DERIVATIONS PRESERVING A MONOMIAL IDEAL

YOHANNES TADESSE

(Communicated by Bernd Ulrich)

ABSTRACT. Let $I$ be a monomial ideal in a polynomial ring $A = \mathbb{k}[x_1, \ldots, x_n]$ over a field $\mathbb{k}$ of characteristic 0. $TA/k(I)$ be the module of $I$-preserving $k$-derivations on $A$ and $G$ be the $n$-dimensional algebraic torus on $\mathbb{k}$. We compute the weight spaces of $TA/k(I)$ considered as a representation of $G$. Using this, we show that $TA/k(I)$ preserves the integral closure of $I$ and the multiplier ideals of $I$.

1. Introduction

Throughout this paper $TA/k$ denotes the module of $k$-linear derivations of the polynomial ring $A = \mathbb{k}[x_1, \ldots, x_n]$ over a field $\mathbb{k}$ of characteristic 0. Let $G = \mathbb{k}^n \times \cdots \times \mathbb{k}^n$ be the $n$-dimensional algebraic torus on $\mathbb{k}$. There is an action of $G$ on a monomial $X^\theta = x_1^{\theta_1} \cdots x_n^{\theta_n}$ defined by $(t_1, \ldots, t_n) \cdot X^\theta = (t_1^{\theta_1} \cdots t_n^{\theta_n}) \cdot X^\theta$, and we say that $X^\theta$ has weight $\theta$. This makes $A$ a representation of $G$ and induces an action of $G$ on $TA/k$ which will make $TA/k$ a representation such that $X^\theta \partial x_j \in TA/k$ has weight $(\theta_1, \ldots, \theta_j - 1, \ldots, \theta_n)$. Let $I \subseteq A$ be a monomial ideal, i.e. $G$-invariant ideal. Let $TA/k(I) = \{ \delta \in TA/k \mid \delta(I) \subseteq I \} \subseteq TA/k$ be the submodule of $I$-preserving derivations, which is a $G$-subrepresentation of $TA/k$.

Our first result is a description of the weight spaces of $TA/k(I)$. This implies in particular [1 Theorem 2.2.1] by Brumatti and Simis. Note that our use of the $G$-action significantly clarifies the structure of $TA/k(I)$ and simplifies the proof.

The action of $G$ also gives a simple argument to the fact that the integral closure $\bar{I}$ is a monomial ideal; one may compare it to the more complicated proof in [7, Proposition 7.3.4]. For any ideal $I$, it is known that $TA/k(I) \subseteq TA/k(\bar{I})$ [3, Theorem 3.2.2]. Again using the action of $G$ and directly employing the integral equation of elements of $\bar{I}$, we prove this inclusion when $I$ is a monomial ideal.

Let $J(r \cdot I)$ be the multiplier ideal of $I$ for the rational number $r > 0$; see [5]. Using Howald’s description [2] of $J(r \cdot I)$ when $I$ is a monomial ideal, we prove:

Theorem 1.1. Let $I$ be a monomial ideal and $J(r \cdot I)$ be any of its multiplier ideals. Then $TA/k(I) \subseteq TA/k(J(r \cdot I))$.

It is a useful fact that $TA/k(I)$ preserves $J(r \cdot I)$, as this radically restricts its form. The proof of the inclusion for any ideal is indicated in [3].

Received by the editors November 25, 2008, and, in revised form, January 5, 2009.

2000 Mathematics Subject Classification. Primary 13A15, 13N15, 14Q99.

Key words and phrases. Derivations, monomial ideals, multiplier ideals.

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2. The structure of $T_{A/k}(I)$

The Lie algebra of the torus is $\nabla_G = \bigoplus_{j=1}^n k\nabla_{x_j}$, where $\nabla_{x_j} = x_j \partial_{x_j}$. The representation $T_{A/k}$ of $G$ decomposes as

$$T_{A/k} = A\nabla_G \oplus s,$$

where $s = \bigoplus_{j=1}^n A_{x_j} \partial_{x_j}$ and $A_{x_j} = k[x_1, \ldots, \hat{x_j}, \ldots, x_n]$. Hence the representation is semi-simple. Since $I$ is a monomial ideal, $T_{A/k}(I)$ is a $G$-subrepresentation of $T_{A/k}$, hence semi-simple, having the decomposition

$$T_{A/k}(I) = A\nabla_G \oplus s(I).$$

Since all the weight spaces of $s(I) = T_{A/k}(I) \cap s \subseteq s$ are 1-dimensional, we easily get the following lemma.

**Lemma 2.1.** Let $I \subseteq A$ be a monomial ideal and $\delta = \sum_{j=1}^n f_j \partial_{x_j} \in T_{A/k}$ be a derivation where $f_j = \sum_{i,j} m_{ij} \in A$ and $m_{ij}$ are distinct monomial terms. Then $\delta \in T_{A/k}(I)$ if and only if $m_{ij} \partial_{x_j} \in T_{A/k}(I)$ for all $i$ and $j = 1, \ldots, n$.

Put $\exp(I) = \{ \theta \mid X^\theta \in I \} \subseteq \mathbb{Z}_{\geq 0}^n$ and let $(e_j)$ denote the standard basis of $\mathbb{R}^n$. Let $\{X^{\alpha_1}, \ldots, X^{\alpha_t}\}$ be the unique minimal generating set of $I$, where $X^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$.

**Theorem 2.2.** Consider the condition

$$C_j^I(\beta) : \quad \beta \in \mathbb{Z}_{\geq 0}^n, \quad \beta_j = 0, \text{ and } \beta + \alpha_i - e_j \in \exp(I) \text{ for all } i \text{ such that } \alpha_{ij} > 0.$$

We have $s(I) = \bigoplus_{j=1}^n I_{x_j} \partial_{x_j}$, where

$$I_{x_j} = (X^\beta \mid C_j^I(\beta)) \subseteq A_{x_j}.$$

If $\beta_j = 0$ and $\alpha_{ij} > 0$ for all $i = 1, \ldots, t$, then $X^\beta \partial_{x_j} \notin s(I)$. Indeed, for a monomial $X^\theta \in I$ such that $\theta_j \leq \alpha_{ij}$ for all $i$, one has $X^\beta \partial_{x_j}(X^\theta) = \theta_j X^{\beta + \theta - e_j} \notin I$. Hence $C_j^I(\beta)$ never holds, and we then put $I_{x_j} = (0)$.

**Proof.** By Lemma 2.1 it suffices to determine when $X^\beta \partial_{x_j}$ belongs to $s(I)$, so in particular $\beta_j = 0$. Since $I_{x_j} = (0)$ if $\alpha_{ij} > 0$ for all $i$, we may assume there exists $1 \leq k \leq t$ such that $\alpha_{kj} = 0$. Then

$$X^\beta \in I_{x_j} \iff \beta - \alpha_i - e_j \in \exp(I) \text{ for all } i \text{ such that } \alpha_{ij} > 0$$

$$\iff X^\beta \partial_{x_j}(X^{\alpha_i}) \in I \text{ for all } i = 1, \ldots, t$$

$$\iff X^\beta \partial_{x_j} \in s(I).$$

\[ \square \]

By [1] we have $T_{A/k}(I) = \bigoplus_{j=1}^n [I : [I : x_j]] \partial_{x_j}$, but the argument there does not profit on the torus action. We now show that $[I : [I : x_j]] = (x_j) + I_{x_j}$. Note
first that colon ideals of monomial ideals are monomial, and \([I : x_j] = (\frac{X^\theta}{x_j} | X^\theta \in I \text{ and } \theta_j > 0)\). Therefore,

\[ [I : [I : x_j]] = (X^\beta | X^\beta \cdot \frac{X^\theta}{x_j} \in I \text{ for all } X^\theta \in I \text{ such that } \theta_j > 0) \]

\[ = (X^\beta | \beta + \theta - e_j \in \exp(I) \text{ for all } \theta \in \exp(I) \text{ such that } \theta_j > 0) \]

\[ = (x_j) + (X^\beta | \beta_j = 0 \text{ and } \beta + \theta - e_j \in \exp(I) \]

for all \( \theta \in \exp(I) \) such that \( \theta_j > 0 \)

\[ = (x_j) + (X^\beta | \beta_j = 0 \text{ and } \beta + \alpha_i - e_j \in \exp(I) \]

for all \( i \) such that \( \alpha_{ij} > 0 \)

\[ = (x_j) + I_{x_j}. \]

We assert that \( C'_j(\beta) \) is equivalent to the following condition:

\[ (2.3) \quad \beta \in \mathbb{Z}_{\geq 0}^n, \quad \beta_j = 0, \text{ and } [(\beta - e_j) + \exp(I)] \cap \mathbb{Z}_{\geq 0}^n \subseteq \exp(I). \]

It is easy to see that \((2.3) \Rightarrow C'_j(\beta)\), so we prove only \( C'_j(\beta) \Rightarrow (2.3) \): If \( \alpha_{ij} > 0 \) for all \( i \), then \( C'_j(\beta) \) does not hold for any \( \beta \). Now assume there exists \( 1 < k \leq t \) such that

\[ (2.4) \quad 0 = \alpha_{1j} = \cdots = \alpha_{k-1,j} < \alpha_{kj} \leq \cdots \leq \alpha_{ij}, \]

and let \( \theta \in \exp(I) \). It is clear that \( \theta + \beta - e_j \notin \mathbb{Z}_{\geq 0}^n \) if \( \theta_j = 0 \); thus assume that \( \theta_j > 0 \). If \( \theta_j < \alpha_{kj} \), then there exists some \( i = 1, \ldots, k-1 \) such that \( \theta = \alpha_i + \gamma \) with \( \gamma_j = \theta_j > 0 \). Hence \( \theta + \beta - e_j = \alpha_i + \beta + \gamma - e_j \in \exp(I) \) since \( \alpha_i + \beta \in \exp(I) \) and \( \gamma - e_j \in \mathbb{Z}_{\geq 0}^n \). If \( \theta_j \geq \alpha_{kj} \), then there exists \( l \geq k \) such that \( \theta = \alpha_l + \gamma' \) for some \( \gamma' \in \mathbb{Z}_{\geq 0}^n \). \( C'_j(\beta) \) implies \( \theta + \beta - e_j = \gamma' + \alpha_l + \beta - e_j \in \exp(I) \). Therefore, \( [(\beta - e_j) + \exp(I)] \cap \mathbb{Z}_{\geq 0}^n \subseteq \exp(I) \).

It is interesting to see the structure of \( T_{A/k}(I) \) when \( A = k[x, y] \). First, if \( I \) is principal, then \( T_{A/k}(I) \) is either of the form \( A \nabla_G, A \nabla_G \oplus A_y \partial_y \) or \( A \nabla_G \oplus A_x \partial_x \). Now assume \( I = (x^a_1 y^b_1, \ldots, x^a_t y^b_t) \) is non-principal with \( a_{i-1} < a_i \) and \( b_{i-1} > b_i \) for \( 1 < i \leq t \). Then define the \textit{widths} \( w(I)_x \) and \( w(I)_y \) of \( I \) in the direction of \( x \) and \( y \) by

\[ w(I)_x = \max\{a_i - a_{i-1}\}_{i=2}^t \quad \text{and} \quad w(I)_y = \max\{b_i - b_{i-1}\}_{i=2}^t. \]

This is illustrated by the following figure:
Corollary 2.3. Assume that $A = \mathbb{k}[x, y]$ and $I = (x^{a_1}y^{b_1}, \ldots, x^{a_t}y^{b_t})$ is a non-principal monomial ideal such that $a_{i-1} < a_i$ and $b_{i-1} > b_i$ for each $1 < i \leq t$. Then

$$
s(I) = \begin{cases} 
(0) & \text{if } a_1 > 0 \text{ and } b_1 > 0 \\
[k|y|y^{w(I)}]_{\partial_x} & \text{if } a_1 = 0 \text{ and } b_1 > 0 \\
k[x|x^{w(I)}]_{\partial_y} & \text{if } a_1 > 0 \text{ and } b_1 = 0 \\
k|y|y^{w(I)}]_{\partial_x} + k[x|x^{w(I)}]_{\partial_y} & \text{if } a_1 = b_1 = 0.
\end{cases}
$$

Proof. It suffices to prove the cases $a_1 = 0$ and $b_1 > 0$ since the other cases are similar.

$$y'_{\partial_x} \in s(I) \iff y'_{\partial_x}(x^n y^b) \in I \text{ for all } x^n y^b \in I$$

$$\iff (0, l) + (a, b) - (1, 0) \in \exp(I) \text{ for all } (a, b) \in \exp(I)$$

$$\text{such that } a > 0$$

$$\iff (a_i - 1, l + b_i) \in \exp(I) \text{ for all } 1 < i \leq t$$

$$\iff l \geq b_{i-1} - b_i \text{ for all } 1 < i \leq t$$

$$\iff l \geq w(I)_y.$$

Hence $s(I) = k|y|y^{w(I)}]_{\partial_x}$. \hfill \Box

Example 2.4. Consider $I = (y^8, x^2y^6, x^5y^4, x^7y^2, x^8y, x^{12}) \subseteq \mathbb{k}[x, y]$. Then $w(I)_x = 4$, $w(I)_y = 2$ and $T_{A/k}(I) = A \nabla_x + A \nabla_y + a^2 k[|y|y^{\partial_x} + x^4 k[|x|y]_{\partial_y}].$

If $I$ is a monomial ideal of $A = \mathbb{k}[x, y]$ and $l > 0$ is an integer, it is not obvious how $w(I)_x$ and $w(I)_y$ depend on $l$. But using Corollary 2.3 and the obvious fact that $T_{A/k}(I) \subseteq T_{A/k}(I^t)$ where equality holds if $I = [I^t : I^{t-1}]$ [2 Remark 3.2.6], we get the following result.

Corollary 2.5. If $I \subseteq \mathbb{k}[x, y]$ is a monomial ideal, then $w(I)_x \geq w(I^t)_x$ and $w(I)_y \geq w(I^t)_y$, and equality holds when $I = [I^t : I^{t-1}].$

We give a guideline on how to use Theorem 2.2 to compute $s(I)$. Since $I_{x_i} = (0)$ if $|\alpha_{ij}| > 0$ for all $i$, we assume that there exists $1 < k \leq t$ such that (2.4) holds. We can assume that the set $\{a_1, \ldots, a_t\}$ is ordered as in (2.4). First compute all $\beta \in \mathbb{Z}_{\geq 0}^n$ satisfying $\beta_i = 0$ and $a_i + \beta - e_j \in \exp(I)$ for each $k \leq i \leq t$. That is, collect all $Z$-linearly independent vectors $\beta$ on the hyperplane plane $x_j = a_{ij} - 1$ with initial point $\alpha_i - e_j$ and terminal point at the boundary of $\exp((\{X^\alpha : a_{ij} < \alpha_j\})_{j=1}^t) \cap (x_j = a_{ij} - 1)$. This gives the monomial ideal

$$I_{x_j}(i) := \{X^{a_i - \alpha_{ij}}e_j : a_{ij} < \alpha_j\}_{j=1}^{t} \subseteq A_{x_j},$$

Since $X^\beta \in I_{x_j}$ if and only if $a_i + \beta - e_j \in \exp(I)$ for all $k \leq i \leq t$, we get

$$I_{x_j} = \bigcap_{i=1}^{t} I_{x_j}(i).$$

Example 2.6. Let $I = (x^4, x^2y^3, xy^4z, z^2) \subset A = \mathbb{k}[x, y, z]$. To compute $I_z$ we take the ordering $a_1 = (4, 0, 0) < a_2 = (2, 3, 0) < a_3 = (1, 4, 1) < a_4 = (0, 0, 2)$. We need to compute the ideals $I_z(3)$ and $I_z(4)$ described in (2.5). That is, $I_z(3) = (xy^4)^{\nabla^2} x^2 y^3 = (x)$ and $I_z(4) = (xy^4, x^2 y^3, x^4) = (x^4, x^2 y^3, x y^4)$. This gives $I_z = (x) \cap (xy^4, x^2 y^3, x^4) = (xy^4, x^2 y^3, x^4)$. We have $T_{A/k}(I) = A \nabla_x + A \nabla_y + A \nabla_z + s(I)$, where

$$s(I) = (y^3 z, z^2)_{\partial_z} + (x^2, z^2)_{\partial_y} + (x^4, y^4, x^2 y^3)_{\partial_z}.$$
3. Preservation of the integral closure and multiplier ideals

It is well known that $T_{\Lambda/k}(I)$ preserves many naturally defined ideals related to $I$, for any ideal $I$. We will investigate this question in the case of monomial ideals in relation to the integral closure and the formation of multiplier ideals.

Given a commutative Noetherian ring $R$ and an ideal $I$ of $R$, an element $f \in R$ is integral over $I$ if $f$ satisfies the equation

$$f^d + g_1 f^{d-1} + \cdots + g_{d-1} f + g_d = 0,$$

where $g_i \in I$ and $i = 1, \ldots, d$. The integral closure $\bar{I}$ consists of all elements in $R$ which are integral over $I$. The following lemma is a standard fact.

Lemma 3.1. Let $I$ be a monomial ideal in $\mathbf{A}$. Then $\bar{I}$ is also a monomial ideal. Furthermore, a monomial $X^\theta$ is in $\bar{I}$ if and only if $(X^\theta)^l \in I^l$ for some integer $l > 0$.

Proof. Let $f$ be integral over $I$. Applying the action of the torus $G$ on $I$ we obtain

$$f^d + g_1 f^{d-1} + \cdots + g_{d-1} f + g_d = 0,$$

where $t = (t_1, \ldots, t_n) \in G$, $X = (x_1, \ldots, x_n)$ and $t \cdot X = (t_1 x_1, \ldots, t_n x_n)$. Since $I$ is a monomial ideal for all $i = 1, \ldots, d$, hence invariant under the action of $G$, we have $g_i(t \cdot X) \in I$ for all $i = 1, \ldots, d$. Thus (3.2) is the integral dependence equation for $f(t \cdot X) \in A$. Therefore $f(t \cdot X) \in \bar{I}$; hence $I$ is invariant under the action of $G$ and it is a monomial ideal. To prove the second statement, assume $X^\theta$ satisfies (3.1). Since each $I^l$ is a monomial ideal, considering terms of weight $d\theta$ in (3.1), we obtain an equation of the form

$$(X^\theta)^d + k_1 X^\theta_1 (X^\theta)^{d-1} + \cdots + k_d X^\theta_d = 0$$

for some $X^\theta_i \in I^l$, $i = 1, \ldots, d$, and $k_1, \ldots, k_d \in k$. Some coefficient $k_l$ must be non-zero; thus $(X^\theta)^d = k_0 X^\theta_1 (X^\theta)^{d-1}$, where $X^\theta_1 \in I^l$ and $k_0 \in k$, so $(X^\theta)^l = k_0 X^\theta_1 \in I^l$. The converse is immediate. $\square$

The inclusion

$$(3.3) \quad T_{\Lambda/k}(I) \subseteq T_{\Lambda/k}(\bar{I})$$

is proved in [3, Theorem 3.2.2] using the blow-up of $I$, for any ideal $I$. But it is difficult to see how this directly follows from equation (3.1). Here is a direct proof of (3.3) when $I$ is a monomial ideal: By Lemma 2.1 it suffices to prove this for derivations of the form $\delta = X^\beta_i \partial_{x_j} \in T_{\Lambda/k}(I)$. Let $X^\theta \in \bar{I}$ such that $\theta_j > 0$. Then $(X^\theta)^l \in I^l$ for some $l > 0$. Since $T_{\Lambda/k}(I) \subseteq T_{\Lambda/k}(I^l)$, we have

$$\delta \in T_{\Lambda/k}(I^l) \Rightarrow \delta((X^\theta)^l) = l (X^\theta)^{l-1} \delta(X^\theta) \in I^l.$$ 

Clearly $\delta((X^\theta)^l)$ is a monomial. We have that

$$\delta^2((X^\theta)^l) = \delta(l (X^\theta)^{l-1} \delta(X^\theta)) = l(l-1)(X^\theta)^{l-2} \delta^2(X^\theta) + l(X^\theta)^{l-1} \delta^2(X^\theta) \in I^l$$

is also a monomial, split into the sum of two monomials of the same weight. It follows that $(X^\theta)^l-2 \delta^2(X^\theta)^2 \in I^l$. Similarly, $\delta^3((X^\theta)^l) = (l-2)(X^\theta)^{l-3} \delta^3(X^\theta)$ is a monomial; hence $(X^\theta)^{l-3} \delta^3(X^\theta)^3 \in I^l$. Continuing in this way, we eventually get $(\delta(X^\theta))^l \in I^l$. Hence by Lemma 3.1 $\delta(X^\theta) \in \bar{I}$. 

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Let \( \text{conv}(I) \subseteq \mathbb{R}^n_{\geq 0} \) be the convex hull of \( \exp(I) \). The following lemma immediately follows from \([2,3]\).

**Lemma 3.2.** Let \( I \) be a monomial ideal and \( \beta \in \mathbb{Z}^n_{\geq 0} \) such that \( \beta_j = 0 \). If \( \theta + \beta - e_j \in \exp(I) \) for all \( \theta \in \exp(I) \) such that \( \theta_j > 0 \), then \([\{\beta - e_j\} + \exp(I)] \cap \mathbb{R}^n_{\geq 0} \subseteq \text{conv}(I)\).

We have \( \exp(\bar{I}) = \exp(I) \cap \mathbb{Z}^n_{\geq 0} \) \([7, \text{Proposition 7.3.4}]\). Using this we can easily prove \([3,3]\) using \([2,3]\) and Lemma \([3,2]\) since \([\{\beta - e_j\} + \exp(I)] \cap \mathbb{R}^n_{\geq 0} \subseteq \exp(\bar{I})\).

Let \( X = \text{Spec} A \). By a log resolution of an ideal \( I \subseteq A \) we mean a proper birational map \( f : Y \to X \) with the property that \( Y \) is smooth and \( f^{-1}(I) = \mathcal{O}_Y(-E) \), where \( E \) is an effective Cartier divisor and \( E + \text{exc}(f) \) has a normal crossing support. Let \( r > 0 \) be a rational number. We define the multiplier ideal of \( I \) with coefficient \( r \) as the ideal

\[
\mathcal{J}(r \cdot I) = f_* \mathcal{O}_Y(K_{Y/X} - \lfloor rE \rfloor).
\]

Here \( K_{Y/X} = K_Y - f^*K_X \) is the relative canonical bundle and \( \lfloor - \rfloor \) is the round down for rational divisors. See \([5]\) for a discussion of \( \mathcal{J}(r \cdot I) \). In general it is difficult to compute \( \mathcal{J}(r \cdot I) \), but for monomial ideals we have a complete description due to Howald.

**Theorem 3.3** (\([2]\)). Let \( I \subseteq A \) be a monomial ideal and \( r > 0 \) be a rational number. Then \( \mathcal{J}(r \cdot I) \) is a monomial ideal generated by the following set of monomials:

\[
\{X^\theta \mid \theta + (1,1,\ldots,1) \in \text{Int}(r \cdot \text{conv}(I)) \cap \mathbb{Z}^n_{\geq 0}\},
\]

where \( \text{Int}(\mathcal{C}) \) denotes the topological interior of a convex set \( \mathcal{C} \subseteq \mathbb{R}^n \).

**Example 3.4.** The figure below contains a graphical description of the ideals \( I = (y^8, x^2y^5, x^3y, x^5) \) and \( \mathcal{J}(r \cdot I) = (y^8, xy^7, x^2y^5, x^3y^4, x^5y^3, x^6y^2, x^8y, x^{10}) \) when \( r = \frac{31}{12} \).

![Diagram of ideals and multiplier ideals](image)

**Proof of Theorem 1.1.** Consider the monomial ideal \( \mathcal{J}^\circ(r \cdot I) \) generated by the following set of monomials:

\[
\{X^\theta \in \mathbb{Z}^n_{\geq 0} \mid \theta \in \text{Int}(r \cdot \text{conv}(I)) \text{ or } \theta \in B(r \cdot \text{conv}(I)) \cap (\bigcup_{j=1}^n H_j)\},
\]
where $B(\mathcal{C})$ is the boundary of a convex set $\mathcal{C} \subseteq \mathbb{R}^n_{\geq 0}$ and $H_i$ is the coordinate hyperplane $x_j = 0$ in $\mathbb{R}^n$. Clearly $X^\theta \in \mathcal{J}(r \cdot I)$ if and only if $x_1 \cdot x_n \cdot X^\theta \in \mathcal{J}^o(r \cdot I)$. It suffices to show that

$$\mathfrak{s}(I) \subseteq \mathfrak{s}(J^o(r \cdot I)) \subseteq \mathfrak{s}(J(r \cdot I)).$$

If $\alpha_{ij} > 0$ for all $i$, then $(0) = I_{x_j} = (J^o(r \cdot I))_{x_j} \subseteq (J(r \cdot I))_{x_j}$. Therefore assume that there exists $1 \leq k \leq t$ such that $\alpha_{kj} = 0$. We first prove the second inclusion. Let $X^\beta \partial_{x_j} \in \mathfrak{s}(J(r \cdot I))$ and $X^\theta \in J(r \cdot I)$ such that $\theta_j > 0$. Then $x_1 \cdot x_n \cdot X^\theta \in J^o(r \cdot I)$ and $X^\beta \partial_{x_j}(x_1 \cdot x_n \cdot X^\theta) = x_1 \cdot x_n \cdot X^{\beta + e_j}$ belong to $J^o(r \cdot I)$. This implies that $X^{\theta + e_j} \in J(r \cdot I)$; hence $X^\beta \partial_{x_j} \in \mathfrak{s}(J(r \cdot I))$.

To prove the first inclusion it suffices, by Theorem 2.2.2, to show that

$$C^I_j(\beta) \Rightarrow C_j^{J^o(r \cdot I)}(\beta).$$

Assume $C^I_j(\beta)$. Then $\theta + \beta - e_j \in \exp(I)$ for all $\theta \in \exp(I)$ such that $\theta_j > 0$. Hence, by Lemma 3.2, $\{\beta - e_j\} + \text{conv}(I) \cap \mathbb{R}^n_{\geq 0} \subseteq \text{conv}(I)$. We divide the proof into two cases.

$r \leq 1$: Consider $X^\theta \in J^o(r \cdot I)$ such that $\theta_j > 0$, so $\theta \in \text{Int}(r \cdot \text{conv}(I))$, and hence $\frac{1}{r}\theta \in \text{Int}(\text{conv}(I))$. By Lemma 3.2, $\frac{1}{r}\theta + \beta - e_j \in \text{Int}(\text{conv}(I))$, and therefore

$$\theta + r\beta - re_j \in \text{Int}(r \cdot \text{conv}(I)).$$

Putting $r = 1$ in (3.4) and using the inclusion $\text{Int}(\text{conv}(I)) \subseteq \text{Int}(r \cdot \text{conv}(I))$, we obtain $\theta + \beta - e_j \in \text{Int}(r \cdot \text{conv}(I))$. This implies $C_j^{J^o(r \cdot I)}(\beta)$.

$r > 1$: If $\theta \in \text{Int}(r \cdot \text{conv}(I))$, then $\frac{1}{r}\theta \in \text{Int}(\text{conv}(I))$; so by Lemma 3.2, it follows that $\frac{1}{r}\theta + \beta - e_j \in \text{Int}(\text{conv}(I))$, since $\frac{1}{r}\theta_j > 0$. By convexity all the points on the line segment joining $\frac{1}{r}\theta$ and $\frac{1}{r}\theta + \beta - e_j$ belong to $\text{Int}(\text{conv}(I))$. In particular

$$\frac{1}{r}(\theta + \beta - e_j) \in \text{Int}(\text{conv}(I)).$$

Hence $\theta + \beta - e_j \in \text{Int}(r \cdot \text{conv}(I))$, implying $C_j^{J^o(r \cdot I)}(\beta)$.

\section*{Acknowledgments}

The author wishes to express his warmest gratitude to his advisor, Rolf Källström, for introducing him to this subject and for his continuing support. This work is financially supported by International Science Programme, Uppsala University.

\section*{References}


Department of Mathematics, Addis Ababa University, P. O. Box 1176, Addis Ababa, Ethiopia

E-mail address: yohannes@math.aau.edu.et

Current address: Department of Mathematics, Stockholm University, SE 106-91, Stockholm, Sweden

E-mail address: tadesse@math.su.se