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EFFECTIVE MINIMAL SUBFLOWS OF BERNOULLI FLOWS

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Abstract. We show that every infinite discrete group G has an infinite minimal subflow in its Bernoulli flow $\{0,1\}^G$. A countably infinite group G has an effective minimal subflow in $\{0,1\}^G$. If G is countable and residually finite, then it has such a subflow which is free. We do not know whether there are groups G with no free subflows in $\{0,1\}^G$.

1. Introduction

A well known theorem of Ellis asserts that every discrete group admits a free compact flow. This was later extended by Veech to the class of locally compact groups [10]. What Ellis and Veech have actually shown is that the "greatest ambit" of the group G is a free G-flow. A *G-flow* is a pair (X, G), where X is a compact Hausdorff space, G is a discrete group, and G acts on X by homeomorphisms. The action is *free* if no element of G but the identity admits a fixed point. An *ambit* is a G-flow (X, G) with a distinguished point $x_0 \in X$ whose orbit is dense: $\overline{Gx_0} = X$. The *greatest ambit* is the Stone–Čech compactification for a discrete group, and it is the Gelfand space of the Banach algebra $BRUC(G)$ of bounded, right uniformly continuous, complex valued functions on G, for a general topological group. For a discrete group G, it is known that the enveloping semigroup of the Bernoulli flow on $\Omega = \{0,1\}^G$, i.e. the action defined on Ω by $(g\omega)(h) = \omega(g^{-1}h)$, coincides with the greatest ambit of G. (The *enveloping semigroup* of a G-flow (X, G) is defined as the closure in X^X of the set of translations defined by the elements of G; for a discussion of the enveloping semigroup including a proof of the above statement, see for example [4, Chapter 1, Section 4].) This implies that, in some sense, the Bernoulli G-flow is sufficiently rich to recapture the universal G-flow, namely the greatest G-ambit. It is thus natural to ask whether for every such G, its Bernoulli flow $(\{0,1\}^G, G)$ admits a free subflow. Recently some variants of this question appeared in other contexts as well. In [2] the authors relate some versions of the above problem to combinatorial group theory via tiling, coloring and other geometrical constructions on groups.

Let e denote the identity element of G. A flow (X, G) is *aperiodic* if it does not contain finite orbits. It is *minimal* if it does not contain proper compact subflows, or, equivalently, the orbit of each point $x \in X$ is dense. The flow (X, G) is *effective*

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if for every $g \neq e$ in G there is some $x \in X$ with $gx \neq x$. It is *strongly effective* if there is a point $x_0 \in X$ such that the map $g \mapsto gx_0,\ g \in G$, is 1-1 from G into X. Finally, the flow is *free* if for every $g \in G \setminus \{e\}$ and every $x \in X$, we have $gx \neq x$.

We introduce the following definitions. Let G be a discrete group. For a positive integer $n \geq 2$ set $\Omega_n = \{0, 1, \ldots, n-1\}^G$ and let G act on Ω_n by left translations:

$$(g\omega)(h) = \omega(g^{-1}h), \qquad \omega \in \Omega_n,\ g, h \in G.$$

Then (Ω_n, G) is the *Bernoulli G-flow on n symbols*. We say that G is *symbolically* $\{$*-aperiodic*$\}$, $\{$*-effective*$\}$, $\{$*-strongly effective*$\}$, $\{$*-free*$\}$ if for some n the Bernoulli flow (Ω_n, G) admits an $\{$infinite minimal$\}$, $\{$effective$\}$, $\{$strongly effective$\}$, $\{$free$\}$ compact subflow, respectively. We denote $\Omega_2 = \{0, 1\}^G$ simply by 2^G.

In this paper we show that every infinite group is symbolically-aperiodic, every countable infinite group is symbolically-effective, and every countable infinite residually finite group is symbolically-free. In all these cases we can take $n = 2$. More precisely: (1) for every infinite G the Bernoulli flow 2^G has an infinite minimal subflow (Theorem 2.1); (2) if additionally G is countable (we do not know if this restriction is essential), then there is a minimal effective subflow of 2^G (Theorem 3.1); (3) if G is countable and residually finite, then there is a minimal subflow of 2^G on which G acts freely (Theorem 4.2). We provide some further examples of symbolically-free groups and give a combinatorial characterization of this property. We do not know whether there are groups G with no free subflows in 2^G. We would like to thank Vladimir Pestov for the elegant formulation of the combinatorial condition in Section 6.

What we call symbolically-free or symbolically-aperiodic groups, were called in [2] groups admitting limit aperiodic colorings or limit weakly aperiodic colorings, respectively. We are grateful to Alexander Dranishnikov for making the paper [2] available to us before its publication and for stimulating discussions.

2. Every group is symbolically-aperiodic

2.1. **Theorem.** *For every infinite G the Bernoulli flow 2^G has an infinite minimal subflow. Thus every infinite group is symbolically-aperiodic.*

We need some preliminaries. We have already mentioned that every discrete group G acts freely on the Stone–Čech compactification $S := \beta G$, which is the greatest ambit of G. This is due to Ellis [3]. The proof is by constructing, for every g in $G \setminus \{e\}$, a three-valued function $v : G \to \mathbb{Z}$ (the integers) such that $v(gh) \neq v(h)$ for every $h \in G$ (an easy exercise), and then extending v to S. By Zorn's lemma there is a minimal G-flow $M \subset S$. Such an M is called the *universal minimal flow*, and it is unique up to an isomorphism of G-spaces. As the action of G on S is free, (M, G) is a minimal free flow.

A topological space X is *extremally disconnected* if the closure of every open set $U \subset X$ is clopen, or, equivalently, if disjoint open sets have disjoint closures.

2.2. **Lemma** (Ellis [3]). *For every infinite discrete group G there exists an extremally disconnected minimal compact G-space X such that the action of G on X is free.*

Proof. The universal minimal compact G-space M is a retract of the greatest ambit βG (see e.g. [9] or [8, Proof of Lemma 6.1.2]). The Stone–Čech compactification βG is extremally disconnected, and it is easy to see that the property of being extremally disconnected is preserved by retracts. We have observed that G acts freely on βG and hence on M. □

2.3. **Lemma.** *If X is an infinite extremally disconnected minimal compact G-space, then there is a clopen susbset $U \subset X$ such that the G-orbit $\{gU : g \in G\}$ of U is infinite.*

Proof. Assume, in order to get a contradiction, that every clopen subset of X has a finite G-orbit. Let B be the (complete) Boolean algebra of all clopen subsets of X. According to our assumption, every $U \in B$ lies in a finite G-invariant subalgebra of B. It follows that the collection $\mathcal{E}$ of all G-invariant finite clopen partitions of X contains "arbitrarily fine" covers: every open cover α of X has a refinement $\beta \in \mathcal{E}$. Consider a sequence $\{\gamma_n : n \in \mathbb{N}\}$ of partitions in $\mathcal{E}$ such that each γ_{n+1} is a refinement of γ_n, and each $U \in \gamma_n$ is the union of at least three members of γ_{n+1}. Construct by induction distinct $U_n, V_n \in \gamma_n$ such that $U_{n+1} \cup V_{n+1} \subset V_n$. Let $U = \bigcup U_n$. We claim that the clopen set $O = \overline{U}$ has an infinite G-orbit.

Indeed, U_0 is the only member of γ_0 contained in O, and U_{n+1} is the only member of γ_{n+1} disjoint from U_n and contained in O. It follows that for every $g \in G$ the only disjoint sequence $\{W_n\}$ such that $\bigcup W_n \subset gO$ and $W_n \in \gamma_n$ is the sequence $\{gU_n\}$. Thus for every $n \in \mathbb{N}$ the map $gO \mapsto gU_n$ from the G-orbit of O onto γ_n is well-defined (this map is onto because X is minimal). Therefore, the cardinality of the G-orbit of O is not less than $|\gamma_n|$. Since $|\gamma_n| \to \infty$, the orbit of O is infinite. □

Proof of Theorem 2.1. According to Lemmas 2.2 and 2.3, there exists a compact minimal G-space X and a clopen set $U \subset X$ such that the collection $\{gU : g \in G\}$ is infinite. Let $\chi_U : X \to \{0,1\}$ be the characteristic function of U. Consider the G-map $\phi : X \to 2^G$ defined by $\phi(x)(g) = \chi_U(g^{-1}x)$ $(x \in X,\ g \in G)$. Since X is compact and minimal, so is its image under ϕ. If $x, y \in X$, then $\phi(x) = \phi(y)$ if and only if x and y cannot be separated by a set of the form gU, $g \in G$. It follows that $\phi(X)$ is infinite. □

3. Every countable group is symbolically-effective

In the proof of the next theorem we will use the following fact. A surjective homomorphism of metric G-flows $\pi : X \to Y$ is called an *almost 1-1 extension* if the subset $\{y \in Y : |\pi^{-1}(y)| = 1\}$ is dense and G_δ. It is an easy exercise to show that if π is an almost 1-1 extension, and if Y is minimal and X is point transitive (i.e. has a point whose orbit is dense), then X is also minimal.

3.1. **Theorem.** *For every discrete infinite countable group G there is a minimal subflow $Y \subset 2^G$ and a point $y_0 \in Y$ such that the map $g \mapsto gy_0$ is* 1-1. *Thus every infinite countable group is symbolically-strongly effective.*

Proof. Let (M, G) be a minimal free flow (see the previous section). Fix a point $m_0 \in M$.

Next consider a metric factor $\pi : (M, G) \to (X, G)$ such that $\pi \restriction Gm_0$ is 1-1; i.e., denoting $x_0 = \pi(m_0)$, the map $g \mapsto gx_0$ from G onto the orbit Gx_0 is 1-1 (this is possible since for a countable G the metric factors separate points on βG). Let $\xi \in X$ be a point which is not in Gx_0. Let $\{g_1, g_2, \dots\}$ be an enumeration of $G \setminus \{e\}$. We construct by induction a sequence of open sets $U_n \subset X$, $n = 0, 1, 2, \dots$, such that:

(i) for each n, $U_{n+1} \subset U_n$,

(ii) $\bigcap_{n=0}^{\infty} U_n = \{\xi\}$,
(iii) for each $n > 0$, $\partial U_n \cap Gx_0 = \emptyset$, and
(iv) for every $g_j \in G \setminus \{e\}$ there are $h_j \in G$ and $n \in \mathbb{N}$ such that U_n distinguishes the points $h_j x_0$ and $h_j g_j x_0$.

Let d be the metric on X. We denote by $B_r(x)$ the closed ball of radius r centered at x ($x \in X$, $r > 0$). Let $U_0 = X$ and let U_1 be an open set containing ξ which separates x_0 and $g_1 x_0$ and such that $\partial U_1 \cap Gx_0 = \emptyset$. Suppose $U_1 \supset U_2 \supset \cdots \supset U_n$ with $\partial U_k \cap Gx_0 = \emptyset$ and that $\operatorname{diam} U_k < 1/k$, $k = 2, \ldots, n$, have been constructed. Let $0 < \delta < \frac{1}{n+1}$ be such that $B_\delta(\xi) \subset U_n$ and such that all the points $h_j x_0$ and $h_j g_j x_0$ for $j \le n$ are not in $B_\delta(\xi)$. By minimality, there is some $g = h_{n+1} \in G$ with $d(gx_0, \xi) < \delta/2$. Consider the point $z = g g_{n+1} x_0$.

A. *Case.* Suppose $z \notin U_n$. Then there is a unique $0 \le t \le n-1$ such that $z \in U_t \setminus U_{t+1}$. Choose radii $0 < 2\, d(gx_0, \xi) < r_2 < r_1 < \delta$ such that $\partial B_{r_i}(\xi) \cap Gx_0 = \emptyset$, $i = 1, 2$, and set $U_{n+1} = B_{r_1}(\xi)$. If $n+1$ and t have the same parity, also set $U_{n+2} = B_{r_2}(\xi)$.

B. *Case.* If $z \in U_n$, choose U_{n+1} to be an open set such that $\xi \in U_{n+1} \subset U_n$, $\partial U_{n+1} \cap Gx_0 = \emptyset$, with diameter $< \frac{1}{n+1}$, and so that $gx_0 \in U_{n+1}$ but $z = g g_{n+1} x_0 \in U_n \setminus U_{n+1}$.

This concludes the inductive construction of the sequence $\{U_n\}_{n=0}^{\infty}$. Next define a function $F : X \setminus \{\xi\} \to \{0,1\}$ by setting F to be 0 and 1 alternately on $U_n \setminus U_{n+1}$.

Note that F is continuous at every point of $Gx_0 = \{gx_0 : g \in G\}$. Set $y_0(g) = f(g) = F(g^{-1}x_0)$; then f is a $\{0,1\}$-valued function on G and the flow it generates in $\{0,1\}^G$ under the shift action, say (Y, G), is minimal. In fact the natural joining $Z = X \vee Y$, obtained as the orbit closure in $X \times Y$ of the point (x_0, y_0), is an almost 1-1 extension of X and, being point transitive, it is minimal. Therefore Y, as a factor of Z, is also minimal. Denote by $P_e : Y \to \{0,1\}$ the restriction to Y of the projection of $\{0,1\}^G$ onto the e-th coordinate. We then have $F(g^{-1}x_0) = f(g) = y_0(g) = (g^{-1}y_0)(e) = P_e(g^{-1}y_0)$ and, by our construction, for every $g_j \in \{g_1, g_2, \ldots\} = G \setminus \{e\}$, there is $g \in G$ with

$$1 = |F(gg_j x_0) - F(gx_0)| = |f(g_j^{-1}g^{-1}) - f(g^{-1})| = |P_e(gg_j y_0) - P_e(gy_0)|.$$

Thus the map $g \mapsto gy_0$, from G into Y, is 1-1. This completes the proof. $\square$

4. Residually finite groups are symbolically-free

We recall that a discrete group G is *residually finite* when the intersection of all its subgroups of finite index is trivial.

4.1. **Lemma.** *Let G be a discrete infinite countable group. There is then a metrizable zero-dimensional free flow (X, G).*

Proof. As we have seen in the proof of Lemma 2.2, the action of G on the universal minimal flow M is free. Let $\{f_g\}_{g \in G \setminus \{e\}}$ be a countable collection of bounded integer valued functions on G with $\inf_{h \in G} |f_g(gh) - f_g(h)| \ge 1$ ("Ellis functions"). Let $\mathcal{A}$ be the uniformly closed G-invariant subalgebra of $\ell^\infty(G)$ containing $\{f_g\}_{g \in G}$ and let $X = |\mathcal{A}|$ be its Gelfand space, with $\pi : \beta G \to X$ denoting the corresponding canonical map. Then (X, G) is a metrizable flow with $C(X) \cong \mathcal{A}$. For $f \in \mathcal{A}$ let $\hat{f}$ be the corresponding function in $C(X)$. Set $x_0 = \pi(e)$.

Given $x \in X$ and $g \in G \setminus \{e\}$, there is a sequence $g_n \in G$ with $\lim_{n\to\infty} g_n x_0 = x$ so that

$$\hat{f}_g(gx) = \lim_{n\to\infty} \hat{f}_g(gg_n x_0) = \lim_{n\to\infty} f_g(gg_n)$$

and

$$\hat{f}_g(x) = \lim_{n\to\infty} \hat{f}_g(g_n x_0) = \lim_{n\to\infty} f_g(g_n).$$

Thus

$$|\hat{f}_g(gx) - \hat{f}_g(x)| = \lim_{n\to\infty} |f_g(gg_n) - f_g(g_n)| \geq 1.$$

In particular $gx \neq x$ and the action (X, G) is free.

Thus we have shown that *for every countable infinite G there is a metrizable free G-flow* (X, G). By [1] (see also [5]) there exists a metrizable zero-dimensional cover $(X', G) \to (X, G)$ and we can therefore assume that X is zero-dimensional. □

Let us denote by $\mathcal{I}$ the class of groups G which admit an effective metric, zero-dimensional, isometric action. If (X, G) is such an action then, since all these properties are preserved in subsystems, we may assume that (X, G) is also minimal. The group $\operatorname{Iso}(X)$ of isometries of the zero-dimensional compact metric space X is itself zero-dimensional and compact. To sum up: G is in $\mathcal{I}$ iff it admits a 1-1, metrizable, zero-dimensional, topological group compactification (see e.g. [4]).

A conspicuous subclass of $\mathcal{I}$ is the class of residually finite countable groups. Let G be a residually finite countable group. Let $\mathcal{H}$ be the collection of subgroups $H < G$ with $[G : H] < \infty$. Pick a decreasing sequence $\{H_n\} \subset \mathcal{H}$ such that $\bigcap H_n = \{e\}$, and consider the associated inverse limit

$$X = \varprojlim G/H_n.$$

The corresponding flow (X, G) is topologically a Cantor set, algebraically a compact zero-dimensional topological group, and dynamically an isometric free G-action with a bi-invariant metric. Thus $G \in \mathcal{I}$. Note that it was necessary to pass to a countable subsequence of $\mathcal{H}$, since the profinite completion

$$\varprojlim_{H\in\mathcal{H}} G/H$$

of G need not be metrizable.

4.2. **Theorem.** *Let G be an infinite countable discrete group in the class $\mathcal{I}$. Then there is a minimal subflow $Y \subset 2^G$ on which G acts freely. Thus every group in $\mathcal{I}$ is symbolically-free. In particular this holds for infinite countable residually finite groups.*

Proof. Let X be a 1-1, metrizable, zero-dimensional, topological group compactification of G (see Lemma 4.1 above).

Repeat the construction in the proof of Theorem 3.1, with the extra property that $\partial U_n = \emptyset$ for every n. This can be done as now X is zero dimensional.

Let $\mathcal{C}(F)$ be the set of continuity points of F. With this additional condition we have $\mathcal{C}(F) = X \setminus \{\xi\}$ and therefore $\mathcal{C}_0(F) = \bigcap\{g\mathcal{C}(F) : g \in G\} = X \setminus G\xi$ is a dense G_δ subset of X with a countable complement.

We can now give a full description of the points of Y. Set $f(g) = f_{x_0}(g) = y_0(g) = F(g^{-1}x_0)$, $(g \in G)$. Then for $h \in G$ we have $(hy_0)(g) = y_0(h^{-1}g) = F(g^{-1}hx_0)$ $(g \in G)$. Suppose now that $\lim_{i\to\infty} h_i x_0 = x$ with $x \in \mathcal{C}_0(F)$. We then have

$$\lim_{i\to\infty} (h_i y_0)(g) = \lim_{i\to\infty} y_0(h_i^{-1}g) = \lim_{i\to\infty} F(g^{-1}h_i x_0) = F(g^{-1}x).$$

Thus in $\{0,1\}^G$, $y_x = f_x = \lim_{i\to\infty} h_i y_0$ exists, with

$$y_x(g) = f_x(g) = F(g^{-1}x)$$

(so that $y_0 = y_{x_0}$). Moreover if $\lim_{i\to\infty} h_i x_0 = \xi$ and $\lim_{i\to\infty} h_i y_0 = y$ exists, then for $g \neq e$,

$$y(g) = \lim_{i\to\infty}(h_i y_0)(g) = \lim_{i\to\infty} y_0(h_i^{-1}g) = \lim_{i\to\infty} F(g^{-1}h_i x_0) = F(g^{-1}\xi),$$

and $y(e)$ is either 0 or 1 and accordingly we denote $y = y^0_\xi$ or $y = y^1_\xi$. It is now clear that, with the notation $gy^\epsilon_\xi = y^\epsilon_{g\xi}$,

$$Y = \{y_x : x \in X_0\} \cup \{y^\epsilon_{g\xi} : g \in G,\ \epsilon = 0,1\}.$$

On the dense G_δ, G-invariant subset $X_0 \subset X$ there is a continuous homomorphism $\phi : x \mapsto y_x$ from X_0 into $Y \subset \{0,1\}^G$ and (with notation as in the proof of Theorem 3.1)

$$Z = X \vee Y = \operatorname{cls}\{(x,\phi(x)) : x \in X_0\}.$$

Given $x_1 \neq x_2$ two points in X_0 we can find a sequence $g_n \in G$ such that $\lim_{n\to\infty} g_n x_1 = \xi$. Since for every $g \in G$, $d(gx_1, gx_2) = d(x_1, x_2)$, we can choose some $g \in G$ for which gx_1 is very close to ξ and

$$|F(gx_1) - F(gx_2)| = |f_{x_1}(g) - f_{x_2}(g)| = 1.$$

This shows that the map $\phi : X_0 \to Y$ is 1-1. It is now easy to check that the natural projection map $\pi_Y : Z \to Y$ is 1-1 (an isomorphism) and it follows that the map $\pi : Y \to X$ defined by $\pi = \pi_X \circ \pi_Y^{-1}$ (with $\pi_X : Z \to X$ denoting the projection on X) is a continuous homomorphism from (Y,G) onto (X,G). (Explicitly we have $\pi(y_x) = x = \phi^{-1}(y_x)$ for $x \in X_0$, and $\pi(y^\epsilon_{g\xi}) = g\xi$ on the complement of $Y_0 = \phi(X_0)$ in Y.)

Thus (Y,G), as an extension of a free action, is itself free and our proof is complete. □

5. Some further examples

5.1. **Theorem.** (1) *Every Abelian group is symbolically-free.*

(2) *Every residually finite group is symbolically-free.*

(3) *Let S^0_∞ be the group of permutations of $\mathbb{N}$ with finite support. Every infinite subgroup of S^0_∞ is symbolically-free.*

(4) *Every torsion free hyperbolic group is symbolically-free.*

Proof. 1. Apply Theorem 3.1 and then note that for G Abelian, an effective minimal flow is already free.

2. This is the statement of Theorem 4.2.

3. This follows from [6] (see also [8, Section 6.3]) where it is shown that S_∞ (the group of all permutations of $\mathbb{N}$) admits a minimal action (actually, the universal minimal flow of S_∞ as a Polish group) on which the subgroup S^0_∞ acts freely.

4. This is a result of Dranishnikov and Schroeder, [2].

□

5.2. *Remark.* A famous open problem is whether every hyperbolic group is residually finite (see e.g. [7]). An affirmative answer will provide, via Theorem 4.2, a proof that every hyperbolic group is symbolically-free, improving the Dranishnikov-Schroeder result.

6. A combinatorial characterization of symbolically-free groups

In the following theorem we use the pictorial term "a 2-coloring of a set X" to describe a partition of X into two disjoint subsets; each has its own "color". Thus elements x and y of X have "different colors" if they belong to different subsets of the partition.

6.1. **Theorem.** *Let G be an infinite countable group. The following conditions are equivalent.*

(1) *G acts freely on some subflow of 2^G.*
(2) *There exists a 2-coloring of G with the following property. For every $g \neq e$, there is a finite set $A = A(g)$ such that for every $h \in G$ there is an $a \in A \cap g^{-1}A$ such that ha and hga have different colors.*

Proof. Suppose first that $Y \subset \Omega_2 = \{0,1\}^G$ is a free subflow. Let ω be any point in Y. We consider ω as a coloring of G and will show that it has the property described in (2). If this is not the case, then there is some $g \in G \setminus \{e\}$ such that for every finite set $A \subset G$ there is some $h = h_A \in G$ with the property that $\omega(hga) = \omega(ha)$ for every $a \in A \cap g^{-1}A$. Let A_n be an increasing sequence of finite subsets of G with $G = \bigcup_{n=1}^{\infty} A_n$. For each n let $h_n = h_{A_n}$ be the corresponding element of G such that

$$h_n^{-1}\omega(ga) = \omega(h_n ga) = \omega(h_n a) = h_n^{-1}\omega(a) \quad \text{for every } a \in A_n \cap g^{-1}A_n.$$

Clearly also $G = \bigcup_{n=1}^{\infty}(A_n \cap g^{-1}A_n)$, and taking a convergent subsequence $h_{n_i}^{-1}\omega \to \xi \in Y$, we have

$$\xi(ga) = \xi(a) \quad \text{for every } a \in G.$$

Thus $g\xi = \xi$, contradicting our assumption that the flow (Y,G) is free.

Conversely, assume now that condition (2) is satisfied. Let $\omega : G \to \{0,1\}$ be the coloring whose existence is ensured by this condition and consider its orbit closure $Y = \mathrm{cls}\,(G\omega)$ in Ω_2. Let ξ be any point in Y and let g be an element of $G \setminus \{e\}$. There exists a sequence $h_n \in G$ with $\xi = \lim_{n\to\infty} h_n^{-1}\omega$. This means that for any finite $A \subset G$, eventually

$$h_n^{-1}\omega(a) = \omega(h_n a) = \xi(a) \quad \text{for every } a \in A. \tag{1}$$

In particular this holds for the finite set $A = A(g^{-1})$ given in condition (2) and we fix some $h = h_n$ for which equation (1) holds with respect to this A. By condition (2) then, there exists some $a \in A \cap g^{-1}A$ with $\omega(ha) \neq \omega(hga)$. But then

$$\xi(a) = h^{-1}\omega(a) = \omega(ha) \neq \omega(hga) = h^{-1}\omega(ga) = \xi(ga) = g^{-1}\xi(a).$$

Thus $g^{-1}\xi \neq \xi$ and we have shown that (Y,G) is a free flow. □

Call the condition (2) in Theorem 6.1 *property P*. With this terminology, each of the groups listed in Theorem 5.1 has property P. The main question now is whether there is any countably infinite group which does not satisfy property P. Stated explicitly we have the following.

6.2. **Problem.** Is there a countable infinite group G with the following coloring property?

> For every 2-coloring of G, there exists $g \in G \setminus \{e\}$ such that for every finite set $A \subset G$, there exists an $h \in G$ for which the pair ha and hga have the same color for every $a \in A \cap g^{-1}A$.

6.3. *Remark.* After this paper was accepted for publication, we learned that our main question (formulated as the coloring problem 6.2) had recently received a negative answer by Su Gao, Steve Jackson, and Brandon Seward. This paper, entitled “A coloring property for countable groups”, is available on Gao's homepage at `http://www.cas.unt.edu/~sgao/pub/paper31.html.`

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