PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 9, September 2009, Pages 3115–3121
S 0002-9939(09)09919-5
Article electronically published on April 20, 2009

ULTRAFILTERS WITH PROPERTY (s)

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(Communicated by Alan Dow)

ABSTRACT. A set $X \subseteq 2^\omega$ has property (s) (Marczewski (Szpilrajn)) iff for every perfect set $P \subseteq 2^\omega$ there exists a perfect set $Q \subseteq P$ such that $Q \subseteq X$ or $Q \cap X = \emptyset$. Suppose $\mathcal{U}$ is a nonprincipal ultrafilter on ω. It is not difficult to see that if $\mathcal{U}$ is preserved by Sacks forcing, i.e., if it generates an ultrafilter in the generic extension after forcing with the partial order of perfect sets, then $\mathcal{U}$ has property (s) in the ground model. It is known that selective ultrafilters or even P-points are preserved by Sacks forcing. On the other hand (answering a question raised by Hrusak) we show that assuming CH (or more generally MA) there exists an ultrafilter $\mathcal{U}$ with property (s) such that $\mathcal{U}$ does not generate an ultrafilter in any extension which adds a new subset of ω.

It is a well known classical result due to Sierpinski (see [1]) that a nonprincipal ultrafilter $\mathcal{U}$ on ω when considered as a subset of $P(\omega) = 2^\omega$ cannot have the property of Baire or be Lebesgue measurable. Here we identify 2^ω and $P(\omega)$ by identifying a subset of ω with its characteristic function. Another very weak regularity property is property (s) of Marczewski (see Miller [7]). A set of reals $X \subseteq 2^\omega$ has property (s) iff for every perfect set P there exists a perfect subset $Q \subseteq P$ such that either $Q \subseteq X$ or $Q \cap X = \emptyset$. Here by perfect we mean homeomorphic to 2^ω.

It is natural to ask:

Question. (Steprans) Can a nonprincipal ultrafilter $\mathcal{U}$ have property (s)?

If $\mathcal{U}$ is an ultrafilter in a model of set theory V and $W \supseteq V$ is another model of set theory, then we say $\mathcal{U}$ generates an ultrafilter in W if for every $z \in P(\omega) \cap W$ there exists $x \in \mathcal{U}$ with $x \subseteq z$ or $x \cap z = \emptyset$. This means that the filter generated by $\mathcal{U}$ (i.e., closing under supersets) is an ultrafilter in W.

We begin with the following result:

Theorem 1. *For $\mathcal{U}$ a nonprincipal ultrafilter on ω in V the following are equivalent:*

(1) *For some Sack's generic real x over V,*
$$V[x] \models \mathcal{U} \text{ generates an ultrafilter.}$$
(2) *In V, for every perfect set $P \subseteq P(\omega)$ there exists a perfect set $Q \subseteq P$ and a $z \in \mathcal{U}$ such that either $\forall x \in Q\ \ z \subseteq x$ or $\forall x \in Q\ \ z \cap x = \emptyset$.*

Received by the editors October 27, 2003, and, in revised form, January 15, 2004.
2000 *Mathematics Subject Classification.* Primary 03E35, 03E17, 03E50.
Thanks to the Fields Institute, Toronto, for their support during the time these results were proved and to Juris Steprans for helpful conversations, and thanks to Boise State University for support during the time this paper was written.

(3) *For some extension $W \supseteq V$ with a new subset of ω,*

$$W \models \mathcal{U} \text{ generates an ultrafilter.}$$

Proof. To see that (3) $\to$ (2), let P be any perfect set coded in V. Since W contains a new subset of ω there exists $x \in (P \cap W) \setminus V$.

Since $\mathcal{U}$ generates an ultrafilter in W there exists $z \in \mathcal{U}$ so that either $z \subseteq x$ or $z \cap x = \emptyset$. Suppose the first happens. In V consider the set

$$Q = \{y \in P : z \subseteq y\}.$$

Note that the new real x is in the closed set Q. It follows that Q must be an uncountable closed set and so it contains a perfect subset. The other case is exactly the same.

One way to see that Q must be uncountable is to note that if (in V) $Q = \{x_n : n < \omega\}$, then the $\mathbf{\Pi}^1_1$ sentence

$$\forall x \in 2^\omega (x \in Q \text{ iff } \exists n < \omega \; x = x_n)$$

would be true in V; and since $\mathbf{\Pi}^1_1$ sentences are absolute (Mostowski absoluteness; see [5]), true in W. Another way to prove it is to apply the standard derivative Cantor argument to the closed set Q, removing isolated points and iterating through the transfinite and noting that each real removed is in V, while the new real is never removed, and hence the kernel of Q is perfect. See Solovay [10] for a similar proof of Mansfield's theorem that a (lightface) Π^1_2 set with a nonconstructible element contains a perfect set.

Now we see that (2) $\to$ (1). A basic property of Sacks forcing is that every $y \in 2^\omega \cap M[x]$ in a Sacks extension is either in M or is itself Sacks generic over M (see Sacks [9]). Hence we need only show that if $y \subseteq \omega$ is Sacks generic over M, then there exists $z \in \mathcal{U}$ with $z \subseteq y$ or $z \cap y = \emptyset$. Recall also that the Sacks real y satisfies that the generic filter G is exactly the set of all perfect sets Q coded in V with $y \in Q$.

Condition (2) says that the set of such Q is dense, and hence there exists Q in the generic filter determined by y and $z \in \mathcal{U}$ such that either $z \subseteq u$ for every $u \in Q$ or $z \cap u = \emptyset$ for every $u \in Q$. But this means that either $z \subseteq y$ or $z \cap y = \emptyset$.

(1) $\to$ (3) is obvious. □

Remark. The above proof also shows that if an ultrafilter is preserved in one Sacks extension, then it is preserved in all Sacks extensions.

Remark. In Baumgartner and Laver [2] it is shown that selective ultrafilters are preserved by Sacks forcing. In Miller [6] it is shown that P-points are preserved by superperfect set forcing (and hence by Sacks forcing also).

We say that an ultrafilter $\mathcal{U}$ is preserved by Sacks forcing iff for some (equivalently all) Sacks generic real x, $\mathcal{U}$ generates an ultrafilter in $V[x]$. Recall that $\mathcal{U} \times \mathcal{V}$ is the ultrafilter on $\omega \times \omega$ defined by

$$A \in \mathcal{U} \times \mathcal{V} \text{ iff } \{n : \{m : (n,m) \in A\} \in \mathcal{U}\} \in \mathcal{V}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters, then $\mathcal{U} \times \mathcal{V}$ is not a P-point. Also recall that $\mathcal{U} \leq_{RK} \mathcal{V}$ (Rudin-Keisler) iff there exists $f \in \omega^\omega$ such that for every $X \subseteq \omega$,

$$X \in \mathcal{U} \text{ iff } f^{-1}(X) \in \mathcal{V}.$$

Proposition 2. *If $\mathcal{U}$ and $\mathcal{V}$ are preserved by Sacks forcing, then so is $\mathcal{U}\times\mathcal{V}$. If $\mathcal{U}\leq_{RK}\mathcal{V}$ and $\mathcal{V}$ is preserved by Sacks forcing, then so is $\mathcal{U}$.*

Proof. Suppose $A\subseteq\omega\times\omega$ and $A\in V[x]$. For each $n<\omega$, let $A_n=\{m:(n,m)\in A\}$. Since $\mathcal{U}$ is preserved, there exists $B_n\in\mathcal{U}$ with $B_n\subseteq A_n$ or $B_n\cap A_n=\emptyset$. By the preservation of $\mathcal{V}$ there exists $C\in\mathcal{V}$ such that either $B_n\subseteq A_n$ for all $n\in C$ or $B_n\cap A_n=\emptyset$ for all $n\in C$. By the Sacks property there exists $(\mathfrak{b}_n\in[\mathcal{U}]^{2^n}:n<\omega)\in V$ such that $B_n\in\mathfrak{b}_n$ for every n. Let $B_n^0=\bigcap\mathfrak{b}_n$. Then

$$\bigcup_{n\in C}\{n\}\times B_n^0\subseteq A \text{ or } \bigcup_{n\in C}\{n\}\times B_n^0\cap A=\emptyset.$$

Suppose $\mathcal{U}\leq_{RK}\mathcal{V}$ via f. If $A\subseteq\omega$, then since $\mathcal{V}$ is preserved, there exists $B\in\mathcal{V}$ such that either $B\subseteq f^{-1}(A)$ or $B\subseteq f^{-1}(\overline{A})$, where $\overline{A}$ is the complement of A. But then $f(B)\subseteq A$ or $f(B)\subseteq\overline{A}$, and since $f(B)\in\mathcal{U}$ we are done. □

Remark. The Rudin-Keisler result is generally true, but the product result depends on the bounding property. For example, if $\mathcal{U}$ is a P-point, then $\mathcal{U}$ is preserved in the superperfect extension (see Miller [6]), but $\mathcal{U}\times\mathcal{U}$ is not.

It is clear that property (2) of Theorem 1 implies that any ultrafilter which is preserved by Sacks forcing has property (s). But what about the converse? The main result of this paper is that the reverse implication is false. This answers a question raised by Hrusak.

Theorem 3. *Suppose the CH is true or even just that the real line cannot be covered by fewer than continuum many meager sets. Then there exists an ultrafilter $\mathcal{U}$ on ω which has property (s) but is not preserved by Sacks forcing.*

Proof. We give the proof in the case of the continuum hypothesis and indicate how to do it under the more general hypothesis.

Let $\mathcal{I}\subseteq[\omega]^\omega$ be an independent perfect family. Independent means that for every m,n and distinct $x_1,\ldots,x_m,y_1,\ldots,y_n\in\mathcal{I}$ the set

$$x_1\cap\ldots\cap x_m\cap\overline{y}_1\cap\ldots\cap\overline{y}_n \text{ is infinite,}$$

where $\overline{y}$ means the complement of y in ω. The usual construction (probably due to Hausdorff) is the following. Let $Q=\{(F,s)\ :\ F\subseteq P(s),\ s\in[\omega]^{<\omega}\}$. Given any $A\subseteq\omega$ define $x_A=\{(F,s)\in Q\ :\ A\cap x\in F\}$. Then $\{x_A\ :\ A\subseteq\omega\}$ is a perfect independent family in the Cantor space $P(Q)$.

We claim that the family

$$\mathcal{I}\cup\{\overline{z}:\ \exists^\infty\ x\in\mathcal{I}\ \ z\subseteq^* x\}$$

has the finite intersection property. ($\exists^\infty$ means there exist infinitely many.) To see this suppose that we are given $x_1,\ldots,x_m\in\mathcal{I}$ and $z_1,\ldots,z_n$ such that $\exists^\infty\ x\in\mathcal{I}\ \ z_i\subseteq^* x$ for each i. Then we can choose $y_i\in\mathcal{I}$ distinct from each other and the x's so that each $z_i\subseteq^* y_i$. But since $\overline{y}_i\subseteq\overline{z}_i$ we have that

$$x_1\cap\ldots\cap x_m\cap\overline{y}_1\cap\ldots\cap\overline{y}_n\subseteq^* x_1\cap\ldots\cap x_m\cap\overline{z}_1\cap\ldots\cap\overline{z}_n.$$

By independence the set on the left is infinite and hence so is the set on the right. Thus this family has the finite intersection property.

Now let $\mathcal{F}_0$ be the filter generated by $\mathcal{I}\cup\{\overline{z}:\exists^\infty\ x\in\mathcal{I}\ \ z\subseteq^* x\}$.

Note that if $\mathcal{U}\supseteq\mathcal{F}_0$ is any ultrafilter, then it cannot be preserved by Sacks forcing. This is because $\mathcal{I}$ is a perfect subset of $\mathcal{U}$; however, there is no $z\in\mathcal{U}$ with

$z \subseteq x$ for all $x \in \mathcal{I}$ or even infinitely many $x \in \mathcal{I}$ or else $\overline{z} \in \mathcal{F}_0 \subseteq \mathcal{U}$. Hence Theorem 1 (2) fails.

Note that since $\mathcal{I}$ was perfect, the filter $\mathcal{F}_0$ is a $\mathbf{\Sigma}^1_1$ subset of $P(\omega)$.

Lemma 4. *Suppose that $P \subseteq [\omega]^\omega$ is a perfect set and $\mathcal{F}$ is a $\mathbf{\Sigma}^1_1$ filter extending the cofinite filter on ω. Then there exists a perfect $Q \subseteq P$ such that either*

(1) *$\mathcal{F} \cup Q$ has the finite intersection property or*
(2) *there exists $z \subseteq \omega$ such that $\mathcal{F} \cup \{z\}$ has the finite intersection property and for every $x \in Q$ we have that there exists $y \in \mathcal{F}$ with $x \cap y \cap z = \emptyset$.*

Proof. The strategy is to try to do a fusion argument to get case (1). If it ever fails, then stop and get case (2).

Claim. Suppose $(Q_i : i < n)$ are disjoint perfect subsets of $[\omega]^\omega$. Then either there exists $(Q'_i \subseteq Q_i : i < n)$ perfect such that for every $(x_i \in Q'_i : i < n)$ and $y \in \mathcal{F}$ we have that

$$|y \cap x_0 \cap x_1 \cap \ldots \cap x_{n-1}| = \omega$$

or there exists $z \subseteq \omega$, $k < n$, and $Q \subseteq Q_k$ perfect such that $\mathcal{F} \cup \{z\}$ has the finite intersection property and for every $x \in Q$ we have that there exists $y \in \mathcal{F}$ with $x \cap y \cap z = \emptyset$.

Proof. Consider

$$A_k = \{(x_i \in [\omega]^\omega : i < k) : \exists y \in \mathcal{F} \ \ |y \cap x_0 \cap x_1 \cap \ldots x_{k-1}| < \omega\}.$$

Since $\mathcal{F}$ is $\mathbf{\Sigma}^1_1$ it is easy to see that each A_k is a $\mathbf{\Sigma}^1_1$ set and hence has the property of Baire relative to the product $\prod_{i<k} Q_i$. By Mycielski [8] (see also Blass [3]) there exist perfect sets $(Q^*_i \subseteq Q_i : i < n)$ so that for every $k \leq n$ either

$$\prod_{i<k} Q^*_i \cap A_k = \emptyset \text{ or } \prod_{i<k} Q^*_i \subseteq A_k.$$

If the first case happens for $k = n$, then we let $Q'_i = Q^*_i$, and the claim is proved. If the second case happens, choose k minimal for which it happens. This means we have that

(1) for all $(x_i : i < k-1) \in \prod_{i<k-1} Q^*_i$ and $y \in \mathcal{F}$ we have
$$|y \cap x_0 \cap x_1 \cap \ldots \cap x_{k-2}| = \omega$$
and
(2) for all $(x_i : i < k) \in \prod_{i<k} Q^*_i$ there exists $y \in \mathcal{F}$ such that
$$|y \cap x_0 \cap x_1 \cap \ldots \cap x_{k-1}| = \emptyset.$$

In this case let $(x_i : i < k-1) \in \prod_{i<k-1} Q^*_i$ be arbitrary and put

$$z = x_0 \cap x_1 \cap \ldots \cap x_{k-2} \text{ and } Q = Q^*_k \subseteq Q_k.$$

This proves the claim.

It is now an easy fusion argument to finish proving the lemma from the claim. $\square$

Now we construct our ultrafilter, proving the theorem under the assumption of CH. We let $(P_\alpha : \alpha < \omega)$ list all perfect subsets of 2^ω. We construct an increasing sequence $\mathcal{F}_\alpha$ for $\alpha < \omega_1$ of $\mathbf{\Sigma}^1_1$ filters as follows.

Let $\mathcal{F}_0$ be the filter generated by

$$\mathcal{I} \cup \{\overline{z} : \exists^\infty x \in \mathcal{I} \ \ z \subseteq^* x\}.$$

At limit ordinals α we let

$$\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$$

and note that it is a $\mathbf{\Sigma}^1_1$ filter. At successor stages $\alpha+1$ we apply the lemma to P_α and $\mathcal{F}_\alpha$. In the first case we find a perfect set $Q \subseteq P_\alpha$ such that $\mathcal{F}_\alpha \cup Q$ has the finite intersection property. In this case we let $\mathcal{F}_{\alpha+1}$ be the filter generated by $\mathcal{F}_\alpha \cup Q$ and note that it is $\mathbf{\Sigma}^1_1$. In the second case we find a perfect set $Q \subseteq P_\alpha$ and $z \subseteq \omega$ so that $\mathcal{F}_\alpha \cup \{z\}$ has the finite intersection property and for every $x \in Q$ we have that there exists $y \in \mathcal{F}_\alpha$ with $x \cap y \cap z = \emptyset$. Here we let $\mathcal{F}_{\alpha+1}$ be the filter generated by $\mathcal{F}_\alpha \cup \{z\}$ and note that $\mathcal{U} \cap Q = \emptyset$ for every ultrafilter $\mathcal{U} \supseteq \mathcal{F}_{\alpha+1}$. This is because we have put $\{\overline{x} : x \in Q\} \subseteq \mathcal{F}_{\alpha+1}$. This ends the proof under CH.

Now we see how to do this construction under the weaker hypothesis that the real line cannot be covered by fewer than continuum many meager sets. We construct an increasing sequence $(\mathcal{F}_\alpha : \alpha < \mathfrak{c})$ of filters such that each $\mathcal{F}_\alpha$ is the union of $\leq |\alpha|$ $\mathbf{\Sigma}^1_1$ sets. In order to prove the corresponding claim and lemma we note that the following is true.

Claim. Suppose the real line cannot be covered by κ many meager sets, $(Q_k : k < n)$ are perfect, and for each $k \leq n$ we have $A_k \subseteq \prod_{i<k}[\omega]^\omega$, which is the union of $\leq \kappa$ many $\mathbf{\Sigma}^1_1$ sets. Then there exists $(Q^*_i \subseteq Q_i : i < n)$ perfect so that for every $k \leq n$ either

$$\prod_{i<k} Q^*_i \subseteq A_k \text{ or } \prod_{i<k} Q^*_i \cap A_k = \emptyset.$$

Proof. Construct $(Q^j_i : i < n)$ perfect by induction so that

(1) $Q^0_i = Q_i$ all $i < n$,
(2) $Q^{j+1}_i \subseteq Q^j_i$ all $i < n$,
(3) $\prod_{i<j} Q^j_i \subseteq A_j$ or $\prod_{i<j} Q^j_i \cap A_j = \emptyset$.

Given $(Q^j_i : i < n)$ and A_{j+1} the union of κ many $\mathbf{\Sigma}^1_1$ sets, say $\bigcup\{B_\alpha : \alpha < \kappa\}$, there are two cases.

Case 1. For some $\alpha < \kappa$ the set $B_\alpha \cap \prod_{i<j+1} Q^j_i$ is not meager.

In this case it must be comeager in some relative interval $\prod_{i<j+1} Q^j_i \cap \prod_{i<j+1}[s_i]$. Now we can find $Q^{j+1}_i \subseteq Q^j_i \cap [s_i]$ such that

$$\prod_{i<j+1} Q^{j+1}_i \subseteq B_\alpha \subseteq A_{j+1}.$$

Case 2. Each B_α is meager in $\prod_{i<j+1} Q^j_i$.

In this case we use the covering of category hypothesis in the form of Martin's axiom for countable posets. For $p \subseteq \omega^{<\omega}$ a finite subtree, define s to be a terminal node of p ($s \in \mathrm{term}(p)$) iff $s \in p$ and for every $t \in p$ if $s \subseteq t$, then $s = t$. For p, q finite subtrees of $2^{<\omega}$ we define $p \supseteq_e q$ (end extension) iff $p \supseteq q$ and every new node of p extends a terminal node of q. Define $T_i = \{s \in 2^{<\omega} : [s] \cap Q_i \neq \emptyset\}$.

Consider the partial order $\mathbb{P}$ consisting of finite approximations to products of perfect trees below the Q_i:

$$\mathbb{P} = \{(p_i : i < n) : p_i \text{ is a finite subtree of } T_i, i < n\}$$

and $p \leq q$ iff $p_i \supseteq_e q_i$ all $i < n$. Our assumption about the covering of the reals by meager sets is equivalent to MA_κ(ctble); i.e., for every countable poset $\mathbb{P}$ and any family $(D_\alpha : \alpha < \kappa)$ of dense subsets of $\mathbb{P}$ there exists a $\mathbb{P}$-filter G such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Note that for any $k < n$ and $C \subseteq \prod_{i<k} Q_i$ which is nowhere dense the set

$$D_C = \{p \in \mathbb{P} \;:\; \forall i < k \;\; \forall (s_i \in \operatorname{term}(p_i) : i < k) \;\; C \cap \prod_{i<k} [s_i] = \emptyset\}$$

is dense in $\mathbb{P}$. Similarly, for any $m < \omega$ the following sets are dense:

$$D_m = \{p \in \mathbb{P} \;:\; \forall i < n \; \forall s \in \operatorname{term}(p_i) \;\; |s| > m\},$$
$$D_m^* = \{p \in \mathbb{P} \;:\; \forall i < n \; \forall s \in p_i \;\; |s| = m \to \exists t \; t \supseteq s \text{ and } t0, t1 \in p_i\}.$$

So a sufficiently generic filter produces a sequence $(T_i' \subseteq T_i : i < n)$ of perfect subtrees such that letting $Q_i' = [T_i']$ we obtain the property that $\prod_{i<n} Q_i' \cap B_\alpha = \emptyset$ for all $\alpha < \kappa$. □

Question 5. Can we prove in ZFC that there exists a nonprincipal ultrafilter with property (s)?

Remark. It is easy to construct a nonprincipal ultrafilter which fails to have property (s). Start with a perfect independent family $I \subseteq P(\omega)$. Choose

$$\{X_\alpha : \alpha < \mathfrak{c}\} \cup \{Y_\alpha : \alpha < \mathfrak{c}\} \subseteq I$$

distinct so that for every perfect $Q \subseteq I$ there exists α with X_α and Y_α both in Q. Then any ultrafilter

$$\mathcal{U} \supseteq \{X_\alpha : \alpha < \mathfrak{c}\} \cup \{\overline{Y}_\alpha : \alpha < \mathfrak{c}\}$$

will fail to have property (s).

Question 6. Can we prove in ZFC that there exists a nonprincipal ultrafilter which is preserved by Sacks forcing?

Note that Shelah (see [1]) has shown it is consistent that there are no nonprincipal P-points. See Brendle [4] for a plethora of ultrafilters weaker than P-points such as Baumgartner's nowhere dense ultrafilters.

Question 7. Suppose $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters in V which generate ultrafilters $\mathcal{U}^*$ and $\mathcal{V}^*$ in $W \supseteq V$. If $\mathcal{U}^* \leq_{RK} \mathcal{V}^*$ holds in W, must $\mathcal{U} \leq_{RK} \mathcal{V}$ be true in V?

Question 8. Suppose $\mathcal{U} \in V$ generates an ultrafilter in $V[x]$ for some (equivalently all) Sacks real x over V. Suppose x is a Sacks real over V and y is a Sacks real over $V[x]$. Must $\mathcal{U}$ generate an ultrafilter in $V[x, y]$?

Question 9. Suppose $V \subseteq W$ and

$$V \models \mathcal{U} \text{ has property (s).}$$

Does

$$W \models \{A \subseteq \omega : \exists B \in \mathcal{U} \; B \subseteq A\} = \mathcal{U}^* \text{ have property (s) ?}$$

Question 10. Is Proposition 2 true for property (s) in place of "preserved by Sacks forcing"?

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