

EFFECTIVE MINIMAL SUBFLOWS OF BERNOULLI FLOWS

ELI GLASNER AND VLADIMIR V. USPENSKIJ

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ABSTRACT. We show that every infinite discrete group G has an infinite minimal subflow in its Bernoulli flow $\{0, 1\}^G$. A countably infinite group G has an effective minimal subflow in $\{0, 1\}^G$. If G is countable and residually finite, then it has such a subflow which is free. We do not know whether there are groups G with no free subflows in $\{0, 1\}^G$.

1. INTRODUCTION

A well known theorem of Ellis asserts that every discrete group admits a free compact flow. This was later extended by Veech to the class of locally compact groups [10]. What Ellis and Veech have actually shown is that the “greatest ambit” of the group G is a free G -flow. A G -flow is a pair (X, G) , where X is a compact Hausdorff space, G is a discrete group, and G acts on X by homeomorphisms. The action is *free* if no element of G but the identity admits a fixed point. An *ambit* is a G -flow (X, G) with a distinguished point $x_0 \in X$ whose orbit is dense: $\overline{Gx_0} = X$. The *greatest ambit* is the Stone–Čech compactification for a discrete group, and it is the Gelfand space of the Banach algebra $BRUC(G)$ of bounded, right uniformly continuous, complex valued functions on G , for a general topological group. For a discrete group G , it is known that the enveloping semigroup of the Bernoulli flow on $\Omega = \{0, 1\}^G$, i.e. the action defined on Ω by $(g\omega)(h) = \omega(g^{-1}h)$, coincides with the greatest ambit of G . (The *enveloping semigroup* of a G -flow (X, G) is defined as the closure in X^X of the set of translations defined by the elements of G ; for a discussion of the enveloping semigroup including a proof of the above statement, see for example [4, Chapter 1, Section 4].) This implies that, in some sense, the Bernoulli G -flow is sufficiently rich to recapture the universal G -flow, namely the greatest G -ambit. It is thus natural to ask whether for every such G , its Bernoulli flow $(\{0, 1\}^G, G)$ admits a free subflow. Recently some variants of this question appeared in other contexts as well. In [2] the authors relate some versions of the above problem to combinatorial group theory via tiling, coloring and other geometrical constructions on groups.

Let e denote the identity element of G . A flow (X, G) is *aperiodic* if it does not contain finite orbits. It is *minimal* if it does not contain proper compact subflows, or, equivalently, the orbit of each point $x \in X$ is dense. The flow (X, G) is *effective*

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if for every $g \neq e$ in G there is some $x \in X$ with $gx \neq x$. It is *strongly effective* if there is a point $x_0 \in X$ such that the map $g \mapsto gx_0$, $g \in G$, is 1-1 from G into X . Finally, the flow is *free* if for every $g \in G \setminus \{e\}$ and every $x \in X$, we have $gx \neq x$.

We introduce the following definitions. Let G be a discrete group. For a positive integer $n \geq 2$ set $\Omega_n = \{0, 1, \dots, n-1\}^G$ and let G act on Ω_n by left translations:

$$(g\omega)(h) = \omega(g^{-1}h), \quad \omega \in \Omega_n, g, h \in G.$$

Then (Ω_n, G) is the *Bernoulli G -flow on n symbols*. We say that G is *symbolically* $\{-aperiodic\}$, $\{-effective\}$, $\{-strongly effective\}$, $\{-free\}$ if for some n the Bernoulli flow (Ω_n, G) admits an $\{\infinite\}$, $\{effective\}$, $\{strongly effective\}$, $\{free\}$ compact subflow, respectively. We denote $\Omega_2 = \{0, 1\}^G$ simply by 2^G .

In this paper we show that every infinite group is symbolically-aperiodic, every countable infinite group is symbolically-effective, and every countable infinite residually finite group is symbolically-free. In all these cases we can take $n = 2$. More precisely: (1) for every infinite G the Bernoulli flow 2^G has an infinite minimal subflow (Theorem 2.1); (2) if additionally G is countable (we do not know if this restriction is essential), then there is a minimal effective subflow of 2^G (Theorem 3.1); (3) if G is countable and residually finite, then there is a minimal subflow of 2^G on which G acts freely (Theorem 4.2). We provide some further examples of symbolically-free groups and give a combinatorial characterization of this property. We do not know whether there are groups G with no free subflows in 2^G . We would like to thank Vladimir Pestov for the elegant formulation of the combinatorial condition in Section 6.

What we call symbolically-free or symbolically-aperiodic groups, were called in [2] groups admitting limit aperiodic colorings or limit weakly aperiodic colorings, respectively. We are grateful to Alexander Dranishnikov for making the paper [2] available to us before its publication and for stimulating discussions.

2. EVERY GROUP IS SYMBOLICALLY-APERIODIC

2.1. Theorem. *For every infinite G the Bernoulli flow 2^G has an infinite minimal subflow. Thus every infinite group is symbolically-aperiodic.*

We need some preliminaries. We have already mentioned that every discrete group G acts freely on the Stone–Čech compactification $S := \beta G$, which is the greatest ambit of G . This is due to Ellis [3]. The proof is by constructing, for every g in $G \setminus \{e\}$, a three-valued function $v : G \rightarrow \mathbb{Z}$ (the integers) such that $v(gh) \neq v(h)$ for every $h \in G$ (an easy exercise), and then extending v to S . By Zorn's lemma there is a minimal G -flow $M \subset S$. Such an M is called the *universal minimal flow*, and it is unique up to an isomorphism of G -spaces. As the action of G on S is free, (M, G) is a minimal free flow.

A topological space X is *extremally disconnected* if the closure of every open set $U \subset X$ is clopen, or, equivalently, if disjoint open sets have disjoint closures.

2.2. Lemma (Ellis [3]). *For every infinite discrete group G there exists an extremally disconnected minimal compact G -space X such that the action of G on X is free.*

Proof. The universal minimal compact G -space M is a retract of the greatest ambit βG (see e.g. [9] or [8, Proof of Lemma 6.1.2]). The Stone–Čech compactification βG is extremally disconnected, and it is easy to see that the property of being

extremally disconnected is preserved by retracts. We have observed that G acts freely on βG and hence on M . \square

2.3. Lemma. *If X is an infinite extremally disconnected minimal compact G -space, then there is a clopen subset $U \subset X$ such that the G -orbit $\{gU : g \in G\}$ of U is infinite.*

Proof. Assume, in order to get a contradiction, that every clopen subset of X has a finite G -orbit. Let B be the (complete) Boolean algebra of all clopen subsets of X . According to our assumption, every $U \in B$ lies in a finite G -invariant subalgebra of B . It follows that the collection \mathcal{E} of all G -invariant finite clopen partitions of X contains “arbitrarily fine” covers: every open cover α of X has a refinement $\beta \in \mathcal{E}$. Consider a sequence $\{\gamma_n : n \in \mathbb{N}\}$ of partitions in \mathcal{E} such that each γ_{n+1} is a refinement of γ_n , and each $U \in \gamma_n$ is the union of at least three members of γ_{n+1} . Construct by induction distinct $U_n, V_n \in \gamma_n$ such that $U_{n+1} \cup V_{n+1} \subset V_n$. Let $O = \bigcup U_n$. We claim that the clopen set $O = \overline{O}$ has an infinite G -orbit.

Indeed, U_0 is the only member of γ_0 contained in O , and U_{n+1} is the only member of γ_{n+1} disjoint from U_n and contained in O . It follows that for every $g \in G$ the only disjoint sequence $\{W_n\}$ such that $\bigcup W_n \subset gO$ and $W_n \in \gamma_n$ is the sequence $\{gU_n\}$. Thus for every $n \in \mathbb{N}$ the map $gO \mapsto gU_n$ from the G -orbit of O onto γ_n is well-defined (this map is onto because X is minimal). Therefore, the cardinality of the G -orbit of O is not less than $|\gamma_n|$. Since $|\gamma_n| \rightarrow \infty$, the orbit of O is infinite. \square

Proof of Theorem 2.1. According to Lemmas 2.2 and 2.3, there exists a compact minimal G -space X and a clopen set $U \subset X$ such that the collection $\{gU : g \in G\}$ is infinite. Let $\chi_U : X \rightarrow \{0, 1\}$ be the characteristic function of U . Consider the G -map $\phi : X \rightarrow 2^G$ defined by $\phi(x)(g) = \chi_U(g^{-1}x)$ ($x \in X, g \in G$). Since X is compact and minimal, so is its image under ϕ . If $x, y \in X$, then $\phi(x) = \phi(y)$ if and only if x and y cannot be separated by a set of the form $gU, g \in G$. It follows that $\phi(X)$ is infinite. \square

3. EVERY COUNTABLE GROUP IS SYMBOLICALLY-EFFECTIVE

In the proof of the next theorem we will use the following fact. A surjective homomorphism of metric G -flows $\pi : X \rightarrow Y$ is called an *almost 1-1 extension* if the subset $\{y \in Y : |\pi^{-1}(y)| = 1\}$ is dense and G_δ . It is an easy exercise to show that if π is an almost 1-1 extension, and if Y is minimal and X is point transitive (i.e. has a point whose orbit is dense), then X is also minimal.

3.1. Theorem. *For every discrete infinite countable group G there is a minimal subflow $Y \subset 2^G$ and a point $y_0 \in Y$ such that the map $g \mapsto gy_0$ is 1-1. Thus every infinite countable group is symbolically-strongly effective.*

Proof. Let (M, G) be a minimal free flow (see the previous section). Fix a point $m_0 \in M$.

Next consider a metric factor $\pi : (M, G) \rightarrow (X, G)$ such that $\pi \upharpoonright Gm_0$ is 1-1; i.e., denoting $x_0 = \pi(m_0)$, the map $g \mapsto gx_0$ from G onto the orbit Gx_0 is 1-1 (this is possible since for a countable G the metric factors separate points on βG). Let $\xi \in X$ be a point which is not in Gx_0 . Let $\{g_1, g_2, \dots\}$ be an enumeration of $G \setminus \{e\}$. We construct by induction a sequence of open sets $U_n \subset X, n = 0, 1, 2, \dots$, such that:

- (i) for each $n, U_{n+1} \subset U_n$,

- (ii) $\bigcap_{n=0}^\infty U_n = \{\xi\}$,
- (iii) for each $n > 0$, $\partial U_n \cap Gx_0 = \emptyset$, and
- (iv) for every $g_j \in G \setminus \{e\}$ there are $h_j \in G$ and $n \in \mathbb{N}$ such that U_n distinguishes the points h_jx_0 and $h_jg_jx_0$.

Let d be the metric on X . We denote by $B_r(x)$ the closed ball of radius r centered at x ($x \in X, r > 0$). Let $U_0 = X$ and let U_1 be an open set containing ξ which separates x_0 and g_1x_0 and such that $\partial U_1 \cap Gx_0 = \emptyset$. Suppose $U_1 \supset U_2 \supset \dots \supset U_n$ with $\partial U_k \cap Gx_0 = \emptyset$ and that $\text{diam } U_k < 1/k, k = 2, \dots, n$, have been constructed. Let $0 < \delta < \frac{1}{n+1}$ be such that $B_\delta(\xi) \subset U_n$ and such that all the points h_jx_0 and $h_jg_jx_0$ for $j \leq n$ are not in $B_\delta(\xi)$. By minimality, there is some $g = h_{n+1} \in G$ with $d(gx_0, \xi) < \delta/2$. Consider the point $z = gg_{n+1}x_0$.

A. *Case.* Suppose $z \notin U_n$. Then there is a unique $0 \leq t \leq n - 1$ such that $z \in U_t \setminus U_{t+1}$. Choose radii $0 < 2d(gx_0, \xi) < r_2 < r_1 < \delta$ such that $\partial B_{r_i}(\xi) \cap Gx_0 = \emptyset, i = 1, 2$, and set $U_{n+1} = B_{r_1}(\xi)$. If $n + 1$ and t have the same parity, also set $U_{n+2} = B_{r_2}(\xi)$.

B. *Case.* If $z \in U_n$, choose U_{n+1} to be an open set such that $\xi \in U_{n+1} \subset U_n, \partial U_{n+1} \cap Gx_0 = \emptyset$, with diameter $< \frac{1}{n+1}$, and so that $gx_0 \in U_{n+1}$ but $z = gg_{n+1}x_0 \in U_n \setminus U_{n+1}$.

This concludes the inductive construction of the sequence $\{U_n\}_{n=0}^\infty$. Next define a function $F : X \setminus \{\xi\} \rightarrow \{0, 1\}$ by setting F to be 0 and 1 alternately on $U_n \setminus U_{n+1}$.

Note that F is continuous at every point of $Gx_0 = \{gx_0 : g \in G\}$. Set $y_0(g) = f(g) = F(g^{-1}x_0)$; then f is a $\{0, 1\}$ -valued function on G and the flow it generates in $\{0, 1\}^G$ under the shift action, say (Y, G) , is minimal. In fact the natural joining $Z = X \vee Y$, obtained as the orbit closure in $X \times Y$ of the point (x_0, y_0) , is an almost 1-1 extension of X and, being point transitive, it is minimal. Therefore Y , as a factor of Z , is also minimal. Denote by $P_e : Y \rightarrow \{0, 1\}$ the restriction to Y of the projection of $\{0, 1\}^G$ onto the e -th coordinate. We then have $F(g^{-1}x_0) = f(g) = y_0(g) = (g^{-1}y_0)(e) = P_e(g^{-1}y_0)$ and, by our construction, for every $g_j \in \{g_1, g_2, \dots\} = G \setminus \{e\}$, there is $g \in G$ with

$$1 = |F(gg_jx_0) - F(gx_0)| = |f(g_j^{-1}g^{-1}) - f(g^{-1})| = |P_e(gg_jy_0) - P_e(gy_0)|.$$

Thus the map $g \mapsto gy_0$, from G into Y , is 1-1. This completes the proof. □

4. RESIDUALLY FINITE GROUPS ARE SYMBOLICALLY-FREE

We recall that a discrete group G is *residually finite* when the intersection of all its subgroups of finite index is trivial.

4.1. Lemma. *Let G be a discrete infinite countable group. There is then a metrizable zero-dimensional free flow (X, G) .*

Proof. As we have seen in the proof of Lemma 2.2, the action of G on the universal minimal flow M is free. Let $\{f_g\}_{g \in G \setminus \{e\}}$ be a countable collection of bounded integer valued functions on G with $\inf_{h \in G} |f_g(gh) - f_g(h)| \geq 1$ (“Ellis functions”). Let \mathcal{A} be the uniformly closed G -invariant subalgebra of $\ell^\infty(G)$ containing $\{f_g\}_{g \in G}$ and let $X = |\mathcal{A}|$ be its Gelfand space, with $\pi : \beta G \rightarrow X$ denoting the corresponding canonical map. Then (X, G) is a metrizable flow with $C(X) \cong \mathcal{A}$. For $f \in \mathcal{A}$ let f be the corresponding function in $C(X)$. Set $x_0 = \pi(e)$.

Given $x \in X$ and $g \in G \setminus \{e\}$, there is a sequence $g_n \in G$ with $\lim_{n \rightarrow \infty} g_n x_0 = x$ so that

$$\hat{f}_g(gx) = \lim_{n \rightarrow \infty} \hat{f}_g(gg_n x_0) = \lim_{n \rightarrow \infty} f_g(gg_n)$$

and

$$\hat{f}_g(x) = \lim_{n \rightarrow \infty} \hat{f}_g(g_n x_0) = \lim_{n \rightarrow \infty} f_g(g_n).$$

Thus

$$|\hat{f}_g(gx) - \hat{f}_g(x)| = \lim_{n \rightarrow \infty} |f_g(gg_n) - f_g(g_n)| \geq 1.$$

In particular $gx \neq x$ and the action (X, G) is free.

Thus we have shown that *for every countable infinite G there is a metrizable free G -flow (X, G)* . By [1] (see also [5]) there exists a metrizable zero-dimensional cover $(X', G) \rightarrow (X, G)$ and we can therefore assume that X is zero-dimensional. \square

Let us denote by \mathcal{J} the class of groups G which admit an effective metric, zero-dimensional, isometric action. If (X, G) is such an action then, since all these properties are preserved in subsystems, we may assume that (X, G) is also minimal. The group $\text{Iso}(X)$ of isometries of the zero-dimensional compact metric space X is itself zero-dimensional and compact. To sum up: G is in \mathcal{J} iff it admits a 1-1, metrizable, zero-dimensional, topological group compactification (see e.g. [4]).

A conspicuous subclass of \mathcal{J} is the class of residually finite countable groups. Let G be a residually finite countable group. Let \mathcal{H} be the collection of subgroups $H < G$ with $[G : H] < \infty$. Pick a decreasing sequence $\{H_n\} \subset \mathcal{H}$ such that $\bigcap H_n = \{e\}$, and consider the associated inverse limit

$$X = \varprojlim G/H_n.$$

The corresponding flow (X, G) is topologically a Cantor set, algebraically a compact zero-dimensional topological group, and dynamically an isometric free G -action with a bi-invariant metric. Thus $G \in \mathcal{J}$. Note that it was necessary to pass to a countable subsequence of \mathcal{H} , since the profinite completion

$$\varprojlim_{H \in \mathcal{H}} G/H$$

of G need not be metrizable.

4.2. Theorem. *Let G be an infinite countable discrete group in the class \mathcal{J} . Then there is a minimal subflow $Y \subset 2^G$ on which G acts freely. Thus every group in \mathcal{J} is symbolically-free. In particular this holds for infinite countable residually finite groups.*

Proof. Let X be a 1-1, metrizable, zero-dimensional, topological group compactification of G (see Lemma 4.1 above).

Repeat the construction in the proof of Theorem 3.1, with the extra property that $\partial U_n = \emptyset$ for every n . This can be done as now X is zero dimensional.

Let $\mathcal{C}(F)$ be the set of continuity points of F . With this additional condition we have $\mathcal{C}(F) = X \setminus \{\xi\}$ and therefore $\mathcal{C}_0(F) = \bigcap \{g\mathcal{C}(F) : g \in G\} = X \setminus G\xi$ is a dense G_δ subset of X with a countable complement.

We can now give a full description of the points of Y . Set $f(g) = f_{x_0}(g) = y_0(g) = F(g^{-1}x_0)$, ($g \in G$). Then for $h \in G$ we have $(hy_0)(g) = y_0(h^{-1}g) = F(g^{-1}hx_0)$ ($g \in G$). Suppose now that $\lim_{i \rightarrow \infty} h_i x_0 = x$ with $x \in \mathcal{C}_0(F)$. We then have

$$\lim_{i \rightarrow \infty} (h_i y_0)(g) = \lim_{i \rightarrow \infty} y_0(h_i^{-1}g) = \lim_{i \rightarrow \infty} F(g^{-1}h_i x_0) = F(g^{-1}x).$$

Thus in $\{0, 1\}^G$, $y_x = f_x = \lim_{i \rightarrow \infty} h_i y_0$ exists, with

$$y_x(g) = f_x(g) = F(g^{-1}x)$$

(so that $y_0 = y_{x_0}$). Moreover if $\lim_{i \rightarrow \infty} h_i x_0 = \xi$ and $\lim_{i \rightarrow \infty} h_i y_0 = y$ exists, then for $g \neq e$,

$$y(g) = \lim_{i \rightarrow \infty} (h_i y_0)(g) = \lim_{i \rightarrow \infty} y_0(h_i^{-1}g) = \lim_{i \rightarrow \infty} F(g^{-1}h_i x_0) = F(g^{-1}\xi),$$

and $y(e)$ is either 0 or 1 and accordingly we denote $y = y_\xi^0$ or $y = y_\xi^1$. It is now clear that, with the notation $gy_\xi^\epsilon = y_{g\xi}^\epsilon$,

$$Y = \{y_x : x \in X_0\} \cup \{y_{g\xi}^\epsilon : g \in G, \epsilon = 0, 1\}.$$

On the dense G_δ , G -invariant subset $X_0 \subset X$ there is a continuous homomorphism $\phi : x \mapsto y_x$ from X_0 into $Y \subset \{0, 1\}^G$ and (with notation as in the proof of Theorem 3.1)

$$Z = X \vee Y = \text{cls} \{(x, \phi(x)) : x \in X_0\}.$$

Given $x_1 \neq x_2$ two points in X_0 we can find a sequence $g_n \in G$ such that $\lim_{n \rightarrow \infty} g_n x_1 = \xi$. Since for every $g \in G$, $d(gx_1, gx_2) = d(x_1, x_2)$, we can choose some $g \in G$ for which gx_1 is very close to ξ and

$$|F(gx_1) - F(gx_2)| = |f_{x_1}(g) - f_{x_2}(g)| = 1.$$

This shows that the map $\phi : X_0 \rightarrow Y$ is 1-1. It is now easy to check that the natural projection map $\pi_Y : Z \rightarrow Y$ is 1-1 (an isomorphism) and it follows that the map $\pi : Y \rightarrow X$ defined by $\pi = \pi_X \circ \pi_Y^{-1}$ (with $\pi_X : Z \rightarrow X$ denoting the projection on X) is a continuous homomorphism from (Y, G) onto (X, G) . (Explicitly we have $\pi(y_x) = x = \phi^{-1}(y_x)$ for $x \in X_0$, and $\pi(y_{g\xi}^\epsilon) = g\xi$ on the complement of $Y_0 = \phi(X_0)$ in Y .)

Thus (Y, G) , as an extension of a free action, is itself free and our proof is complete. □

5. SOME FURTHER EXAMPLES

- 5.1. Theorem.**
- (1) *Every Abelian group is symbolically-free.*
 - (2) *Every residually finite group is symbolically-free.*
 - (3) *Let S_∞^0 be the group of permutations of \mathbb{N} with finite support. Every infinite subgroup of S_∞^0 is symbolically-free.*
 - (4) *Every torsion free hyperbolic group is symbolically-free.*

Proof. 1. Apply Theorem 3.1 and then note that for G Abelian, an effective minimal flow is already free.

2. This is the statement of Theorem 4.2.

3. This follows from [6] (see also [8, Section 6.3]) where it is shown that S_∞ (the group of all permutations of \mathbb{N}) admits a minimal action (actually, the universal minimal flow of S_∞ as a Polish group) on which the subgroup S_∞^0 acts freely.

4. This is a result of Dranishnikov and Schroeder, [2]. □

5.2. Remark. A famous open problem is whether every hyperbolic group is residually finite (see e.g. [7]). An affirmative answer will provide, via Theorem 4.2, a proof that every hyperbolic group is symbolically-free, improving the Dranishnikov-Schroeder result.

6. A COMBINATORIAL CHARACTERIZATION OF SYMBOLICALLY-FREE GROUPS

In the following theorem we use the pictorial term “a 2-coloring of a set X ” to describe a partition of X into two disjoint subsets; each has its own “color”. Thus elements x and y of X have “different colors” if they belong to different subsets of the partition.

6.1. Theorem. *Let G be an infinite countable group. The following conditions are equivalent.*

- (1) G acts freely on some subflow of 2^G .
- (2) There exists a 2-coloring of G with the following property. For every $g \neq e$, there is a finite set $A = A(g)$ such that for every $h \in G$ there is an $a \in A \cap g^{-1}A$ such that ha and hga have different colors.

Proof. Suppose first that $Y \subset \Omega_2 = \{0, 1\}^G$ is a free subflow. Let ω be any point in Y . We consider ω as a coloring of G and will show that it has the property described in (2). If this is not the case, then there is some $g \in G \setminus \{e\}$ such that for every finite set $A \subset G$ there is some $h = h_A \in G$ with the property that $\omega(hga) = \omega(ha)$ for every $a \in A \cap g^{-1}A$. Let A_n be an increasing sequence of finite subsets of G with $G = \bigcup_{n=1}^{\infty} A_n$. For each n let $h_n = h_{A_n}$ be the corresponding element of G such that

$$h_n^{-1}\omega(ga) = \omega(h_nga) = \omega(h_na) = h_n^{-1}\omega(a) \quad \text{for every } a \in A_n \cap g^{-1}A_n.$$

Clearly also $G = \bigcup_{n=1}^{\infty} (A_n \cap g^{-1}A_n)$, and taking a convergent subsequence $h_{n_i}^{-1}\omega \rightarrow \xi \in Y$, we have

$$\xi(ga) = \xi(a) \quad \text{for every } a \in G.$$

Thus $g\xi = \xi$, contradicting our assumption that the flow (Y, G) is free.

Conversely, assume now that condition (2) is satisfied. Let $\omega : G \rightarrow \{0, 1\}$ be the coloring whose existence is ensured by this condition and consider its orbit closure $Y = \text{cls}(G\omega)$ in Ω_2 . Let ξ be any point in Y and let g be an element of $G \setminus \{e\}$. There exists a sequence $h_n \in G$ with $\xi = \lim_{n \rightarrow \infty} h_n^{-1}\omega$. This means that for any finite $A \subset G$, eventually

$$(1) \quad h_n^{-1}\omega(a) = \omega(h_na) = \xi(a) \quad \text{for every } a \in A.$$

In particular this holds for the finite set $A = A(g^{-1})$ given in condition (2) and we fix some $h = h_n$ for which equation (1) holds with respect to this A . By condition (2) then, there exists some $a \in A \cap g^{-1}A$ with $\omega(ha) \neq \omega(hga)$. But then

$$\xi(a) = h^{-1}\omega(a) = \omega(ha) \neq \omega(hga) = h^{-1}\omega(ga) = \xi(ga) = g^{-1}\xi(a).$$

Thus $g^{-1}\xi \neq \xi$ and we have shown that (Y, G) is a free flow. \square

Call the condition (2) in Theorem 6.1 *property P*. With this terminology, each of the groups listed in Theorem 5.1 has property *P*. The main question now is whether there is any countably infinite group which does not satisfy property *P*. Stated explicitly we have the following.

6.2. Problem. Is there a countable infinite group G with the following coloring property?

For every 2-coloring of G , there exists $g \in G \setminus \{e\}$ such that for every finite set $A \subset G$, there exists an $h \in G$ for which the pair ha and hga have the same color for every $a \in A \cap g^{-1}A$.

6.3. *Remark.* After this paper was accepted for publication, we learned that our main question (formulated as the coloring problem 6.2) had recently received a negative answer by Su Gao, Steve Jackson, and Brandon Seward. This paper, entitled “A coloring property for countable groups”, is available on Gao’s homepage at <http://www.cas.unt.edu/~sgao/pub/paper31.html>.

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DEPARTMENT OF MATHEMATICS, TEL-AVIV UNIVERSITY, TEL AVIV, ISRAEL

E-mail address: glasner@math.tau.ac.il

URL: <http://www.math.tau.ac.il/~glasner>

DEPARTMENT OF MATHEMATICS, 321 MORTON HALL, OHIO UNIVERSITY, ATHENS, OHIO 45701

E-mail address: uspensk@math.ohiou.edu