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QUANTUM ISOMETRY GROUP OF THE n-TORI

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Abstract. We show that the quantum isometry group (introduced by Goswami) of the n-tori $\mathbb{T}^n$ coincides with its classical isometry group; i.e. there does not exist any faithful isometric action on $\mathbb{T}^n$ by a genuine (noncommutative as a C^*-algebra) compact quantum group. Moreover, using an earlier result, we conclude that the quantum isometry group of the noncommutative n tori is a Rieffel deformation of the quantum isometry group of the commutative n-tori.

1. Introduction

In [4], Goswami has defined the quantum isometry group of a noncommutative manifold (given by a spectral triple), motivated by the definition and study of quantum permutation groups of finite sets and finite graphs by a number of mathematicians (see, e.g., [1], [2], [6], and the references therein) and using some ideas of Woronowicz and Soltan (see [10]).

The main ingredient of this theory is the Laplacian $\mathcal{L}$ coming from the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ satisfying certain regularity conditions (see [4] for its construction), which coincides with the Hodge Laplacian $-d^*d$ (restricted on a space of smooth functions) in the classical case, where d denotes the de Rham differential.

The linear span of eigenvectors of $\mathcal{L}$, which is a subspace of $\mathcal{A}^\infty$, is denoted by $\mathcal{A}_0^\infty$, and it is assumed that $\mathcal{A}_0^\infty$ is norm-dense in the C^*-algebra $\mathcal{A}$ obtained by completing $\mathcal{A}^\infty$. The $*$-subalgebra of $\mathcal{A}^\infty$ generated by $\mathcal{A}_0^\infty$ is denoted by $\mathcal{A}_0$. Then $\mathcal{L}(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty$, and a compact quantum group $(\mathcal{G}, \Delta)$ which has an action α on $\mathcal{A}$ is said to act smoothly and isometrically on the noncommutative manifold $(\mathcal{A}^\infty, \mathcal{H}, D)$ if for every state ϕ on $\mathcal{G}$, $(\mathrm{id} \otimes \phi) \circ \alpha(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty$ for all states ϕ on $\mathcal{G}$, and also $(\mathrm{id} \otimes \phi) \circ \alpha$ commutes with $\mathcal{L}$ on $\mathcal{A}_0^\infty$.

One can then consider the category of all compact quantum groups acting smoothly and isometrically on $\mathcal{A}$, where the morphisms are quantum group morphisms which intertwine the actions on $\mathcal{A}$. It is proved in [4] (under some regularity assumptions, which are valid for any compact connected Riemannian spin manifold with the usual Dirac operator) that there exists a universal object in this category, and this universal object is defined to be the quantum isometry group of $(\mathcal{A}^\infty, \mathcal{H}, D)$, denoted by $QISO(\mathcal{A}^\infty, \mathcal{H}, D)$, or simply as $QISO(\mathcal{A}^\infty)$ or even $QISO(\mathcal{A})$ if the spectral triple is understood.

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It is important to explicitly describe quantum isometry groups of sufficiently many classical and noncommutative manifolds. In [5], the quantum isometry group of the two-torus has been explicitly computed.

Although there are similarities between the basic principle of the computation of [5] and that of the present article, which is to exploit the eigenspaces of the Laplacian, there is a key difference between the two proofs, which is as follows.

The computation of the quantum isometry group of $\mathcal{A}_\theta$ in Section 2.3 of [5] is valid for all θ; i.e. have all θ has been treated on the same footing. On the other hand, we have adopted a different strategy in the present article based on the observation that the proof of [5] can be simplified quite a bit for the special case of $\theta = 0$, i.e. the commutative case, and this simplification even goes through easily for any $n \geq 3$. Thus, in this article we first compute the quantum isometry group of the commutative C^*-algebra $C(\mathbb{T}^n)$. This commutativity assumption decreases a lot of computations because it suffices to show that the quantum isometry group is a commutative C^*-algebra and hence has to coincide with the classical isometry group. Then the structure of the quantum isometry group of $\mathbb{T}^n_\theta$ for an arbitrary value of θ follows immediately from Theorem 3.13 of [5].

Throughout the paper, we have denoted by $\mathcal{A}_1 \otimes \mathcal{A}_2$ the minimal (injective) C^*-tensor product between two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. The symbol $\otimes_{\text{alg}}$ has been used to denote the algebraic tensor product between vector spaces or algebras.

For a compact quantum group $\mathcal{G}$, let the dense unital $*$-subalgebra generated by the matrix coefficients of irreducible unitary representations be denoted by $\mathcal{G}_0$. The coproduct of $\mathcal{G}$, say Δ, maps $\mathcal{G}_0$ into the algebraic tensor product $\mathcal{G}_0 \otimes_{\text{alg}} \mathcal{G}_0$, and there exist a canonical antipode and a counit defined on $\mathcal{G}_0$ which make it into a Hopf $*$-algebra (see [7] for the details).

Remark 1.1. In [4], it was assumed that the compact quantum groups are separable. But in [9], the separability assumption for the C^*-algebra of the underlying compact quantum group was removed. It can be easily seen that the separability was not at all used in the proofs of [4], and hence all the results in [4] proceed verbatim in the nonseparable case.

1.1. **Quantum isometry group of the commutative n-tori.** Consider $C(\mathbb{T}^n)$ as the universal commutative C^*-algebra generated by n commuting unitaries $U_1, U_2, \ldots, U_n$. It is clear that the set $\{U_i^m U_j^n : m, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(C(\mathbb{T}^n), \tau)$, where τ denotes the unique faithful normalized trace on $C(\mathbb{T}^n)$ given by $\tau(\sum a_{mn} U_i^m U_j^n) = a_{00}$, which is just the integration against the Haar measure. We shall denote by $\langle A, B \rangle = \tau(A^* B)$ the inner product on $\mathcal{H}_0 := L^2(C(\mathbb{T}^n), \tau)$. Let $C(\mathbb{T}^n)^{\text{fin}}$ be the unital $*$-subalgebra generated by finite complex linear combinations of $U^m V^n$, $m, n \in \mathbb{Z}$. The Laplacian $\mathcal{L}$ is given by $\mathcal{L}(U_1^{m_1}, \ldots, U_n^{m_n}) = -(m_1^2 + \ldots m_n^2) U_1^{m_1}, \ldots, U_n^{m_n}$, and it is also easy to see that the algebraic span of eigenvectors of $\mathcal{L}$ is nothing but the space $C(\mathbb{T}^n)^{\text{fin}}$ and, moreover, all the assumptions in [4] required for defining the quantum isometry group are satisfied.

Let $\mathcal{Q}$ be the quantum isometry group coming from the above Laplacian, with the smooth isometric action of $\mathcal{Q}$ on $C(\mathbb{T}^n)$ given by $\alpha : C(\mathbb{T}^n) \to C(\mathbb{T}^n) \otimes \mathcal{Q}$. By definition, α must keep invariant the eigenspace of $\mathcal{L}$ corresponding to the eigenvalue

-1 , spanned by $U_1, \ldots, U_n, U_1^{-1}, \ldots, U_n^{-1}$. Thus, the action α is given by

$$\alpha(U_i) = \sum_{j=1}^{n} U_j \otimes A_{ij} + \sum_{j=1}^{n} U_j^{-1} \otimes B_{ij},$$

where $A_{ij}, B_{ij} \in \mathcal{Q}, i, j = 1, 2, \ldots, n$. By faithfulness of the action of the quantum isometry group (see [4]), the norm-closure of the unital $*$-algebra generated by $A_{ij}, B_{ij}; i, j = 1, 2, \ldots, n$ must be the whole of $\mathcal{Q}$.

Next we derive a number of conditions on $A_{ij}, B_{ij}, i, j = 1, 2, ...n$ using the fact that α is a $*$-homomorphism.

Lemma 1.2. *The condition $U^*U = 1 = UU^*$ gives:*

$$\sum_j (A_{ij}^* A_{ij} + B_{ij}^* B_{ij}) = 1 \tag{1}$$

$$B_{ij}^* A_{ik} + B_{ik}^* A_{ij} = 0 \ \forall j \neq k \tag{2}$$

$$A_{ij}^* B_{ik} + A_{ik}^* B_{ij} = 0 \ \forall j \neq k \tag{3}$$

$$A_{ij}^* B_{ij} = B_{ij}^* A_{ij} = 0 \tag{4}$$

$$\sum_j (A_{ij} A_{ij}^* + B_{ij} B_{ij}^*) = 1 \tag{5}$$

$$A_{ik} B_{ij}^* + A_{ij} B_{ik}^* = 0 \ \forall j \neq k \tag{6}$$

$$B_{ik} A_{ij}^* + B_{ij} A_{ik}^* = 0 \ \forall j \neq k \tag{7}$$

$$A_{ij} B_{ij}^* = B_{ij} A_{ij}^* = 0. \tag{8}$$

Proof. We get (1) - (4) by using the condition $U_i^* U_i = 1$ along with the fact that α is a homomorphism and then comparing the coefficients of $1, U_j U_k, U_j^{-1} U_k^{-1}$ (for $j \neq k$), U_j^{-2}, U_k^{-2}.

Similarly the condition $U_i U_i^* = 1$ gives (5) - (8). $\square$

Now, $\forall i \neq j, U_i^* U_j, U_i U_j^*$ and $U_i U_j$ belong to the eigenspace of the Laplacian with eigenvalue -2, while U_k^2, U_k^{-2} belong to the eigenspace corresponding to the eigenvalue -4. As α preserves the eigenspaces of the Laplacian, the coefficients of U_k^2, U_k^{-2} are zero $\forall k$ in $\alpha(U_i^* U_j), \alpha(U_i U_j^*), \alpha(U_i U_j)$ $\forall i \neq j$.

We use this observation in the next lemma.

Lemma 1.3. $\forall k$ *and* $\forall i \neq j$,

$$B_{ik}^* A_{jk} = A_{ik}^* B_{jk} = 0 \tag{9}$$

$$A_{ik} B_{jk} = B_{ik} A_{jk}^* = 0 \tag{10}$$

$$A_{ik} A_{jk} = B_{ik} B_{jk} = 0. \tag{11}$$

Proof. The equation (9) is obtained from the coefficients of U_k^2 and U_k^{-2} in $\alpha(U_i^* U_j)$, while (10) and (11) are obtained from the same coefficients in $\alpha(U_i U_j^*)$ and $\alpha(U_i U_j)$, respectively. $\square$

Now, by Lemma 2.12 in [4] it follows that $\tilde{\alpha} : C(\mathbb{T}^n) \otimes \mathcal{Q} \to C(\mathbb{T}^n) \otimes \mathcal{Q}$ defined by $\tilde{\alpha}(X \otimes Y) = \alpha(X)(1 \otimes Y)$ extends to a unitary of the Hilbert $\mathcal{Q}$-module $L^2(C(\mathbb{T}^n), \tau) \otimes \mathcal{Q}$ (or in other words, α extends to a unitary representation of $\mathcal{Q}$ on $L^2(C(\mathbb{T}^n), \tau)$). But α keeps $W = \mathrm{Sp}\{U_i, U_i^* : 1 \leq i \leq n\}$ invariant. So α is a unitary representation of $\mathcal{Q}$ on W. Hence, the matrix (say M) corresponding to the $2n$-dimensional representation of $\mathcal{Q}$ on W is a unitary in $M_{2n}(\mathcal{Q})$.

From the definition of the action it follows that $M = \begin{pmatrix} A_{ij} & B_{ij}^* \\ B_{ij} & A_{ij}^* \end{pmatrix}$.

Since M is the matrix corresponding to a finite dimensional unitary representation, $\kappa(M_{kl}) = M_{kl}^{-1}$, where κ denotes the antipode of $\mathcal{Q}$ (see [8]).

But M is a unitary, $M^{-1} = M^*$. So, $(k(M_{kl})) = \begin{pmatrix} A_{ji}^* & B_{ji}^* \\ B_{ji} & A_{ji} \end{pmatrix}$.

Now we apply the antipode κ to get some more relations.

Lemma 1.4. *$\forall k$ and $i \neq j$,*

$$A_{kj}^* A_{ki}^* = B_{kj} B_{ki} = A_{kj}^* B_{ki}^* = B_{kj} A_{ki} = B_{kj} A_{ki}^* = A_{kj} B_{ki} = 0. \tag{12}$$

Proof. The result follows by applying κ on the equations $A_{ik} A_{jk} = B_{ik} B_{jk} = B_{ik}^* A_{jk} = A_{ik}^* B_{jk} = A_{ik} B_{jk} = B_{ik} A_{jk}^* = 0$ obtained from Lemma 1.3. □

Lemma 1.5. *A_{li} is a normal partial isometry $\forall\, l, i$ and hence has the same domain and range.*

Proof. From the relation (1) in Lemma 1.2, we have by applying κ, $\sum(A_{ji}^* A_{ji} + B_{ji} B_{ji}^*) = 1$. Applying A_{li} on the right of this equation, we have

$$A_{li}^* A_{li} A_{li} + \sum_{j \neq l} (A_{ji}^* A_{ji} A_{li} + B_{li} B_{li}^* A_{li}) + \sum_{j \neq l} B_{ji} B_{ji}^* A_{li} = A_{li}.$$

From Lemma 1.3, we have $A_{ji} A_{li} = 0$ and $B_{ji}^* A_{li} = 0\ \forall j \neq l$. Moreover, from Lemma 1.2, we have $B_{li}^* A_{li} = 0$. Applying these to the above equation, we have

$$A_{li}^* A_{li} A_{li} = A_{li}. \tag{13}$$

Again, from the relation $\sum_j (A_{ij} A_{ij}^* + B_{ij} B_{ij}^*) = 1\ \forall i$ in Lemma 1.2, applying κ and multiplying by A_{li}^* on the right, we have $A_{li} A_{li}^* A_{li}^* + \sum_{j \neq l} A_{ji} A_{ji}^* A_{li}^* + B_{li}^* B_{li} A_{li}^* + \sum_{j \neq l} B_{ji}^* B_{ji} A_{li}^* = A_{li}^*$. From Lemma 1.3, we have $A_{li} A_{ji} = 0\ \forall\ j \neq l$ (hence $A_{ji}^* A_{li}^* = 0$) and $B_{ji} A_{li}^* = 0$. Moreover, we have $B_{li} A_{li}^* = 0$ from Lemma 1.2. Hence, we have

$$A_{li} A_{li}^* A_{li}^* = A_{li}^*. \tag{14}$$

From (13), we have

$$(A_{li}^* A_{li})(A_{li} A_{li}^*) = A_{li} A_{li}^*. \tag{15}$$

By taking $*$ on (14), we have

$$A_{li} A_{li} A_{li}^* = A_{li}. \tag{16}$$

Using this in (15), we have

$$A_{li} A_{li}^* A_{li} = A_{li} A_{li}^*, \tag{17}$$

and hence A_{li} is normal.

So, $A_{li} = A_{li}^* A_{li} A_{li}$ (from (13)) $= A_{li} A_{li}^* A_{li}$.

Therefore, A_{li} is a partial isometry which is normal and hence has the same domain and range. □

Lemma 1.6. *B_{li} is a normal partial isometry and hence has the same domain and range.*

Proof. First we note that A_{ji} is a normal partial isometry and that $A_{ji}B_{li} = 0$ $\forall\ j \neq l$ (obtained from Lemma 1.3) implies that $Ran(A_{ji}^*) \subseteq Ker(B_{li}^*)$ and hence $Ran(A_{ji}) \subseteq Ker(B_{li}^*)$, which means that $B_{li}^* A_{ji} = 0\ \forall j \neq l$.

To obtain $B_{li}^* B_{li} B_{li} = B_{li}$, we apply κ and multiply by B_{li} on the right of (5) and then use $A_{li}^* B_{li} = 0$ from Lemma 1.2, $A_{ji}B_{li} = 0\ \forall\ j \neq l$ (from Lemma 1.3, which implies $B_{li}^* A_{ji} = 0\ \forall j \neq l$ as above) and $B_{ji}B_{li} = 0\ \forall j \neq l$ from Lemma 1.3 .

Similarly, we have $B_{li}B_{li}^* B_{li}^* = B_{li}^*$ by applying κ and multiplying by B_{li}^* on the right of (1) obtained from Lemma 1.2 and then using $A_{li}B_{li}^* = 0$ (Lemma 1.2), $B_{li}A_{ji}^* = 0\ \forall\ j \neq l$, and $B_{li}B_{ji} = 0\ \forall j \neq l$ (Lemma 1.3).

Using $B_{li}^* B_{li} B_{li} = B_{li}$ and $B_{li}B_{li}^* B_{li}^* = B_{li}^*$ as in Lemma 1.5, we have that B_{li} is a normal partial isometry. □

Now, we use the condition $\alpha(U_i)\alpha(U_j) = \alpha(U_j)\alpha(U_i) \forall i, j$.

Lemma 1.7. $\forall k \neq l$,

$$A_{ik}A_{jl} + A_{il}A_{jk} = A_{jl}A_{ik} + A_{jk}A_{il} \tag{18}$$

$$A_{ik}B_{jl} + B_{il}A_{jk} = B_{jl}A_{ik} + A_{jk}B_{il} \tag{19}$$

$$B_{ik}A_{jl} + A_{il}B_{jk} = A_{jl}B_{ik} + B_{jk}A_{il} \tag{20}$$

$$B_{ik}B_{jl} + B_{il}B_{jk} = B_{jl}B_{ik} + B_{jk}B_{il}. \tag{21}$$

Proof. The result follows by equating the coefficients of $U_kU_l, U_kU_l^{-1}, U_k^{-1}U_l$ and $U_k^{-1}U_l^{-1}$ (where $k \neq l$) in $\alpha(U_i)\alpha(U_j) = \alpha(U_j)\alpha(U_i) \forall i, j$. □

Lemma 1.8. $A_{ik}B_{jl} = B_{jl}A_{ik} \forall i \neq j, k \neq l$.

Proof. From Lemma 1.7, we have $\forall k \neq l, A_{ik}B_{jl} - B_{jl}A_{ik} = A_{jk}B_{il} - B_{il}A_{jk}$. We consider the case where $i \neq j$.

We have $Ran(A_{ik}B_{jl} - B_{jl}A_{ik}) \subseteq Ran(A_{ik}) + Ran(B_{jl}) \subseteq Ran(B_{jl}^* B_{jl} + A_{ik}^* A_{ik})$ (using the facts that A_{ik} and B_{jl} are normal partial isometries by Lemma 1.5 and 1.6 and also that $B_{jl}^* B_{jl}$ and $A_{ik}^* A_{ik}$ are projections).

Similarly, $Ran(A_{jk}B_{il} - B_{il}A_{jk}) \subseteq Ran(B_{il}^* B_{il} + A_{jk}^* A_{jk})$.

Let

$$\begin{aligned} T_1 &= A_{ik}B_{jl} - B_{jl}A_{ik} \\ T_2 &= A_{jk}B_{il} - B_{il}A_{jk} \\ T_3 &= B_{jl}^* B_{jl} + A_{ik}^* A_{ik} \\ T_4 &= B_{il}^* B_{il} + A_{jk}^* A_{jk}. \end{aligned}$$

Hence, $T_1 = T_2, RanT_1 \subseteq RanT_3\ , RanT_2 \subseteq RanT_4$.

We claim that $T_4T_3 = 0$. Then $Ran(T_3) \subseteq Ker(T_4)$. But $RanT_1 \subseteq RanT_3$ will imply that $RanT_1 \subseteq KerT_4$. Hence, $Ran(T_2) \subseteq Ker(T_4) = \overline{Ran(T_4^*)}^{\perp} = \overline{Ran(T_4)}^{\perp}$. But $Ran(T_2) \subseteq Ran(T_4)$, which implies that $Ran(T_2) = 0$ and hence that both T_2 and T_1 are zero. Thus, the proof of the lemma will be complete if we can prove the claim

$$\begin{aligned} T_4T_3 &= (B_{il}^* B_{il} + A_{jk}^* A_{jk})(B_{jl}^* B_{jl} + A_{ik}^* A_{ik}) \\ &= B_{il}^* B_{il} B_{jl}^* B_{jl} + B_{il}^* B_{il} A_{ik}^* A_{ik} + A_{jk}^* A_{jk} B_{jl}^* B_{jl} + A_{jk}^* A_{jk} A_{ik}^* A_{ik}. \end{aligned}$$

From Lemma 1.3, we have $\forall i \neq j, B_{il}B_{jl} = 0$, implying $B_{il}B_{jl}^* = 0$ as B_{jl} is a normal partial isometry.

Again, from Lemma 1.4 $\forall k \neq l, B_{il}A_{ik} = 0$. Then A_{ik} is a normal partial isometry implies that $B_{il}A_{ik}^* = 0 \ \forall \ k \neq l$.

Similarly, by taking the adjoint of the relation $B_{jl}A_{jk}^* = 0 \ \forall k \neq l$ obtained from Lemma 1.4, we have $A_{jk}B_{jl}^* = 0$.

From Lemma 1.3, we have $A_{jk}A_{ik} = 0 \ \forall \ i \neq j$. A_{ik} is a normal partial isometry implies that $A_{jk}A_{ik}^* = 0 \ \forall i \neq j$.

Using these, we note that $T_4T_3 = 0$, which proves the claim and hence the lemma. □

Lemma 1.9.

$$A_{ik}B_{jk} = 0 = B_{jk}A_{ik}$$
$$A_{ki}B_{kj} = 0 = B_{kj}A_{ki}$$

$\forall i \neq j$ *and* $\forall k$.

Proof. By Lemma 1.3, we have $A_{ik}B_{jk} = 0$ and $B_{jk}A_{ik}^* = 0 \ \forall i \neq j$. The second relation along with the fact that A_{ik} is a normal partial isometry implies that $B_{jk}Aik = 0 \ \forall i \neq j$.

Thus, $A_{ik}B_{jk} = 0 = B_{jk}A_{ik} \ \forall i \neq j$.

Applying κ to the above equation and using that B_{kj} and A_{ki} are normal partial isometries, we have $A_{ki}B_{kj} = 0 = B_{kj}A_{ki}$. □

Lemma 1.10. $A_{ik}B_{ik} = B_{ik}A_{ik} \ \forall i, k$.

Proof. We have $A_{ij}^*B_{ij} = 0 = B_{ij}^*A_{ij}$ from Lemma 1.2. Using the fact that B_{ij} and A_{ij} are normal partial isometries we have $A_{ij}^*B_{ij}^* = 0 = B_{ij}^*A_{ij}^*$ and hence $A_{ij}B_{ij} = B_{ij}A_{ij}$. □

Lemma 1.11. $A_{ik}A_{jl} = A_{jl}A_{ik} \ \forall i \neq j, k \neq l$.

Proof. Using (18) in Lemma 1.7, we proceed as in Lemma 1.8 to get $Ran(A_{ik}A_{jl} - A_{jl}A_{ik}) \subseteq Ran(A_{jl}A_{jl}^* + A_{ik}A_{ik}^*)$ and $Ran(A_{jk}A_{il} - A_{il}A_{jk}) \subseteq Ran(A_{il}A_{il}^* + A_{jk}A_{jk}^*)$.

We claim that $(A_{ik}A_{ik}^* + A_{jl}A_{jl}^*)(A_{jk}A_{jk}^* + A_{il}A_{il}^*) = 0$.

Then by the same reasoning as given in Lemma 1.8 we will have $A_{jk}A_{il} = A_{il}A_{jk}$.

To prove the claim, we use $A_{ik}A_{jk} = 0 \ \forall \ i \neq j$ from Lemma 1.3 (which implies $A_{jk}^*A_{ik} = 0 \ \forall \ i \neq j$ as A_{ik} is a normal partial isometry), $A_{il}^*A_{ik}^* = 0 \ \forall \ k \neq l$ from Lemma 1.4 (which implies $A_{il}^*A_{ik} = 0 \ \forall k \neq l$ as A_{ik} is a normal partial isometry) and $A_{il}A_{jl} = 0 \ \forall i \neq j$ from Lemma 1.3 (which implies $A_{jl}^*A_{il} = 0 \ \forall i \neq j$ as A_{il}^* is a normal partial isometry). □

Lemma 1.12.

$$A_{ik}A_{il} = A_{il}A_{ik} \forall k \neq l$$
$$A_{ik}A_{jk} = A_{jk}A_{ik} \forall i \neq j.$$

Proof. From Lemma 1.3, we have $A_{ki}A_{li} = 0 \ \forall k \neq l$.

Applying κ and taking the adjoint, we have $A_{ik}A_{il} = 0 \ \forall k \neq l$. Interchanging k and l, we get $A_{il}A_{ik} = 0 \ \forall k \neq l$. Hence, $A_{ik}A_{il} = A_{il}A_{ik} \forall k \neq l$.

From Lemma 1.3, we have $A_{ik}A_{jk} = 0 \ \forall \ i \neq j$. Interchanging i and j, we have $A_{jk}A_{ik} = 0 \ \forall i \neq j$. □

Remark 1.13. Proceeding in a similar way, we have that the B_{ij}'s commute among themselves.

Theorem 1.14. *The quantum isometry group of $\mathbb{T}^n$ is commutative as a C^*-algebra and hence coincides with the classical isometry group.*

Proof. The proof follows from the results in Lemmas 1.8 - 1.12 and the remark following them. □

Corollary 1.15. *Using Theorem* 3.13 *of* [5]*, we conclude that the quantum isometry group of the noncommutative n-tori $\mathbb{T}^n_\theta$ is a Rieffel deformation of the quantum isometry group of $\mathbb{T}^n$.*

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