PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 9, September 2009, Pages 2969–2978
S 0002-9939(09)09925-0
Article electronically published on April 23, 2009

ON THE SLICE MAP PROBLEM FOR $H^\infty(\Omega)$ AND THE REFLEXIVITY OF TENSOR PRODUCTS

MICHAEL DIDAS

(Communicated by Marius Junge)

This paper is dedicated to Christine and Tim

ABSTRACT. Let $\Omega \subset \mathbb{C}^n$ be a bounded convex or strictly pseudoconvex open subset. Given a separable Hilbert space K and a weak* closed subspace $\mathcal{T} \subset B(K)$, we show that the space $H^\infty(\Omega, \mathcal{T})$ of all bounded holomorphic $\mathcal{T}$-valued functions on Ω possesses the tensor product representation $H^\infty(\Omega, \mathcal{T}) = H^\infty(\Omega)\overline{\otimes}\mathcal{T}$ with respect to the normal spatial tensor product. As a consequence we deduce that $H^\infty(\Omega)$ has property S_σ. This implies that, if $S \in B(H)^n$ is a subnormal tuple of class $\mathbb{A}$ on a strictly pseudoconvex or bounded symmetric domain and $T \in B(K)^m$ is a commuting tuple satisfying $\mathrm{AlgLat}(T) = \mathcal{A}_T$ (where $\mathcal{A}_T$ denotes the unital dual operator algebra generated by T), then the tensor product tuple $(S \otimes 1, 1 \otimes T)$ is reflexive.

1. Property S_σ and the reflexivity of tensor products

Given a complex Hilbert space H and an arbitrary family $\mathcal{S} \subset B(H)$ of bounded linear operators, we define $\mathcal{W}_\mathcal{S}$ to be the smallest WOT-closed subalgebra of $B(H)$ containing $\mathcal{S}$ and the identity 1_H. As usual, we write $\mathrm{Lat}(\mathcal{S})$ for the set of all closed subspaces of H that are invariant under each member of $\mathcal{S}$ and we define $\mathrm{AlgLat}(\mathcal{S})$ to be the set of all operators $C \in B(H)$ with $\mathrm{Lat}(C) \supset \mathrm{Lat}(\mathcal{S})$. Obviously $\mathrm{AlgLat}(\mathcal{S})$ is a WOT-closed unital subalgebra of $B(H)$ containing $\mathcal{S}$ (and hence $\mathcal{W}_\mathcal{S}$). The family $\mathcal{S}$ is called *reflexive* if the identity

$$\mathrm{AlgLat}(\mathcal{S}) = \mathcal{W}_\mathcal{S}$$

holds. For many concrete examples of reflexive systems $\mathcal{S}$, the algebra $\mathcal{W}_\mathcal{S}$ coincides with the unital dual operator algebra

$$\mathcal{A}_\mathcal{S} = \overline{\mathrm{Alg}(\mathcal{S} \cup \{1_H\})}^{w^*} \subset B(H)$$

generated by $\mathcal{S}$, e.g. if $\mathcal{S} = \{S\}$ consists of a single von Neumann operator $S \in B(H)$, and hence in particular if S is subnormal (Conway and Dudziak [1], Corollary 3.2), or if $\mathcal{S}$ is the set of all distinct components of a von Neumann n-tuple $(S_1, \dots, S_n) \in B(H)^n$ of class $\mathbb{A} \cap \mathbb{A}_{1,\aleph_0}$ on a strictly pseudoconvex domain

Received by the editors February 16, 2007, and, in revised form, August 26, 2007.
2000 *Mathematics Subject Classification.* Primary 47A15, 47B20, 47L45; Secondary 46B28, 46K50.
Key words and phrases. Property S_σ, strictly pseudoconvex domains, subnormal tuples, tensor products.

(see [2], Corollary 4.4.4). In what follows, such a family $\mathcal{S} \subset B(H)$ which is reflexive and satisfies the identity $\mathcal{A}_\mathcal{S} = \mathcal{W}_\mathcal{S}$ will be called *strongly reflexive*, for short. Observe that a family $\mathcal{S}$ of operators is strongly reflexive if and only if the equality $\text{AlgLat}(\mathcal{S}) = \mathcal{A}_\mathcal{S}$ holds or, equivalently, if $\text{AlgLat}(\mathcal{A}_\mathcal{S}) = \mathcal{A}_\mathcal{S}$.[1] Clearly, the family $\mathcal{S}$ is strongly reflexive if and only if so is the dual operator algebra $\mathcal{A}_\mathcal{S}$.

For the rest of the paper, let us simplify the notation in the following way: Given an operator tuple $S \in B(H)^n$ we use the same symbol S to denote the subset of $B(H)$ consisting of the different components of the tuple. For example, if $S = (S_1, \dots, S_n) \in B(H)^n$, then we simply write $\text{AlgLat}(S)$ ($\mathcal{A}_S$, resp.) when we really mean $\text{AlgLat}(\mathcal{S})$ ($\mathcal{A}_\mathcal{S}$, resp.), where the elements of the set $\mathcal{S} \subset B(H)$ are the different components of $(S_1, \dots, S_n)$.

If two commuting Hilbert-space multi-operators $S \in B(H)^n$ and $T \in B(K)^m$ are (strongly) reflexive, then it is natural to ask for the (strong) reflexivity of the tensor product tuple

$$(S \otimes 1, 1 \otimes T) \in B(H \otimes K)^{n+m}.$$

We will focus on the strong-reflexivity version of this question, namely if $\mathcal{A}_{(S\otimes 1, 1\otimes T)}$ is strongly reflexive whenever $\mathcal{A}_S$ and $\mathcal{A}_T$ are also. Answering this question turns out to be equivalent to solving a reflexivity problem for tensor products of dual algebras. To point this out, we have to recall that the normal spatial tensor product of two arbitrary weak* closed subspaces $\mathcal{S} \subset B(H)$ and $\mathcal{T} \subset B(K)$ is defined by

$$\mathcal{S} \overline{\otimes} \mathcal{T} = \overline{\mathcal{S} \otimes \mathcal{T}}^{w^*} \subset B(H \otimes K),$$

where $\mathcal{S} \otimes \mathcal{T} = LH\{x \otimes y : x \in \mathcal{S}, y \in \mathcal{T}\}$ stands for the algebraic tensor product of $\mathcal{S}$ and $\mathcal{T}$. It is a simple observation that the dual algebra generated by any tensor product tuple splits with respect to the normal spatial tensor product.

Lemma 1.1. *For arbitrary commuting tuples $S \in B(H)^n$ and $T \in B(K)^m$, we have $\mathcal{A}_{(S\otimes 1, 1\otimes T)} = \mathcal{A}_S \overline{\otimes} \mathcal{A}_T$.*

Proof. Since the set on the right-hand side is a unital dual operator algebra containing $S \otimes 1$ and $1 \otimes T$, the inclusion "$\subset$" follows by the minimality of the algebra on the left. To prove the non-trivial inclusion "$\supset$" first note that all elementary tensors $A \otimes B$ with $A \in \mathbb{C}[S]$ and $B \in \mathbb{C}[T]$ are clearly contained in the set on the left-hand side. The weak* continuity of the mapping $B(H) \to B(H \otimes K)$, $A \mapsto A \otimes B$, for each fixed $B \in B(K)$, therefore implies that the set of all elementary tensors $A \otimes B$ with $A \in \mathcal{A}_S$ and $B \in \mathbb{C}[T]$ is contained in $\mathcal{A}_{(S\otimes 1, 1\otimes T)}$. By the same argument with the roles of the first and second factor exchanged, we may also replace the condition $B \in \mathbb{C}[T]$ by $B \in \mathcal{A}_T$. Passing to the linear hull, we deduce that the algebraic tensor product $\mathcal{A}_S \otimes \mathcal{A}_T$ is contained in $\mathcal{A}_{(S\otimes 1, 1\otimes T)}$. This observation finishes the proof. □

The reflexivity problem for tensor products of dual operator algebras has been studied intensively in a series of papers by Jon Kraus (see e.g. [6] and [7]). Kraus showed that the tensor product of two strongly reflexive dual operator algebras

[1]It should be remarked that the notion of reflexivity is not uniformly defined in the literature. Operator algebraists often use the identity $\text{AlgLat}(\mathcal{A}_\mathcal{S}) = \mathcal{A}_\mathcal{S}$ as the definition for the reflexivity of the operator algebra $\mathcal{A}_\mathcal{S}$ (see e.g. [7]) while our definition, which is commonly used in operator theory, says that the family $\mathcal{A}_\mathcal{S}$ is reflexive if the weaker condition $\text{AlgLat}(\mathcal{A}_\mathcal{S}) = \mathcal{W}_{\mathcal{A}_\mathcal{S}} = \mathcal{W}_\mathcal{S}$ holds (being equivalent to the reflexivity of the family $\mathcal{S}$ itself). Therefore we introduced the new notion of strong reflexivity hoping to avoid a misunderstanding.

remains strongly reflexive if one of the factors satisfies a certain (natural) splitting property (called S_σ).

Towards a precise formulation of property S_σ we have to recall the definition of Tomiyama's slice maps. Let $C^1(H)$ denote the space of all trace-class operators on the Hilbert space H. Recall that we may identify $B(H)$ with the dual space of $C^1(H)$ via the bilinear form $C^1(H) \times B(H) \to \mathbb{C}$, $(C, A) \mapsto \mathrm{trace}(CA)$. The right slice map R_C associated with a given element $C \in C^1(H)$ now can be defined as the adjoint of the continuous linear map

$$(R_C)_* : C^1(K) \longrightarrow C^1(H \otimes K), \qquad D \mapsto C \otimes D.$$

The mapping R_C obtained in this way is the unique weak* continuous linear operator

$$R_C : B(H \otimes K) \to B(K) \quad \text{satisfying} \quad R_C(A \otimes B) = \langle C, A\rangle B$$

for every $A \in B(H)$ and $B \in B(K)$, where $\langle C, A\rangle = \mathrm{trace}(CA)$. Given $D \in C^1(K)$, the assignment $L_D(A \otimes B) = \langle D, B\rangle A$, where $A \otimes B \in B(H \otimes K)$, can in a completely analogous manner be extended to a weak* continuous linear map $L_D : B(H \otimes K) \to B(H)$, called the left slice map induced by D. For further properties of slice maps, see Kraus [6] and the references therein. Let us, for later use, just mention the intertwining property explicitly, which can be easily verified by the reader and (in the context of right slice maps) says that

$$R_C((1 \otimes V)X(1 \otimes W)) = V R_C(X) W$$

whenever $V, W \in B(K)$ and $X \in B(H \otimes K)$. Analogously, for the left slice maps we have $L_D((V \otimes 1)X(W \otimes 1)) = V L_D(X) W$ for $V, W \in B(H)$ and $X \in B(H \otimes K)$ (see the formulas (1.3) and (1.4) in Kraus [6]).

In order to define property S_σ, we associate with each pair of weak* closed subspaces $\mathcal{S} \subset B(H)$ and $\mathcal{T} \subset B(K)$ the so-called Fubini product

$$F(\mathcal{S}, \mathcal{T}) = \left\{ A \in B(H \otimes K) \,\middle|\, \begin{matrix} R_C(A) \in \mathcal{T} \quad \text{and} \quad L_D(A) \in \mathcal{S} \\ \text{whenever } C \in C^1(H) \text{ and } D \in C^1(K) \end{matrix} \right\},$$

which is easily seen to be a weak* closed subspace of $B(H \otimes K)$ containing $\mathcal{S} \overline{\otimes} \mathcal{T}$. Now following Kraus [7] we say that a weak* closed subspace $\mathcal{S} \subset B(H)$ satisfies property S_σ if the subspace tensor product formula

$$F(\mathcal{S}, \mathcal{T}) = \mathcal{S} \overline{\otimes} \mathcal{T}$$

holds whenever $\mathcal{T} \subset B(K)$ is a weak* closed subspace of $B(K)$ for any Hilbert space K. As shown by Kraus in [6], it suffices to consider the case where K is separable and infinite dimensional.

For later reference we remark that the Fubini product can be expressed using right slice maps only. Theorem 1.9 in [6] guarantees that $B(K)$ has property S_σ and, consequently, we have $F(\mathcal{S}, \mathcal{T}) \subset F(\mathcal{S}, B(K)) = \mathcal{S} \overline{\otimes} B(K)$. Using this and the fact that $L_D(\mathcal{S} \overline{\otimes} B(K)) \subset \mathcal{S}$, we obtain the desired representation

$$F(\mathcal{S}, \mathcal{T}) = \{A \in \mathcal{S} \overline{\otimes} B(K) : R_C(A) \in \mathcal{T} \text{ for all } C \in C^1(H)\}.$$

Let us now turn back to the reflexivity problem for tensor product tuples. In Section 3 of [6], Kraus settles a link between property S_σ and the reflexivity of tensor products which, in our context, reads as follows.

Proposition 1.2 (Kraus). *Let $S \in B(H)^n$ and $T \in B(K)^m$ be commuting tuples of bounded linear Hilbert-space operators which are strongly reflexive in the sense that* $\mathrm{AlgLat}(S) = \mathcal{A}_S$ *and* $\mathrm{AlgLat}(T) = \mathcal{A}_T$*. If $\mathcal{A}_S$ has property S_σ, then the tensor product tuple $(S \otimes 1, 1 \otimes T) \in B(H \otimes K)^{n+m}$ satisfies*

$$\mathrm{AlgLat}(S \otimes 1, 1 \otimes T) = F(\mathcal{A}_S, \mathcal{A}_T) = \mathcal{A}_{(S\otimes 1, 1 \otimes T)}.$$

Proof. Let $M \in \mathrm{Lat}(T)$ and let $P \in B(K)$ denote the orthogonal projection with range M. Since $H \otimes M$ is $(S \otimes 1, 1 \otimes T)$-invariant, an operator $A \in \mathrm{AlgLat}(S \otimes 1, 1 \otimes T)$ clearly satisfies $(1 \otimes P)A(1 \otimes P) = A(1 \otimes P)$. Using the intertwining property of the right slice-map R_C, we deduce that

$$PR_C(A)P = R_C((1 \otimes P)A(1 \otimes P)) = R_C(A(1 \otimes P)) = R_C(A)P$$

and hence $R_C(A) \in \mathrm{AlgLat}(T)$, for every $C \in C^1(H)$. In a completely analogous fashion it can be shown that $L_D(A) \in \mathrm{AlgLat}(S)$ $(D \in C^1(K))$. Now, a look at the definition of the Fubini product immediately yields the inclusion

$$\mathrm{AlgLat}(S \otimes 1, 1 \otimes T) \subset F(\mathrm{AlgLat}(S), \mathrm{AlgLat}(T)),$$

where, by hypothesis, the right-hand side can be written as $F(\mathcal{A}_S, \mathcal{A}_T)$. Using property S_σ we further obtain that $F(\mathcal{A}_S, \mathcal{A}_T) = \mathcal{A}_S \overline{\otimes} \mathcal{A}_T$. By Lemma 1.1, the latter space coincides with $\mathcal{A}_{(S\otimes 1, 1\otimes T)}$, as desired. □

Due to Kraus [7], Theorem 4.1, we know that the dual operator algebra $\mathcal{A}_S$ generated by a single subnormal operator $S \in B(H)$ has property S_σ. By a classical theorem of Olin and Thomson, $\mathcal{A}_S$ is strongly reflexive in this case. Hence Proposition 1.2 applies to every subnormal operator S. In the special case that S is the unilateral shift, i.e. $S = M_z$ on the Hardy space $H^2(\mathbb{D})$ over the unit disc, short proofs of the above proposition using elementary arguments have been given by M. Ptak ([9], Theorem $2'$) and J.E. McCarthy ([8], Lemma 6).

Our aim is to extend Kraus' result to the setting of subnormal tuples $S \in B(H)^n$ of class $\mathbb{A}$ on sufficiently nice sets Ω for which $\mathcal{A}_S \cong H^\infty(\Omega)$. The dual algebra generated by a tensor product tuple of the form $(S \otimes 1, 1 \otimes T)$ then corresponds to some space of vector-valued H^∞-functions. The next section is therefore devoted to this kind of function space.

2. A tensor product formula for $H^\infty(\Omega, \mathcal{T})$

From now on suppose that $\emptyset \neq \Omega \subset X$ is either a bounded convex open subset of $X = \mathbb{C}^n$ or a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$. By the latter we mean that there exist an open subset $U \subset X$ containing the boundary $\partial\Omega$ and a strictly plurisubharmonic C^2-function $\rho : U \to \mathbb{R}$ such that $\Omega \cap U = \{z \in U : \rho(z) < 0\}$. To abbreviate the description of these two cases, let us simply say in the following that Ω is a "bounded convex or strictly pseudoconvex open set".

Let us fix such a set $\Omega \subset X$ now. As a relatively compact submanifold of $\mathbb{C}^n$, the set Ω carries a natural volume measure. After normalization and trivial extension we obtain a Borel probability measure λ on $\overline{\Omega}$ with $\lambda(\Omega) = 1$, $\lambda(\partial\Omega) = 0$ and the property that $\lambda(W) > 0$ for every non-empty open set $W \subset \Omega$. By $H^\infty(\Omega)$ we denote the Banach algebra of all bounded holomorphic functions on Ω equipped with the supremum norm $\|f\|_{\infty,\Omega} = \sup_{z\in\Omega} |f(z)|$. As a consequence of Montel's theorem, the isometric embedding $H^\infty(\Omega) \hookrightarrow L^\infty(\lambda)$ has weak* closed range and thus turns $H^\infty(\Omega)$ into a dual algebra. Via the representation

$$\gamma : L^\infty(\lambda) \to B(L^2(\lambda)), \qquad \varphi \mapsto M_\varphi \qquad \text{with } M_\varphi f = \varphi \cdot f \quad (f \in L^2(\lambda)),$$

which is a weak* continuous isometric $*$-homomorphism, we may identify $H^\infty(\Omega)$ with the dual operator algebra

$$\mathscr{H}^\infty(\Omega) = \gamma(H^\infty(\Omega)) = \{M_\varphi : \varphi \in H^\infty(\Omega)\} \subset B(L^2(\lambda))$$

of all multiplication operators with H^∞-symbol.

Towards the vector-valued case, fix a separable Banach space E and consider the Banach space $L^\infty(\lambda, E')$ of all equivalence classes of bounded weak*-measurable functions $f : \Omega \to E'$ equipped with the essential supremum norm. Via the bilinear form $\langle g, f\rangle = \int_\Omega \langle g, f\rangle d\lambda$, we can identify $L^\infty(\lambda, E')$ with the dual of the space $L^1(\lambda, E)$ of all equivalence classes of Bochner-integrable functions $g : \Omega \to E$ with $\|g\|_{1,\lambda} = \int_\Omega \|g\| d\lambda < \infty$. In analogy with the $\mathbb{C}$-valued case, the Banach space of all E'-valued bounded holomorphic functions $H^\infty(\Omega, E')$ with the supremum norm can, via the canonical embedding, be thought of as a weak*-closed subspace of $L^\infty(\lambda, E')$ (for details, see [3], Lemma 5.3). It should be mentioned that a sequence (f_k) in $H^\infty(\Omega, E')$ is a weak* zero sequence if and only if (f_k) is norm-bounded and $(f_k(z))$ is a weak* zero sequence in E' for every $z \in \Omega$.

If K is a separable Hilbert space and $\mathcal{T} \subset B(K)$ is a weak* closed subspace, then $H^\infty(\Omega, \mathcal{T})$ and $L^\infty(\lambda, \mathcal{T})$ fit into the above context, since $\mathcal{T}$ can then be identified with the dual space of the separable Banach space $E = C^1(K)/{}^\perp\mathcal{T}$. If, in addition, $\mathcal{T}$ is a subalgebra of $B(K)$, then $H^\infty(\Omega, \mathcal{T})$ and $L^\infty(\lambda, \mathcal{T})$ are dual algebras in a canonical way. If $\mathcal{T} \subset B(K)$ is even a W^*-algebra, then so is $L^\infty(\lambda, \mathcal{T})$.

Again in analogy with the scalar-valued case we obtain a representation of the W^*-algebra $L^\infty(\lambda, B(K))$ via the weak* continuous and isometric $*$-homomorphism

$$\Gamma : L^\infty(\lambda, B(K)) \to B(L^2(\lambda) \otimes K)), \quad \varphi \mapsto M_\varphi,$$

where the operator M_φ acts on the space $L^2(\lambda, K) \cong L^2(\lambda) \otimes K$ as multiplication with symbol φ.

Proposition 2.1. *Let Ω be a bounded convex or strictly pseudoconvex open set, and let $\mathcal{T} \subset B(K)$ be a weak* closed subspace. Then there is a unique dual algebra isomorphism*

$$\Gamma_{\mathcal{T}} : H^\infty(\Omega, \mathcal{T}) \longrightarrow \mathscr{H}^\infty(\Omega)\overline{\otimes}\mathcal{T} \subset B(L^2(\lambda) \otimes K)$$

mapping $\varphi \cdot T$ to $M_\varphi \otimes T$ whenever $\varphi \in H^\infty(\Omega)$ and $T \in \mathcal{T}$. In fact, $\Gamma_{\mathcal{T}}$ can be obtained by restricting the map Γ from above to $H^\infty(\Omega, \mathcal{T})$.

Towards a proof of this result, consider the set

$$M = LH\{\varphi \cdot T : \varphi \in H^\infty(\Omega),\, T \in \mathcal{T}\}$$

of elementary functions. From the properties of the map Γ described above and the trivial fact that $\Gamma(M) \subset \mathscr{H}^\infty(\Omega) \otimes \mathcal{T}$ we deduce that the assertion follows as soon as we know that M is dense in $H^\infty(\Omega, \mathcal{T})$.

To realize this claim we first derive an intermediate result which is interesting in its own right. In the following proposition $\mathscr{O}(\overline{\Omega}, E')$ stands for the space of all E'-valued functions that are holomorphic in some open neighbourhood of $\overline{\Omega}$ in $\mathbb{C}^n$.

Proposition 2.2. *Suppose that Ω is a convex or strictly pseudoconvex open set and that E is a separable complex Banach space. Then*

$$\mathscr{O}(\overline{\Omega}, E')|\Omega \subset H^\infty(\Omega, E')$$

is sequentially weak dense. More precisely, there is a constant $c \geq 1$, such that every function f in the unit ball of $H^\infty(\Omega, E')$ can be approximated (with respect to the weak* topology) by a sequence (f_k) with $f_k \in \mathscr{O}(\overline{\Omega}, E')|\Omega$ and $\|f_k\|_{\infty,\Omega} \leq c$ $(k \geq 1)$.*

Proof. Elementary arguments show that the assertion holds in the convex case. (Translate Ω in such a way that it contains the origin and use radial limits.) To treat the strictly pseudoconvex case we use the embedding theorem of Fornaess saying that, up to a biholomorphic identification, the set Ω can be represented as the intersection $\Omega \cong Y \cap C$ of some closed complex submanifold $Y \subset \mathbb{C}^m$ and a C^2-strictly convex open subset $C \subset \mathbb{C}^m$ for some suitably chosen $m \geq 1$ (see Theorem 10 in [4]).

Theorem 5.11 in [3] says that our assertion holds in the special case where $E' = H^\infty(\Omega_2)$. Following the proof of the cited theorem (setting there $D_1 = \Omega$ and replacing E by E'), we deduce that it suffices to show that the restriction map

$$H^\infty(C, E') \longrightarrow H^\infty(Y \cap C, E')$$

is onto. Towards this end, note that the mapping

$$B : E \times H^\infty(Y \cap C, E') \to H^\infty(Y \cap C), \qquad (x, f) \mapsto \langle x, f(\cdot)\rangle$$

is (norm-) continuous and bilinear.

By the remark following Theorem 4.11.1 in Henkin-Leiterer [5], there is a bounded linear extension operator

$$\theta : H^\infty(Y \cap C) \to H^\infty(C).$$

In order to lift this operator to the E'-valued setting, we start with an arbitrary function $f \in H^\infty(Y \cap C, E')$. By defining

$$\hat{f}(z) : E \to \mathbb{C}, \qquad x \mapsto \mathcal{E}_z \theta B(x, f) \qquad (\text{for every } z \in C),$$

where $\mathcal{E}_z : H^\infty(\Omega, E') \to E'$ denotes the (weak* continuous) point evaluation at z, we obtain a family of vectors $\hat{f}(z) \in E'$ $(z \in C)$ satisfying

$$\langle x, \hat{f}(z)\rangle = \theta(\langle x, f(\cdot)\rangle)(z) \qquad (x \in E,\, z \in C).$$

The function $\hat{f} : C \to E'$ constructed this way clearly extends f and is weak* holomorphic and hence holomorphic. Since the estimate

$$\|\hat{f}(z)\| \leq \|\mathcal{E}_z\| \|\theta\| \sup_{\|x\| \leq 1} \|B(x, f)\|_{\infty,\Omega} \leq \|\theta\| \|f\|_{\infty,\Omega} \qquad (z \in C)$$

holds, the assignment

$$\hat{\theta} : H^\infty(Y \cap C, E') \to H^\infty(C, E'), \qquad f \mapsto \hat{f}$$

yields a bounded linear extension operator in the vector-valued case. In particular, the corresponding restriction $H^\infty(C, E') \to H^\infty(Y \cap C, E')$ is onto, as desired. Hence the assertion of the proposition holds with approximation constant $c = \|\theta\|$. □

Now we are able to finish the proof of Proposition 2.1. We use the notation $\mathscr{O}(W)$ ($\mathscr{O}(W,\mathcal{T})$, resp.) to denote the set of all $\mathbb{C}$-valued ($\mathcal{T}$-valued, resp.) holomorphic functions on an open set $W \subset X$.

Proof of Proposition 2.1. As pointed out above, it remains to check that the set $M = LH\{\varphi \cdot T : \varphi \in H^\infty(\Omega), T \in \mathcal{T}\}$ is weak* dense in $H^\infty(\Omega,\mathcal{T})$. Towards this end, fix an arbitrary function $f \in H^\infty(\Omega,\mathcal{T})$. Then, by the preceding proposition, there is a sequence (f_k) in $\mathscr{O}(\overline{\Omega},\mathcal{T})|\Omega$ such that $f_k \xrightarrow{k} f$ pointwise weak* on Ω and $\sup_k \|f_k\|_{\infty,\Omega} \leq c\|f\|_{\infty,\Omega}$. For each $k \geq 1$ we may choose an open neighborhood U_k of $\overline{\Omega}$ in such a way that f_k can be extended to a function in $\mathscr{O}(U_k,\mathcal{T})$, again denoted by f_k. In view of the well-known identification $\mathscr{O}(U_k,\mathcal{T}) \cong \mathscr{O}(U_k)\widehat{\otimes}\mathcal{T}$, there are elementary functions

$$g_k = \sum_{i=1}^{r_k} h_i^{(k)} \otimes A_i^{(k)} \in M \qquad \text{with } h_i^{(k)} \in H^\infty(\Omega),\ A_i^{(k)} \in \mathcal{T} \quad (k \geq 1)$$

satisfying $\|g_k - f_k\|_{\infty,\Omega} < 1/k$. The sequence $(g_k)_k$ is norm-bounded and converges to f pointwise weak*. Therefore (g_k) is the desired sequence in M approximating f in the weak* topology of $H^\infty(\Omega,\mathcal{T})$. □

3. Property S_σ for $\mathscr{H}^\infty(\Omega)$ and applications

In the special case that Ω is the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, Kraus remarked in [7] (Example 3.3, p. 399) that $H^\infty(\mathbb{D})$ has property S_σ. Since Kraus' original proof involves W^*-dynamical systems and makes use of the group structure of $\partial\mathbb{D} = \mathbb{T}$, it cannot be extended to our situation. Via the tensor product formula established in Proposition 2.1, we show directly and with elementary arguments that $\mathscr{H}^\infty(\Omega)$ satisfies property S_σ.

Theorem 3.1. *For every bounded convex or strictly pseudoconvex open set $\Omega \subset X$, the dual operator algebra $\mathscr{H}^\infty(\Omega)$ has property S_σ.*

Proof. Fix an arbitrary separable complex Hilbert space K and a weak* closed subspace $\mathcal{T} \subset B(K)$. We have to show that

$$F(\mathscr{H}^\infty(\Omega),\mathcal{T}) \subset \mathscr{H}^\infty(\Omega)\overline{\otimes}\mathcal{T}.$$

Towards this end, we start with an arbitrary element $A \in F(\mathscr{H}^\infty(\Omega),\mathcal{T})$, which means by definition that

$$A \in \mathscr{H}^\infty(\Omega)\overline{\otimes}B(K) \quad \text{and} \quad R_C(A) \in \mathcal{T} \quad \text{for all} \quad C \in C^1(L^2(\lambda)).$$

Proposition 2.1 says that, using the canonical weak* continuous isometry

$$\Gamma : L^\infty(\lambda, B(K)) \to B(L^2(\lambda) \otimes K),$$

the operator A can be written as $A = \Gamma(f_A)$ with some bounded holomorphic function $f_A \in H^\infty(\Omega, B(K))$. Suppose for a moment that f_A takes its values in $\mathcal{T}$ only. Then we could again use Proposition 2.1 to finish the proof with the observation that

$$\Gamma(f_A) \in \Gamma(H^\infty(\Omega,\mathcal{T})) \subset \mathscr{H}^\infty(\Omega)\overline{\otimes}\mathcal{T}.$$

Therefore, our aim is to show that $f_A(z) \in \mathcal{T}$ for all $z \in \Omega$.

In view of the dual algebra isomorphism $\gamma : H^\infty(\Omega) \to \mathscr{H}^\infty(\Omega)$, $\varphi \mapsto M_\varphi$, each $g \in L^1(\lambda)$ induces a weak* continuous linear form

$$\mathscr{H}^\infty(\Omega) \to \mathbb{C}, \quad M_\varphi \mapsto \langle g, \varphi\rangle = \int_\Omega g\varphi d\lambda,$$

which, by the Hahn-Banach theorem, can be extended from $\mathscr{H}^\infty(\Omega)$ to a weak* continuous linear form on all of $B(L^2(\lambda))$. Hence via trace-duality we find an operator $C_g \in C^1(L^2(\lambda))$ satisfying

$$\langle C_g, M_\varphi\rangle = \int_\Omega g\varphi d\lambda \qquad (\varphi \in H^\infty(\Omega)).$$

From the very definition of the right slice map associated with C_g we deduce that, for every $D \in C^1(K)$, every $\varphi \in H^\infty(\Omega)$ and every $T \in B(K)$, the identity

$$\langle D, R_{C_g}(\Gamma(\varphi T))\rangle = \langle D, \langle C_g, M_\varphi\rangle T\rangle = \langle D, \left(\int_\Omega g\varphi d\lambda\right) T\rangle = \int_\Omega g\langle D, \varphi T\rangle d\lambda$$

holds. Since, according to Proposition 2.1, the linear span of $H^\infty(\Omega) \cdot B(K)$ is weak* dense in $H^\infty(\Omega, B(K))$, this implies that

$$\langle D, R_{C_g}(\Gamma(f_A))\rangle = \int_\Omega g\langle D, f_A(\cdot)\rangle d\lambda \qquad (D \in C^1(L^2(\lambda)),\, g \in L^1(\lambda)).$$

By hypothesis, we have $R_C(A) \in \mathcal{T}$ for every $C \in C^1(L^2(\lambda))$, and consequently

$$0 = \langle D, R_{C_g}\Gamma(f_A)\rangle = \int_\Omega g\langle D, f_A(\cdot)\rangle d\lambda \qquad (D \in {}^\perp\mathcal{T},\, g \in L^1(\lambda)).$$

From this we conclude that the scalar-valued H^∞-function $\langle D, f_A(\cdot)\rangle$ vanishes identically on Ω for every $D \in {}^\perp\mathcal{T}$. But this means precisely that

$$f_A(z) \in ({}^\perp\mathcal{T})^\perp = \overline{\mathcal{T}}^{w^*} = \mathcal{T} \qquad (z \in \Omega),$$

as was to be shown. □

For the rest of this article, we specialize to the case where $\Omega \subset \mathbb{C}^n$ is a bounded symmetric and circled domain or a relatively compact strictly pseudoconvex open subset $\Omega \subset X$ of a Stein submanifold $X \subset \mathbb{C}^n$ and assume that the closure $\overline{\Omega} \subset \mathbb{C}^n$ is polynomially convex.

Fix a subnormal tuple $S \in B(H)^n$ of class $\mathbb{A}$ over Ω. This means by definition that S possesses an extension to a commuting tuple $\hat{S} \in B(\hat{H})^n$ of normal operators on some Hilbert space $\hat{H} \supset H$ and that there exists an isometric and weak* continuous functional calculus $\Phi : H^\infty(\Omega) \to B(H)$ for S. Furthermore, the normal extension $\hat{S}$ can be chosen to be minimal in the sense that if $M \subset \hat{H}$ is any reducing subspace for $\hat{S}$ containing H, then $M = \hat{H}$.

The spectral theory for the minimal normal extension $\hat{S}$ of S then yields a regular Borel probability measure μ on $\overline{\Omega}$ having the following properties (see e.g. [3]):

(a) There is an isometric and weak* continuous algebra homomorphism

$$r_\mu : H^\infty(\Omega) \to L^\infty(\mu)$$

extending the canonical map $\mathbb{C}[z] \to L^\infty(\mu)$, $p \mapsto [p|\overline{\Omega}]$. In other words, μ is a faithful Henkin measure.

(b) The normal tuple $\hat{S}$ possesses an isometric, weak* continuous and involutive functional calculus

$$\Psi : L^\infty(\mu) \to B(\hat{H}).$$

The mappings Φ, r_μ and Ψ will be used now to show that $\mathcal{A}_S$ has property S_σ.

Corollary 3.2. *Suppose that Ω is a bounded symmetric and circled domain in $\mathbb{C}^n$ or a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$ possessing polynomially convex closure $\overline{\Omega} \subset \mathbb{C}^n$. Then the dual algebra $\mathcal{A}_S$ generated by a subnormal tuple $S \in B(H)^n$ of class $\mathbb{A}$ over Ω has property S_σ.*

Proof. From the hypothesis that Ω has polynomially convex closure, we deduce that the polynomials $\mathbb{C}[z]|\overline{\Omega}$ are dense in $\mathscr{O}(\overline{\Omega})$ with respect to the supremum norm $\|\cdot\|_{\infty,\overline{\Omega}}$. Combining this with the assertion of Proposition 2.2 (with $E' = \mathbb{C}$) we see that $\mathbb{C}[z]|\Omega \subset H^\infty(\Omega)$ is weak* dense. Consequently, we have $\Phi(H^\infty(\Omega)) = \mathcal{A}_S$ and $\Psi \circ r_\mu(H^\infty(\Omega)) = \mathcal{A}_{\hat{S}}$. In particular, the composition $\Phi \circ r_\mu^{-1} \circ \Psi^{-1}|\mathcal{A}_{\hat{S}}$ yields a dual algebra isomorphism

$$\tau : \mathcal{A}_{\hat{S}} \to \mathcal{A}_S \quad \text{with} \quad \Psi(r_\mu(f)) \mapsto \Phi(f) \qquad (f \in H^\infty(\Omega)).$$

In view of the identity

$$\Phi(f) = \Psi(r_\mu(f))|H,$$

extending from $f \in \mathbb{C}[z]$ to all of $H^\infty(\Omega)$ by a weak* density argument, the mapping τ is nothing other than the restriction map $\tau(A) = A|H$, for $A \in \mathcal{A}_{\hat{S}}$. This shows that τ is completely bounded. Since the range of τ^{-1} is contained in the abelian C^*-algebra $W^*(\hat{S})$, the inverse of τ is also completely bounded.

Next observe that the two isometric and weak* continuous embeddings

$$\psi : H^\infty(\Omega) \xrightarrow{\Psi \circ r_\mu} \mathcal{A}_{\hat{S}} \subset W^*(\hat{S}) \quad \text{and} \quad \gamma_0 : H^\infty(\Omega) \xrightarrow{\gamma} \mathscr{H}^\infty(\Omega) \subset W^*(M_z)$$

both induce the same operator space structure on $H^\infty(\Omega)$ as their ranges are both contained in abelian C^*-algebras. The composition

$$\Delta : \mathscr{H}^\infty(\Omega) \xrightarrow{\gamma_0^{-1}} H^\infty(\Omega) \xrightarrow{\psi} \mathcal{A}_{\hat{S}} \xrightarrow{\tau} \mathcal{A}_S$$

therefore is a completely bounded dual algebra isomorphism having a completely bounded inverse. Proposition 4.2 in Kraus [7] now guarantees that, via Δ, property S_σ carries over from $\mathscr{H}^\infty(\Omega)$ to $\mathcal{A}_S$. □

Corollary 3.3. *Suppose that Ω is a bounded symmetric and circled domain in $\mathbb{C}^n$ or a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$ possessing polynomially convex closure $\overline{\Omega} \subset \mathbb{C}^n$. Given a subnormal tuple $S \in B(H)^n$ of class $\mathbb{A}$ over Ω and a commuting tuple $T \in B(K)^m$ which is strongly reflexive (i.e.* $\operatorname{AlgLat}(T) = \mathcal{A}_T$*), the tensor product tuple*

$$(S \otimes 1, 1 \otimes T) \in B(H \otimes K)^{n+m}$$

is strongly reflexive.

Proof. Theorem 1.4 in [3] says that $\mathcal{A}_S$ is strongly reflexive. Hence the assertion follows from the preceding corollary and Proposition 1.2. □

The last corollary in particular applies to the tuple $S = (M_{z_1}, \ldots, M_{z_n})$ of multiplication with the coordinate functions on the classical Hardy or Bergman spaces, $H = H^2(\Omega)$ or $H = A^2(\Omega)$, on a strictly pseudoconvex or a bounded symmetric and circled domain Ω.

References

1. J. B. Conway, J. J. Dudziak, *Von Neumann operators are reflexive.* J. Reine Angew. Math. **408** (1990), 34–56. MR1058983 (91f:47003)
2. M. Didas, *Dual algebras generated by von Neumann n-tuples over strictly pseudoconvex sets.* Dissertationes Math. **425** (2004). MR2067612 (2005d:47009)
3. M. Didas, J. Eschmeier, *Subnormal tuples on strictly pseudoconvex and bounded symmetric domains.* Acta Sci. Math. (Szeged) **71** (2005), 691–731. MR2206604 (2006m:47036)
4. J. E. Fornaess, *Embedding strictly pseudoconvex domains in convex domains.* Amer. J. Math. **98** (1976), 529–569. MR0422683 (54:10669)
5. G. M. Henkin, J. Leiterer, *Theory of functions on complex manifolds.* Monogr. Math. 79, Birkhäuser, Basel, 1984. MR0774049 (86a:32002)
6. J. Kraus, *The slice map problem for σ-weakly closed subspaces of von Neumann algebras.* Trans. Amer. Math. Soc. **279** (1983), 357–376. MR704620 (85e:46036)
7. J. Kraus, *Abelian operator algebras and tensor products.* J. Operator Theory **14** (1985), 391–407. MR808298 (87f:47064)
8. J. E. McCarthy, *Reflexivity of subnormal operators.* Pacific J. Math. **161**(2) (1993), 359–370. MR1242204 (94h:47042)
9. M. Ptak, *On the reflexivity of pairs of isometries and of tensor products of some operator algebras.* Studia Math. **83** (1986), 47–55. MR829898 (87f:47066)
10. M. Takesaki, *Theory of operator algebras, I.* Springer-Verlag, Berlin-Heidelberg-New York, 2002. MR1873025 (2002m:46083)

Fachrichtung Mathematik, Universität des Saarlandes, Postfach 151150, D-66041 Saarbrücken, Germany

E-mail address: didas@math.uni-sb.de