on  $K_X$ . Moreover, Bourgain's construction has occasionally been used in the computer science literature. The strong form of Bourgain's theorem is also used in [8] to prove an inequality that is valid for all metric spaces.

### 3. LINEAR OPERATORS

In this section we show that the Lipschitz  $p$ -summing norm of a linear operator is the same as its  $p$ -summing norm. This justifies that the notion of Lipschitz  $p$ -summing operator is really a generalization of the concept of linear  $p$ -summing operator.

**Theorem 2.** *Let  $u$  be a bounded linear operator from  $X$  into  $Y$  and  $1 \leq p < \infty$ . Then  $\pi_p^L(u) = \pi_p(u)$ .*

*Proof.* Note that we can assume, without loss of generality, that  $\dim Y \leq \dim X = N < \infty$ . Indeed, it is clear from the definition that  $\pi_p^L(u)$  is the supremum of  $\pi_p^L(u|_E)$  as  $E$  ranges over finite dimensional subspaces of  $X$  and similarly for  $\pi_p^L(u)$ . That we can assume  $\dim Y \leq \dim X$  is clear from the linearity of  $u$ .

Since  $\dim Y \leq N$ , there is an embedding  $J$  of  $Y$  into  $\ell_\infty^m$  with  $m \leq (\frac{3}{\epsilon})^N$  so that  $\|J\| = 1$  and  $\|J^{-1}\| \leq 1 + \epsilon$ . We then get the following nonlinear Pietsch factorization:

$$\begin{array}{ccc} L_\infty(\mu) & \xrightarrow{i_{\infty,p}} & L_p(\mu) \\ \alpha \uparrow & & \downarrow \beta \\ X & \xrightarrow{u} Y \xrightarrow{J} & \ell_\infty^m \end{array}$$

where  $\text{Lip}(\alpha) = 1$ ,  $\text{Lip}(\beta) \leq \pi_p^L(Ju) \leq \pi_p^L(u)$ . We can also assume, without loss of generality, that the probability  $\mu$  is a separable measure.

We now use some nonlinear theory that can be found in the book [3].

- (1) The mapping  $\alpha$  is weak\* differentiable almost everywhere. This means that for (Lebesgue) almost every  $x_0$  in  $X$ , there is a linear operator

$D_{x_0}^{w*}(\alpha) : X \rightarrow L_\infty(\mu)$  such that for all  $f \in L_1(\mu)$  and for every  $y \in X$ ,

$$\lim_{t \rightarrow 0} \left\langle \frac{\alpha(x_0 + ty) - \alpha(x_0)}{t}, f \right\rangle = \langle D_{x_0}^{w*}(\alpha)(y), f \rangle.$$

- (2) The operator  $i_{\infty,p}\alpha$  is differentiable almost everywhere. This means that for almost every  $x_0$  in  $X$ , there is a linear operator  $D_{x_0}(i_{\infty,p}\alpha) : X \rightarrow L_p(\mu)$  such that

$$\sup_{\|y\| \leq 1} \left\| \frac{i_{\infty,p}\alpha(x_0 + ty) - i_{\infty,p}\alpha(x_0)}{t} - D_{x_0}(i_{\infty,p}\alpha)(y) \right\|_p \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

When  $1 < p < \infty$ , statement (2) follows from the reflexivity of  $L_p$  (see [3, Corollary 5.12 & Proposition 6.1]). For  $p = 1$ , just use (2) for  $p = 2$  and compose with  $i_{2,1}$ .

The mapping  $i_{\infty,p}$  is weak\* to weak continuous, so  $D_{x_0}(i_{\infty,p}\alpha) = i_{\infty,p}D_{x_0}^{w*}(\alpha)$  whenever both derivatives exist. Since they both exist almost everywhere, by making several translations we can assume without loss of generality that this equation is true for  $x_0 = 0$  and also that  $\alpha(0) = 0$ .

Next we show that in the factorization diagram the nonlinear map  $\alpha$  can be replaced by the linear operator  $D_0^{w*}(\alpha)$  by constructing a mapping  $\tilde{\beta} : L_p(\mu) \rightarrow \ell_\infty^m$  so that  $\tilde{\beta}i_{\infty,p}D_0^{w*}(\alpha) = Ju$  and  $\text{Lip}(\tilde{\beta}) \leq \text{Lip}(\beta)$ . To do this, define  $\beta_n : L_p(\mu) \rightarrow \ell_\infty^m$  by  $\beta_n(y) := n\beta(\frac{y}{n})$  and note that  $\text{Lip}(\beta_n) = \text{Lip}(\beta)$ . We have for each  $x$  in  $X$ ,

$$\begin{aligned} \|Ju(x) - \beta_n i_{\infty,p} D_0^{w*}(\alpha)(x)\| &= \|\beta_n n i_{\infty,p} \alpha(x/n) - \beta_n D_0(i_{\infty,p}\alpha)(x)\| \\ &\leq \text{Lip}(\beta) \|n i_{\infty,p} \alpha(x/n) - D_0(i_{\infty,p}\alpha)(x)\|, \end{aligned}$$

which tends to zero as  $n \rightarrow \infty$ . For  $\tilde{\beta}$  we can take any cluster point of  $\beta_n$  in the space of functions from  $L_p(\mu)$  into  $\ell_\infty^m$ ; such points exist because  $\beta_n$  is uniformly Lipschitz and  $\beta_n(0) = 0$ .

Summarizing, we see that we have a factorization

$$\begin{array}{ccc} L_\infty(\mu) & \xrightarrow{i_{\infty,p}} & L_p(\mu) \\ \tilde{\alpha} \uparrow & & \downarrow \tilde{\beta} \\ X & \xrightarrow{u} Y \xrightarrow{J} & \ell_\infty^m \end{array}$$

with  $\tilde{\alpha}$  linear,  $\|\tilde{\alpha}\| \leq \text{Lip}(\alpha)$ , and  $\text{Lip}(\tilde{\beta}) \leq \text{Lip}(\beta)$ .

The final step involves replacing  $\tilde{\beta}$  with a linear operator. Since the restriction  $\bar{\beta}$  of  $\tilde{\beta}$  to the linear subspace  $i_{\infty,p}\tilde{\alpha}[X]$  is linear and  $\ell_\infty^m$  is reflexive, this follows from [3, Theorem 7.2], which is proved by a simple invariant means argument. Alternatively, one can use the injectivity of  $\ell_\infty^n$  to extend  $\bar{\beta}$  to  $L_p(\mu)$ .  $\square$

#### 4. OPEN PROBLEMS AND CONCLUDING REMARKS

**Problem 1.** Is there a composition formula for Lipschitz  $p$ -summing operators? That is, do we have  $\pi_p^L(TS) \leq \pi_r^L(T)\pi_s^L(S)$ , when  $\frac{1}{p} \leq (\frac{1}{r} + \frac{1}{s}) \wedge 1$ ?

Say that a Lipschitz mapping  $T : X \rightarrow Y$  is Lipschitz  $p$ -integral if it satisfies a factorization diagram as in condition (3) of Theorem 1, except with  $J$  being the canonical isometry from  $Y$  into  $(Y^\#)^*$ . We then define the Lipschitz  $p$ -integral norm  $I_p^L(T)$  of  $T$  to be the infimum of  $\text{Lip}(A) \cdot \text{Lip}(B)$ , the infimum being taken over all such factorizations. When  $T$  is a linear operator, this is the same as the usual  $p$ -integral norm of  $T$ . Indeed, in this case one can use for  $J$  the canonical

isometry from  $Y$  into  $Y^{**}$  because  $Y^{**}$  is norm one complemented in  $(Y^\#)^*$ . Then the proof that  $I_p(T) \leq I_p^L(T)$  is identical to the proof of Theorem 2.

**Problem 2.** Is every Lipschitz 2-summing operator Lipschitz 2-integral?

In the case where the target space  $Y$  is a Hilbert space, problem 2 has an affirmative answer by Kirszbraun's theorem [3, p. 18]. If  $Y$  has K. Ball's Markov cotype 2 property [1], it follows from Ball's work that the answer is still positive, although his result does not yield that  $I_p^L(T)$  and  $\pi_p^L(T)$  are equal. It is worth mentioning that the work of Naor, Peres, Schramm, and Sheffield [9] combines with Ball's result to yield that for  $2 \leq p < \infty$ , every Lipschitz  $p$ -summing operator into  $L_r$ ,  $1 < r \leq 2$ , is Lipschitz  $p$ -integral.

We mentioned in section 2 that  $\Pi_p^L(X, Y)$ , the class of Lipschitz  $p$ -summing operators from  $X$  into  $Y$ , is a Banach space under the norm  $\pi_p^L(\cdot)$  when  $Y$  is a Banach space.

**Problem 3.** When  $Y$  is a Banach space and  $X$  is finite, what is the dual of  $\Pi_p^L(X, Y)$ ?

In section 2 we noted that there is a version of Grothendieck's theorem that is true in the nonlinear setting. Are there other versions? In particular, we ask the following:

**Problem 4.** Is every Lipschitz mapping from an  $L_1$  space to a Hilbert space Lipschitz 1-summing? Is every Lipschitz mapping from a  $C(K)$  space to a Hilbert space Lipschitz 2-summing?

It is elementary that for a linear operator  $T: X \rightarrow Y$ ,  $\pi_p(T)$  is the supremum of  $\pi_p(TS)$  as  $S$  ranges over all operators from  $\ell_{p'}$  into  $X$  of norm at most one. This leads us to ask:

**Problem 5.** If  $T: X \rightarrow Y$  is Lipschitz, is  $\pi_p^L(T)$  is the supremum of  $\pi_p^L(TS)$  as  $S$  ranges over all mappings from finite subsets of  $\ell_{p'}$  into  $X$  having Lipschitz constant at most one?

Since all finite metric spaces embed isometrically into  $\ell_\infty$ , the answer to problem 5 is yes for  $p = 1$ .

Of course, all of the above problems are special cases of the general

**Problem 6.** What results about  $p$ -summing operators have analogues for Lipschitz  $p$ -summing operators?

*Added in proof.* Problem 3 has been solved by J. A. Chávez Domínguez (unpublished).

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