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LIMIT-LIKE PREDICTABILITY FOR DISCONTINUOUS FUNCTIONS

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Abstract. Our starting point is the following question: To what extent is a function's value at a point x of a topological space determined by its values in an arbitrarily small (deleted) neighborhood of x? For continuous functions, the answer is typically "always" and the method of prediction of $f(x)$ is just the limit operator. We generalize this to the case of an arbitrary function mapping a topological space to an arbitrary set. We show that the best one can ever hope to do is to predict correctly except on a scattered set. Moreover, we give a predictor whose error set, in T_0 spaces, is always scattered.

1. Topological preliminaries

Among the topological spaces that we will be interested in are the ones arising from a partial ordering by either declaring a set to be open if it is closed upward in the ordering (the *upward topology*) or closed downward in the ordering (the *downward topology*). For example, the interval $(-\infty, 0]$ is open in the downward topology on the reals. These topologies are not typically T_1, but they are always T_0.

We will be asserting that certain things happen except on a set that is "topologically small." On the real line $\mathbf{R}$ with the downward topology, we want these small sets to be the well-ordered subsets of $\mathbf{R}$, and for the upward topology on an ordinal, we want these small sets to be the finite sets. The following well-known notions achieve both.

Definition 1.1. A set S in a topological space X is *weakly scattered* if for every nonempty $T \subseteq S$ there exists some $x \in T$ and some neighborhood V of x such that $V \cap T$ is finite. We call such points *weakly isolated points of* T. The set S is *scattered* if the conclusion can be strengthened to $V \cap T = \{x\}$, in which case these are called *isolated points of* T.

What we are calling "weakly scattered" is called "separated" by Morgan [6]; he attributes the definition to Cantor. Every scattered set is weakly scattered, and it is straightforward to show that the two notions are equivalent in T_0 spaces. The concept of a weakly scattered set is a smallness notion in the sense that these sets are closed under the formation of subsets and finite unions.

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With the downward topology on any partial order, the scattered sets are the well-founded subsets (so with the upward topology, the scattered sets are the co-well-founded subsets). In particular, with the downward topology on $\mathbf{R}$, the scattered sets are the well-ordered subsets, and with the upward topology on any ordinal, the scattered sets are the finite subsets. The usual topology of the reals is not of much interest to us here, but the scattered sets there are countable and nowhere dense; in fact, they are precisely the countable G_δs. This observation goes back to the early 20th century, but for a readily accessible proof, see [1].

If S is a scattered set in a space X, then S gives rise to what we call the *canonical decomposition* $\langle A_\alpha : \alpha < \lambda \rangle$ *of* X *associated with* S. In this decomposition, $A_0 = X - S$ and, for $\alpha > 0$, A_α is the set of isolated points of $X - \bigcup\{ A_\beta : \beta < \alpha \}$. This decomposition is directly related to the Cantor-Bendixson derivative.

2. Main results

In what follows we fix a topological space X, and we assume Y is a set with two or more elements. We let $\mathrm{Hom}(X,Y)$ denote the set of all functions mapping X to Y. The following is closely related to the idea of a germ of functions; this connection will be discussed in Section 4.

Definition 2.1. For each $x \in X$ we define an equivalence relation $\approx_x$ on $\mathrm{Hom}(X,Y)$ by $f \approx_x g$ iff f and g agree on a deleted neighborhood $V - \{x\}$ of x. We let $[f]_x$ denote the equivalence class of f under $\approx_x$.

We will be trying to guess $f(x)$ from the values of f near, but not at, x. The equivalence class $[f]_x$ precisely describes how much information we have about f when we produce this guess. That is, our guess must be based only on $[f]_x$; if we could use f directly when guessing, we could cheat and use $f(x)$ as our guess. The following formalizes this notion of guessing.

Definition 2.2. Let $\mathrm{Hom}(X,Y)/{\approx_x} = \{ [f]_x : f \in \mathrm{Hom}(X,Y) \}$. A *predictor* is a collection $M = \{ M_x : x \in X \}$, where $M_x : \mathrm{Hom}(X,Y)/{\approx_x} \to Y$ for each $x \in X$. Given $f : X \to Y$ we say that the predictor M *guesses correctly for* f *at* x if $M_x([f]_x) = f(x)$. A set $E \subseteq X$ is an *error set for* M if there exists a function $f : X \to Y$ such that M guesses incorrectly for f at x for every $x \in E$.

Error sets are small in the sense that singleton sets are always error sets and error sets are (by definition) closed under subset formation.

The following predictor, essentially a generalization of the μ-strategy from [5], will be of particular interest. We actually find it convenient to throw away a certain amount of information, by disregarding finite differences.

Definition 2.3. For $f, g : X \to Y$ and $V \subseteq X$, we say that f and g *differ only finitely* on V if $\{ x \in V : f(x) \neq g(x) \}$ is finite. For $x \in X$, we define the equivalence relation $\approx^*_x$ on $\mathrm{Hom}(X,Y)$ by $f \approx^*_x g$ iff f and g differ only finitely on some neighborhood V of x. Let $[f]^*_x$ denote the equivalence class of f under $\approx^*_x$.

Fix a well-ordering $\preceq$ of $\mathrm{Hom}(X,Y)$. The μ^**-predictor* is the predictor $\mu^* = \{ \mu^*_x : x \in X \}$ defined by letting $\mu^*_x([f]_x) = \langle f \rangle_x(x)$, where $\langle f \rangle_x$ is the $\preceq$-least element of $[f]^*_x$.

In other words, at any point x, the μ^*-predictor guesses according to the $\preceq$-least element of $\mathrm{Hom}(X,Y)$ that, overlooking finite differences, is consistent with the information available. One can remember this more easily by noting the similarity to Occam's razor, which would have us guess according to the simplest theory consistent with what can be observed.

Our main results are the following. The first shows that the μ^*-predictor guesses correctly except on weakly scattered sets of points, and the second shows that one cannot improve on this, at least for T_0 spaces.

Theorem 2.4. *For every $f : X \to Y$, the μ^*-predictor guesses incorrectly for f only at a set of points that is weakly scattered. If X is T_0, then this error set is scattered.*

Proof. Take any $f : X \to Y$ and let S be the set of points where the μ^*-predictor guesses incorrectly for f. Suppose $T \subseteq S$ is nonempty. We will exhibit $x \in T$ and a neighborhood V of x such that $V \cap T$ is finite. Choose $x \in T$ such that $\langle f \rangle_x$ is the $\preceq$-least element of $\{ \langle f \rangle_y : y \in T \}$, and let V be a neighborhood of x such that $\{ y \in V : f(y) \neq \langle f \rangle_x(y) \}$ is finite. Take any $y \in V \cap T$. Since V is a neighborhood of y, we have $\langle f \rangle_x \approx^*_y f$, so $\langle f \rangle_y \preceq \langle f \rangle_x$. Then $\langle f \rangle_y = \langle f \rangle_x$, by the minimality of $\langle f \rangle_x$. Since $y \in S$, $f(y) \neq \langle f \rangle_y(y) = \langle f \rangle_x(y)$. Therefore, $f(y) \neq \langle f \rangle_x(y)$ for all $y \in V \cap T$. It follows that $V \cap T$ is finite, since $\{ y \in V : f(y) \neq \langle f \rangle_x(y) \}$ is finite. Therefore S is weakly scattered.

When X is T_0, the notions of scattered and weakly scattered coincide, so S is scattered. $\square$

Theorem 2.5. *For every predictor M and every scattered set $S \subseteq X$, there exists some $f : X \to Y$ such that M guesses incorrectly for f at every $x \in S$.*

Proof. Suppose that M is a predictor and that $S \subseteq X$ is scattered. Let $\langle A_\alpha : \alpha < \lambda \rangle$ be the canonical decomposition of X associated with S. So $A_0 = X - S$ and for $\alpha > 0$ and $x \in A_\alpha$ there exists a neighborhood V of x such that $V \cap \bigcup \{ A_\beta : \beta \geq \alpha \} = \{x\}$. Let f be defined arbitrarily on A_0. For $\alpha > 0$, assuming f has been defined on A_β for each $\beta < \alpha$, we define f on A_α as follows. Given $x \in A_\alpha$, let V be a neighborhood of x such that $V \cap \bigcup \{ A_\beta : \beta \geq \alpha \} = \{x\}$. Then f has already been defined on $V - \{x\}$, so $[f]_x$ has been determined. We can now define $f(x) \in Y$ so that $f(x) \neq M_x([f]_x)$, ensuring that M guesses incorrectly for f at x. $\square$

3. Consequences

In this section we derive three known theorems from our main result and extend two others. The first known result is a so-called hat problem (see [4]) and goes as follows.

Suppose we have a set X of people (infinite, for our purposes) and hats of various colors that are to be placed on their heads. Each person can see the hats of the others, but not his own. The question is whether or not they can devise a strategy ensuring that no matter how the hats are placed, only finitely many people will guess their own hat color incorrectly. Yuval Gabay and Michael O'Connor answered this in the affirmative (included in [4] with permission). Their proof easily generalizes to the context of ideals, where an *ideal* on a set X is a collection of subsets of X that contains all singletons and is closed under finite unions and subset formation. Sets in an ideal $\mathcal{I}$ are said to be of *$\mathcal{I}$-measure zero*, sets not in $\mathcal{I}$ of *positive $\mathcal{I}$-measure*, and sets whose complement is in $\mathcal{I}$ of *$\mathcal{I}$-measure one.* The Gabay-O'Connor theorem is the special case of the following in which the ideal $\mathcal{I}$ is the collection of finite sets.

Theorem 3.1. *Suppose that $\mathcal{I}$ is an ideal on the set X and that we have the hat problem in which each person in X can see the hats of a collection of people of $\mathcal{I}$-measure one. Then there exists a strategy ensuring that the set of people guessing their own hat color incorrectly is of $\mathcal{I}$-measure zero.*

This follows from our main result by considering the topology on X in which each set of $\mathcal{I}$-measure one containing x is a basic neighborhood of x. This topology is T_0 because singletons are in $\mathcal{I}$. It is easy to see that the scattered sets are precisely the sets of $\mathcal{I}$-measure zero, and thus the result follows.

The second known result we derive concerns the extent to which "the present is determined by the past." Here, the exact characterization of the error sets occurs in Theorems 3.1 and 3.5 in [5]; some philosophical implications are contained in Alexander George's article [3]. The result is the following.

Theorem 3.2. *There exists a predictor that will, for each function $f : \mathbf{R} \to Y$, correctly guess the value of $f(x)$ from the values of f on $(-\infty, x)$, for all x except those in a well-ordered subset of $\mathbf{R}$.*

The derivation of this from our main result uses the downward topology on $\mathbf{R}$ in which, as we mentioned, the scattered sets are the well-ordered subsets of the reals.

The third known result is from Section 7 of [5]. It is the following.

Theorem 3.3. *Consider the hat problem in which the set of people is an ordinal α, and each person sees the hats of all higher-numbered people. Then there exists a strategy ensuring that the set of people incorrectly guessing their hat color is finite.*

The derivation of this from our main result uses the upward topology on α in which, as we mentioned, the scattered sets are the finite subsets of α.

A known result that we extend here is Theorem 5.1 from [5] in which the present is predicted from an "infinitesimal" piece of the past, and the predictor is correct except on a countable set that is nowhere dense. In terms of our framework here, we have the topology on $\mathbf{R}$ in which the basic open sets are half-open intervals $(w, x]$ (so $f \approx_x g$ if f and g agree on (w, x) for some $w < x$). It is known that the scattered sets here are countable and nowhere dense. The exact characterization of the error sets in this example (as scattered sets) was absent in [5].

Finally, for our second extension, suppose that we have a graph on ω that is transitive in the sense that if $x < y < z$ and x is adjacent to y and y is adjacent to z, then x is adjacent to z. Consider the topology on ω in which the only basic neighborhood of n is the set $V_n = \{n\} \cup \{\, m : n < m \text{ and } n \text{ is adjacent to } m \,\}$. Treating this as a hat problem, each player n can see the hats of players in $V_n - \{n\}$. In this context it is known that there exists a finite-error predictor iff the graph contains no infinite independent set [5]. But what if the graph does contain an infinite independent set? It is easy to see that the scattered sets in this topology (which is T_0) are precisely the ones that contain no infinite complete subgraph, so our main results give us the following.

Theorem 3.4. *Consider the hat problem on ω in which each person sees some of the hats to his right and the visibility relation is transitive. Then there is a strategy in which the error set E will never contain an infinite subset W such that, for all $m, n \in W$ with $m < n$, m can see n. Furthermore, for any strategy and any set E containing no infinite W as above, there is a way to color the hats so that the error set contains E.*

4. Concluding remarks

There is a game-theoretic characterization of weakly scattered sets, special cases of which occur in [2]. Given a space X and a set $S \subseteq X$, Players I and II take turns, with Player I choosing elements of S and Player II choosing open sets. Player I must choose his point in the last open set chosen by Player II, and Player II must choose his open set to be a neighborhood of Player I's last chosen point. Player I wins iff all his choices are distinct. The set S is weakly scattered iff Player II has a winning strategy and not weakly scattered iff Player I has a winning strategy. This characterization generalizes the fact that a set is well ordered iff it has no infinite descending chains.

Our second remark concerns the equivalence relation in Section 2 that was defined on the set $\mathrm{Hom}(X, Y)$ of functions mapping the topological space X to the set Y. A similar equivalence relation, denoted here by $\approx'_x$, can be defined by setting $f \approx'_x g$ iff $f|V = g|V$ for some neighborhood V of x. Letting $[f]'_x$ denote the equivalence class of f under $\approx'_x$, we get the traditional notion of a *germ of functions at x*.

There is a version of Theorem 2.5 for the case in which the space is not T_0. However, it turns out to be somewhat of a disjoint union of what we have done here and the finite case in [4].

Finally, in the event that Y is a topological space, one might wonder whether the μ^*-predictor, or any predictor that guarantees an error set that is weakly scattered, can be an extension of the limit operator. In general, this is not possible, as the following example shows. Let $X = Y = \mathbf{R}$ and consider the familiar function

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ in lowest terms,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

It is straightforward to show that $\lim_{x \to a} f(x) = 0$ for every $a \in \mathbf{R}$. Any predictor that extends the limit operator would guess $f(x) = 0$ at each x and would be wrong at every rational x. However, the set of rationals is not weakly scattered in the usual topology on $\mathbf{R}$. So, predictors that guarantee that error sets are weakly scattered cannot extend the limit operator.

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