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DERIVATIONS PRESERVING A MONOMIAL IDEAL

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Abstract. Let I be a monomial ideal in a polynomial ring $\mathbf{A} = \mathbf{k}[x_1, \dots, x_n]$ over a field $\mathbf{k}$ of characteristic 0, $T_{\mathbf{A}/\mathbf{k}}(I)$ be the module of I-preserving $\mathbf{k}$-derivations on $\mathbf{A}$ and G be the n-dimensional algebraic torus on $\mathbf{k}$. We compute the weight spaces of $T_{\mathbf{A}/\mathbf{k}}(I)$ considered as a representation of G. Using this, we show that $T_{\mathbf{A}/\mathbf{k}}(I)$ preserves the integral closure of I and the multiplier ideals of I.

1. Introduction

Throughout this paper $T_{\mathbf{A}/\mathbf{k}}$ denotes the module of $\mathbf{k}$-linear derivations of the polynomial ring $\mathbf{A} = \mathbf{k}[x_1, \dots, x_n]$ over a field $\mathbf{k}$ of characteristic 0. Let $G = \mathbf{k}^* \times \cdots \times \mathbf{k}^*$ be the n-dimensional algebraic torus on $\mathbf{k}$. There is an action of G on a monomial $X^\theta = x_1^{\theta_1} \cdots x_n^{\theta_n}$ defined by $(t_1, \dots, t_n) \cdot X^\theta = (t_1^{\theta_1} \cdots t_n^{\theta_n}) \cdot X^\theta$, and we say that X^θ has weight θ. This makes $\mathbf{A}$ a representation of G and induces an action of G on $T_{\mathbf{A}/\mathbf{k}}$ which will make $T_{\mathbf{A}/\mathbf{k}}$ a representation such that $X^\theta \partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}$ has weight $(\theta_1, \dots, \theta_j - 1, \dots \theta_n)$. Let $I \subseteq \mathbf{A}$ be a monomial ideal, i.e. a G-invariant ideal. Let $T_{\mathbf{A}/\mathbf{k}}(I) = \{\delta \in T_{\mathbf{A}/\mathbf{k}} \mid \delta(I) \subseteq I\} \subset T_{\mathbf{A}/\mathbf{k}}$ be the submodule of I-preserving derivations, which is a G-subrepresentation of $T_{\mathbf{A}/\mathbf{k}}$. Our first result is a description of the weight spaces of $T_{\mathbf{A}/\mathbf{k}}(I)$. This implies in particular [1, Theorem 2.2.1] by Brumatti and Simis. Note that our use of the G-action significantly clarifies the structure of $T_{\mathbf{A}/\mathbf{k}}(I)$ and simplifies the proof.

The action of G also gives a simple argument to the fact that the integral closure $\bar{I}$ is a monomial ideal; one may compare it to the more complicated proof in [7, Proposition 7.3.4]. For any ideal I, it is known that $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\bar{I})$ [3, Theorem 3.2.2]. Again using the action of G and directly employing the integral equation of elements of $\bar{I}$, we prove this inclusion when I is a monomial ideal.

Let $\mathcal{J}(r \cdot I)$ be the multiplier ideal of I for the rational number $r > 0$; see [5]. Using Howald's description [2] of $\mathcal{J}(r \cdot I)$ when I is a monomial ideal, we prove:

Theorem 1.1. *Let I be a monomial ideal and $\mathcal{J}(r \cdot I)$ be any of its multiplier ideals. Then $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\mathcal{J}(r \cdot I))$.*

It is a useful fact that $T_{\mathbf{A}/\mathbf{k}}(I)$ preserves $\mathcal{J}(r \cdot I)$, as this radically restricts its form. The proof of the inclusion for any ideal is indicated in [3].

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2. THE STRUCTURE OF $T_{\mathbf{A}/\mathbf{k}}(I)$

The Lie algebra of the torus is $\nabla_G = \bigoplus_{j=1}^n \mathbf{k}\nabla_{x_j}$, where $\nabla_{x_j} = x_j\partial_{x_j}$. The representation $T_{\mathbf{A}/\mathbf{k}}$ of G decomposes as

$$T_{\mathbf{A}/\mathbf{k}} = \mathbf{A}\nabla_G \oplus \mathfrak{s}, \tag{2.1}$$

where $\mathfrak{s} = \bigoplus_{j=1}^n \mathbf{A}_{x_j}\partial_{x_j}$ and $\mathbf{A}_{x_j} = \mathbf{k}[x_1, \dots, \hat{x}_j, \dots, x_n]$. Hence the representation is semi-simple. Since I is a monomial ideal, $T_{\mathbf{A}/\mathbf{k}}(I)$ is a G-subrepresentation of $T_{\mathbf{A}/\mathbf{k}}$, hence semi-simple, having the decomposition

$$T_{\mathbf{A}/\mathbf{k}}(I) = \mathbf{A}\nabla_G \oplus \mathfrak{s}(I). \tag{2.2}$$

Since all the weight spaces of $\mathfrak{s}(I) = T_{\mathbf{A}/\mathbf{k}}(I) \cap \mathfrak{s} \subseteq \mathfrak{s}$ are 1-dimensional, we easily get the following lemma.

Lemma 2.1. *Let $I \subseteq \mathbf{A}$ be a monomial ideal and $\delta = \sum_{j=1}^n f_j\partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}$ be a derivation where $f_j = \sum_{ij} m_{ij} \in \mathbf{A}$ and m_{ij} are distinct monomial terms. Then $\delta \in T_{\mathbf{A}/\mathbf{k}}(I)$ if and only if $m_{ij}\partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(I)$ for all i and $j = 1, \dots, n$.*

Put $\exp(I) = \{\theta \mid X^\theta \in I\} \subseteq \mathbb{Z}^n_{\geq 0}$ and let (e_j) denote the standard basis of $\mathbb{R}^n$. Let $\{X^{\alpha_1}, \dots, X^{\alpha_t}\}$ be the unique minimal generating set of I, where $X^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$.

Theorem 2.2. *Consider the condition*

$$\begin{aligned} \mathcal{C}^I_j(\beta): \quad & \beta \in \mathbb{Z}^n_{\geq 0}, \quad \beta_j = 0, \ \textit{and} \quad \beta + \alpha_i - e_j \in \exp(I) \\ & \textit{for all } i \textit{ such that } \alpha_{ij} > 0. \end{aligned}$$

We have $\mathfrak{s}(I) = \bigoplus_{j=1}^n I_{x_j}\partial_{x_j}$, where

$$I_{x_j} = (X^\beta \mid \mathcal{C}^I_j(\beta)) \subseteq \mathbf{A}_{x_j}.$$

If $\beta_j = 0$ and $\alpha_{ij} > 0$ for all $i = 1, ..., t$, then $X^\beta\partial_{x_j} \notin \mathfrak{s}(I)$. Indeed, for a monomial $X^\theta \in I$ such that $\theta_j \leq \alpha_{ij}$ for all i, one has $X^\beta\partial_{x_j}(X^\theta) = \theta_j X^{\beta+\theta-e_j} \notin I$. Hence $\mathcal{C}^I_j(\beta)$ never holds, and we then put $I_{x_j} = (0)$.

Proof. By Lemma 2.1 it suffices to determine when $X^\beta\partial_{x_j}$ belongs to $\mathfrak{s}(I)$, so in particular $\beta_j = 0$. Since $I_{x_j} = (0)$ if $\alpha_{ij} > 0$ for all i, we may assume there exists $1 \leq k \leq t$ such that $\alpha_{kj} = 0$. Then

$$\begin{aligned} X^\beta \in I_{x_j} &\Leftrightarrow \beta + \alpha_i - e_j \in \exp(I) \text{ for all } i \text{ such that } \alpha_{ij} > 0 \\ &\Leftrightarrow X^\beta\partial_{x_j}(X^{\alpha_i}) \in I \text{ for all } i = 1, \dots, t \\ &\Leftrightarrow X^\beta\partial_{x_j} \in \mathfrak{s}(I). \end{aligned}$$

□

By [1] we have $T_{\mathbf{A}/\mathbf{k}}(I) = \bigoplus_{j=1}^n [I : [I : x_j]]\partial_{x_j}$, but the argument there does not profit on the torus action. We now show that $[I : [I : x_j]] = (x_j) + I_{x_j}$. Note first that colon ideals of monomial ideals are monomial, and $[I : x_j] = (\frac{X^\theta}{x_j} \mid X^\theta \in I \text{ and } \theta_j > 0)$. Therefore,

$$\begin{aligned}[I:[I:x_j]] &= (X^\beta \mid X^\beta \cdot \frac{X^\theta}{x_j} \in I \text{ for all } X^\theta \in I \text{ such that } \theta_j > 0) \\ &= (X^\beta \mid \beta + \theta - e_j \in \exp(I) \text{ for all } \theta \in \exp(I) \text{ such that } \theta_j > 0) \\ &= (x_j) + (X^\beta \mid \beta_j = 0 \text{ and } \beta + \theta - e_j \in \exp(I) \\ &\quad \text{for all } \theta \in \exp(I) \text{ such that } \theta_j > 0) \\ &= (x_j) + (X^\beta \mid \beta_j = 0 \text{ and } \beta + \alpha_i - e_j \in \exp(I) \\ &\quad \text{for all } i \text{ such that } \alpha_{ij} > 0) \\ &= (x_j) + I_{x_j}.\end{aligned}$$

We assert that $\mathcal{C}_j^I(\beta)$ is equivalent to the following condition:

$$\beta \in \mathbb{Z}^n_{\geq 0}, \quad \beta_j = 0, \text{ and } [\{\beta - e_j\} + \exp(I)] \cap \mathbb{Z}^n_{\geq 0} \subseteq \exp(I). \tag{2.3}$$

It is easy to see that (2.3)$\Rightarrow \mathcal{C}_j^I(\beta)$, so we prove only $\mathcal{C}_j^I(\beta) \Rightarrow$ (2.3): If $\alpha_{ij} > 0$ for all i, then $\mathcal{C}_j^I(\beta)$ does not hold for any β. Now assume there exists $1 < k \leq t$ such that

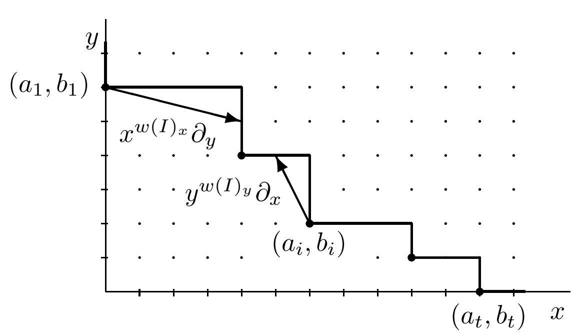
$$0 = \alpha_{1j} = \cdots = \alpha_{k-1,j} < \alpha_{kj} \leq \cdots \leq \alpha_{tj}, \tag{2.4}$$

and let $\theta \in \exp(I)$. It is clear that $\theta + \beta - e_j \notin \mathbb{Z}^n_{\geq 0}$ if $\theta_j = 0$; thus assume that $\theta_j > 0$. If $\theta_j < \alpha_{kj}$, then there exists some $i = 1, \ldots, k-1$ such that $\theta = \alpha_i + \gamma$ with $\gamma_j = \theta_j > 0$. Hence $\theta + \beta - e_j = \alpha_i + \beta + \gamma - e_j \in \exp(I)$ since $\alpha_i + \beta \in \exp(I)$ and $\gamma - e_j \in \mathbb{Z}^n_{\geq 0}$. If $\theta_j \geq \alpha_{kj}$, then there exists $l \geq k$ such that $\theta = \alpha_l + \gamma'$ for some $\gamma' \in \mathbb{Z}^n_{\geq 0}$. $\mathcal{C}_j^I(\beta)$ implies $\theta + \beta - e_j = \gamma' + \alpha_l + \beta - e_j \in \exp(I)$. Therefore, $[\{\beta - e_j\} + \exp(I)] \cap \mathbb{Z}^n_{\geq 0} \subseteq \exp(I)$.

It is interesting to see the structure of $T_{\mathbf{A}/\mathbf{k}}(I)$ when $\mathbf{A} = \mathbf{k}[x, y]$. First, if I is principal, then $T_{\mathbf{A}/\mathbf{k}}(I)$ is either of the form $\mathbf{A}\nabla_G$, $\mathbf{A}\nabla_G \oplus \mathbf{A}_y\partial_y$ or $\mathbf{A}\nabla_G \oplus \mathbf{A}_x\partial_x$. Now assume $I = (x^{a_1}y^{b_1}, \ldots, x^{a_t}y^{b_t})$ is non-principal with $a_{i-1} < a_i$ and $b_{i-1} > b_i$ for $1 < i \leq t$. Then define the *widths* $w(I)_x$ and $w(I)_y$ of I in the direction of x and y by

$$w(I)_x = \max\{a_i - a_{i-1}\}_{i=2}^t \quad \text{and} \quad w(I)_y = \max\{b_{i-1} - b_i\}_{i=2}^t.$$

This is illustrated by the following figure:



Corollary 2.3. *Assume that $\mathbf{A} = \mathbf{k}[x, y]$ and $I = (x^{a_1}y^{b_1}, \dots, x^{a_t}y^{b_t})$ is a non-principal monomial ideal such that $a_{i-1} < a_i$ and $b_{i-1} > b_i$ for each $1 < i \leq t$. Then*

$$\mathfrak{s}(I) = \begin{cases} (0) & \textit{if} \quad a_1 > 0 \textit{ and } b_t > 0 \\ \mathbf{k}[y]y^{w(I)_y}\partial_x & \textit{if} \quad a_1 = 0 \textit{ and } b_t > 0 \\ \mathbf{k}[x]x^{w(I)_x}\partial_y & \textit{if} \quad a_1 > 0 \textit{ and } b_t = 0 \\ \mathbf{k}[y]y^{w(I)_y}\partial_x \oplus \mathbf{k}[x]x^{w(I)_x}\partial_y & \textit{if} \quad a_1 = b_t = 0. \end{cases}$$

Proof. It suffices to prove the cases $a_1 = 0$ and $b_t > 0$ since the other cases are similar.

$$\begin{aligned} y^l\partial_x \in \mathfrak{s}(I) &\Leftrightarrow y^l\partial_x(x^a y^b) \in I \text{ for all } x^a y^b \in I \\ &\Leftrightarrow (0, l) + (a, b) - (1, 0) \in \exp(I) \text{ for all } (a, b) \in \exp(I) \\ &\quad\ \text{such that } a > 0 \\ &\Leftrightarrow (a_i - 1, l + b_i) \in \exp(I) \text{ for all } 1 < i \leq t \\ &\Leftrightarrow l \geq b_{i-1} - b_i \text{ for all } 1 < i \leq t \\ &\Leftrightarrow l \geq w(I)_y. \end{aligned}$$

Hence $\mathfrak{s}(I) = \mathbf{k}[y]y^{w(I)_y}\partial_x$. □

Example 2.4. Consider $I = (y^8, x^2y^6, x^5y^4, x^7y^2, x^8y, x^{12}) \subseteq \mathbf{k}[x, y]$. Then $w(I)_x = 4$, $w(I)_y = 2$ and $T_{\mathbf{A}/\mathbf{k}}(I) = \mathbf{A}\nabla_x \oplus \mathbf{A}\nabla_y \oplus y^2\mathbf{k}[y]\partial_x \oplus x^4\mathbf{k}[x]\partial_y$.

If I is a monomial ideal of $\mathbf{A} = \mathbf{k}[x, y]$ and $l > 0$ is an integer, it is not obvious how $w(I^l)_x$ and $w(I^l)_y$ depend on l. But using Corollary 2.3 and the obvious fact that $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(I^l)$ where equality holds if $I = [I^l : I^{l-1}]$ [3, Remark 3.2.6], we get the following result.

Corollary 2.5. *If $I \subseteq \mathbf{k}[x, y]$ is a monomial ideal, then $w(I)_x \geq w(I^l)_x$ and $w(I)_y \geq w(I^l)_y$, and equality holds when $I = [I^l : I^{l-1}]$.*

We give a guideline on how to use Theorem 2.2 to compute $\mathfrak{s}(I)$. Since $I_{x_j} = (0)$ if $\alpha_{ij} > 0$ for all i, we assume that there exists $1 < k \leq t$ such that (2.4) holds. We can assume that the set $\{\alpha_1 \dots, \alpha_t\}$ is ordered as in (2.4). First compute all $\beta \in \mathbb{Z}^n_{\geq 0}$ satisfying $\beta_j = 0$ and $\alpha_i + \beta - e_j \in \exp(I)$ for each $k \leq i \leq t$. That is, collect all $\mathbb{Z}$-linearly independent vectors β on the hyperplane plane $x_j = \alpha_{ij} - 1$ with initial point $\alpha_i - e_j$ and terminal point at the boundary of $\exp((\{X^{\alpha_l} \mid \alpha_{lj} < \alpha_{ij}\}^t_{l=1})) \cap (x_j = \alpha_{ij} - 1)$. This gives the monomial ideal

$$I_{x_j}(i) := \frac{(X^{\alpha_i - \alpha_{ij}e_j}) \cap (\{X^{\alpha_l - \alpha_{lj}e_j} : \alpha_{lj} < \alpha_{ij}\}^{i-1}_{l=1})}{X^{\alpha_i - \alpha_{ij}e_j}} \subseteq \mathbf{A}_{x_j}. \tag{2.5}$$

Since $X^\beta \in I_{x_j}$ if and only if $\alpha_i + \beta - e_j \in \exp(I)$ for all $k \leq i \leq t$, we get $I_{x_j} = \bigcap_{i=k}^t I_{x_j}(i)$.

Example 2.6. Let $I = (x^4, x^2y^3, xy^4z, z^2) \in \mathbf{A} = \mathbf{k}[x, y, z]$. To compute I_z take the ordering $\alpha_1 = (4, 0, 0) \prec \alpha_2 = (2, 3, 0) \prec \alpha_3 = (1, 4, 1) \prec \alpha_4 = (0, 0, 2)$. We need to compute the ideals $I_z(3)$ and $I_z(4)$ described in (2.5). That is, $I_z(3) = \frac{(xy^4) \cap (x^4, x^2y^3)}{xy^4} = (x)$ and $I_z(4) = \frac{(1) \cap (x^4, x^2y^3, xy^4)}{1} = (x^4, x^2y^3, xy^4)$. This gives $I_z = (x) \cap (xy^4, x^2y^3, x^4) = (xy^4, x^2y^3, x^4)$. We have $T_{\mathbf{A}/\mathbf{k}}(I) = \mathbf{A}\nabla_x \oplus \mathbf{A}\nabla_y \oplus \mathbf{A}\nabla_z \oplus \mathfrak{s}(I)$, where

$$\mathfrak{s}(I) = (y^3z, z^2)\partial_x \oplus (x^2, z^2)\partial_y \oplus (x^4, xy^4, x^2y^3)\partial_z.$$

3. Preservation of the integral closure and multiplier ideals

It is well known that $T_{\mathbf{A}/\mathbf{k}}(I)$ preserves many naturally defined ideals related to I, for any ideal I [3, 4, 6]. We will investigate this question in the case of monomial ideals in relation to the integral closure and the formation of multiplier ideals.

Given a commutative Noetherian ring R and an ideal I of R, an element $f \in R$ is integral over I if f satisfies the equation

$$f^d + g_1 f^{d-1} + \cdots + g_{d-1} f + g_d = 0, \text{ where } g_i \in I^i \text{ and } i = 1, ..., d. \tag{3.1}$$

The integral closure $\bar{I}$ consists of all elements in R which are integral over I. The following lemma is a standard fact.

Lemma 3.1. *Let I be a monomial ideal in $\mathbf{A}$. Then $\bar{I}$ is also a monomial ideal. Furthermore, a monomial X^θ is in $\bar{I}$ if and only if $(X^\theta)^l \in I^l$ for some integer $l > 0$.*

Proof. Let f be integral over I. Applying the action of the torus G on (3.1) we obtain

$$(f^d(t \cdot X) + g_1(t \cdot X) f^{d-1}(t \cdot X) + \cdots + g_{d-1}(t \cdot X) f(t \cdot X) + g_d(t \cdot X) = 0), \tag{3.2}$$

where $t = (t_1, \ldots, t_n) \in G$, $X = (x_1, \ldots, x_n)$ and $t \cdot X = (t_1 x_1, \ldots, t_n x_n)$. Since I^i is a monomial ideal for all $i = 1, \ldots, d$, hence invariant under the action of G, we have $g_i(t \cdot X) \in I^i$ for all $i = 1, \ldots, d$. Thus (3.2) is the integral dependence equation for $f(t \cdot X) \in \mathbf{A}$. Therefore $f(t \cdot X) \in \bar{I}$; hence $\bar{I}$ is invariant under the action of G and it is a monomial ideal. To prove the second statement, assume X^θ satisfies (3.1). Since each I^i is a monomial ideal, considering terms of weight $d\theta$ in (3.1), we obtain an equation of the form

$$(X^\theta)^d + k_1 X^{\theta_1} (X^\theta)^{d-1} + \cdots + k_d X^{\theta_d} = 0$$

for some $X^{\theta_i} \in I^i$, $i = 1, \ldots, d$, and $k_1, \ldots, k_d \in \mathbf{k}$. Some coefficient k_l must be non-zero; thus $(X^\theta)^d = k_0 X^{\theta_l} (X^\theta)^{d-l}$, where $X^{\theta_l} \in I^l$ and $k_0 \in \mathbf{k}$, so $(X^\theta)^l = k_0 X^{\theta_l} \in I^l$. The converse is immediate. □

The inclusion

$$T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\bar{I}) \tag{3.3}$$

is proved in [3, Theorem 3.2.2] using the blow-up of I, for any ideal I. But it is difficult to see how this directly follows from equation (3.1). Here is a direct proof of (3.3) when I is a monomial ideal: By Lemma 2.1, it suffices to prove this for derivations of the form $\delta = X^{\beta_j} \partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(I)$. Let $X^\theta \in \bar{I}$ such that $\theta_j > 0$. Then $(X^\theta)^l \in I^l$ for some $l > 0$. Since $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(I^l)$, we have

$$\delta \in T_{\mathbf{A}/\mathbf{k}}(I^l) \Rightarrow \delta((X^\theta)^l) = l(X^\theta)^{l-1} \delta(X^\theta) \in I^l.$$

Clearly $\delta((X^\theta)^l)$ is a monomial. We have that

$$\delta^2((X^\theta)^l) = \delta[l(X^\theta)^{l-1}\delta(X^\theta)] = l(l-1)(X^\theta)^{l-2}(\delta(X^\theta))^2 + l(X^\theta)^{l-1}\delta^2(X^\theta) \in I^l$$

is also a monomial, split into the sum of two monomials of the same weight. It follows that $(X^\theta)^{l-2}(\delta(X^\theta))^2 \in I^l$. Similarly,

$$\delta[(X^\theta)^{l-2}(\delta(X^\theta))^2] = (l-2)(X^\theta)^{l-3}(\delta(X^\theta)^3) + 2\delta(X^\theta)\delta^2(X^\theta) \in I^l$$

is a monomial; hence $(X^\theta)^{l-3}(\delta(X^\theta))^3 \in I^l$. Continuing in this way, we eventually get $(\delta(X^\theta))^l \in I^l$. Hence by Lemma 3.1 $\delta(X^\theta) \in \bar{I}$.

Let $\mathrm{conv}(I) \subseteq \mathbb{R}^n_{\geq 0}$ be the convex hull of $\exp(I)$. The following lemma immediately follows from (2.3).

Lemma 3.2. *Let I be a monomial ideal and $\beta \in \mathbb{Z}^n_{\geq 0}$ such that $\beta_j = 0$. If $\theta + \beta - e_j \in \exp(I)$ for all $\theta \in \exp(I)$ such that $\theta_j > 0$, then $[\{\beta - e_j\} + \mathrm{conv}(I)] \cap \mathbb{R}^n_{\geq 0} \subseteq \mathrm{conv}(I)$.*

We have $\exp(\bar{I}) = \mathrm{conv}(I) \cap \mathbb{Z}^n_{\geq 0}$ [7, Proposition 7.3.4]. Using this we can easily prove (3.3) using (2.3) and Lemma 3.2, since $[\{\beta - e_j\} + \mathrm{conv}(I)] \cap \mathbb{R}^n_{\geq 0} \subseteq \mathrm{conv}(I) \Rightarrow [\{\beta - e_j\} + \exp(\bar{I})] \cap \mathbb{Z}^n_{\geq 0} \subseteq \exp(\bar{I})$.

Let $X = \mathrm{Spec}\,\mathbf{A}$. By a log resolution of an ideal $I \subset \mathbf{A}$ we mean a proper birational map $f : Y \to X$ with the property that Y is smooth and $f^{-1}(I) = \mathcal{O}_Y(-E)$, where E is an effective Cartier divisor and $E + exc(f)$ has a normal crossing support. Let $r > 0$ be a rational number. We define the multiplier ideal of I with coefficient r as the ideal

$$\mathcal{J}(r \cdot I) = f_* \mathcal{O}_Y(K_{Y/X} - \lfloor rE \rfloor).$$

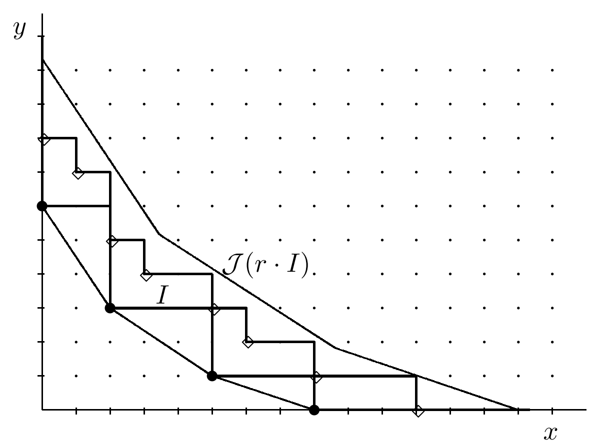
Here $K_{Y/X} = K_Y - f^* K_X$ is the relative canonical bundle and $\lfloor - \rfloor$ is the round down for rational divisors. See [5] for a discussion of $\mathcal{J}(r \cdot I)$. In general it is difficult to compute $\mathcal{J}(r \cdot I)$, but for monomial ideals we have a complete description due to Howald.

Theorem 3.3 ([2])**.** *Let $I \subset \mathbf{A}$ be a monomial ideal and $r > 0$ be a rational number. Then $\mathcal{J}(r \cdot I)$ is a monomial ideal generated by the following set of monomials:*

$$\{X^\theta \mid \theta + (1, 1, \ldots, 1) \in \mathrm{Int}(r \cdot \mathrm{conv}(I)) \cap \mathbb{Z}^n_{\geq 0}\},$$

where $\mathrm{Int}(\mathfrak{C})$ denotes the topological interior of a convex set $\mathfrak{C} \subseteq \mathbb{R}^n$.

Example 3.4. The figure below contains a graphical description of the ideals $I = (y^6, x^2y^3, x^5y, x^8)$ and $\mathcal{J}(r \cdot I) = (y^8, xy^7, x^2y^5, x^3y^4, x^5y^3, x^6y^2, x^8y, x^{10})$ when $r = \frac{31}{18}$.



Proof of Theorem 1.1. Consider the monomial ideal $\mathcal{J}^o(r \cdot I)$ generated by the following set of monomials:

$$\{X^\theta \in \mathbb{Z}^n_{\geq 0} \mid \theta \in \mathrm{Int}(r \cdot \mathrm{conv}(I)) \text{ or } \theta \in \mathcal{B}(r \cdot \mathrm{conv}(I)) \cap (\bigcup_{j=1}^{n} H_j)\},$$

where $\mathcal{B}(\mathfrak{C})$ is the boundary of a convex set $\mathfrak{C} \subseteq \mathbb{R}^n_{\geq 0}$ and H_j is the coordinate hyperplane $x_j = 0$ in $\mathbb{R}^n$. Clearly $X^\theta \in \mathcal{J}(r \cdot I)$ if and only if $x_1 \cdots x_n \cdot X^\theta \in \mathcal{J}^o(r \cdot I)$. It suffices to show that

$$\mathfrak{s}(I) \subseteq \mathfrak{s}(\mathcal{J}^o(r \cdot I)) \subseteq \mathfrak{s}(\mathcal{J}(r \cdot I)).$$

If $\alpha_{ij} > 0$ for all i, then $(0) = I_{x_j} = (\mathcal{J}^o(r \cdot I))_{x_j} \subseteq (\mathcal{J}(r \cdot I))_{x_j}$. Therefore assume that there exists $1 < k \leq t$ such that $\alpha_{kj} = 0$. We first prove the second inclusion. Let $X^\beta \partial_{x_j} \in \mathfrak{s}(\mathcal{J}(r \cdot I))$ and $X^\theta \in \mathcal{J}(r \cdot I)$ such that $\theta_j > 0$. Then $x_1 \cdots x_n \cdot X^\theta \in \mathcal{J}^o(r \cdot I)$ and $X^\beta \partial_{x_j}(x_1 \cdots x_n \cdot X^\theta) = x_1 \cdots x_n \cdot X^{\beta+\theta-e_j} \in \mathcal{J}^o(r \cdot I)$. This implies that $X^{\theta+\beta-e_j} \in \mathcal{J}(r \cdot I)$; hence $X^\beta \partial_{x_j} \in \mathfrak{s}(\mathcal{J}(r \cdot I))$.

To prove the first inclusion it suffices, by Theorem 2.2, to show that

$$\mathcal{C}^I_j(\beta) \Rightarrow \mathcal{C}^{\mathcal{J}^o(r \cdot I)}_j(\beta).$$

Assume $\mathcal{C}^I_j(\beta)$. Then $\theta + \beta - e_j \in \exp(I)$ for all $\theta \in \exp(I)$ such that $\theta_j > 0$. Hence, by Lemma 3.2 $[\{\beta - e_j\} + \operatorname{conv}(I)] \cap \mathbb{R}^n_{\geq 0} \subseteq \operatorname{conv}(I)$. We divide the proof into two cases.

$r \leq 1$: Consider $X^\theta \in \mathcal{J}^o(r \cdot I)$ such that $\theta_j > 0$, so $\theta \in \operatorname{Int}(r \cdot \operatorname{conv}(I))$, and hence $\frac{1}{r}\theta \in \operatorname{Int}(\operatorname{conv}(I))$. By Lemma 3.2, $\frac{1}{r}\theta + \beta - e_j \in \operatorname{Int}(\operatorname{conv}(I))$, and therefore

$$\theta + r\beta - re_j \in \operatorname{Int}(r \cdot \operatorname{conv}(I)). \tag{3.4}$$

Putting $r = 1$ in (3.4) and using the inclusion $\operatorname{Int}(\operatorname{conv}(I)) \subseteq \operatorname{Int}(r \cdot \operatorname{conv}(I))$, we obtain $\theta + \beta - e_j \in \operatorname{Int}(r \cdot \operatorname{conv}(I))$. This implies $\mathcal{C}^{\mathcal{J}^o(r \cdot I)}_j(\beta)$.

$r > 1$: If $\theta \in \operatorname{Int}(r \cdot \operatorname{conv}(I))$, then $\frac{1}{r}\theta \in \operatorname{Int}(\operatorname{conv}(I))$; so by Lemma 3.2, it follows that $\frac{1}{r}\theta + \beta - e_j \in \operatorname{Int}(\operatorname{conv}(I))$, since $\frac{1}{r}\theta_j > 0$. By convexity all the points on the line segment joining $\frac{1}{r}\theta$ and $\frac{1}{r}\theta + \beta - e_j$ belong to $\operatorname{Int}(\operatorname{conv}(I))$. In particular

$$\frac{1}{r}(\theta + \beta - e_j) \in \operatorname{Int}(\operatorname{conv}(I)).$$

Hence $\theta + \beta - e_j \in \operatorname{Int}(r \cdot \operatorname{conv}(I))$, implying $\mathcal{C}^{\mathcal{J}^o(r \cdot I)}_j(\beta)$. □

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