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A q-ANALOGUE OF NON-STRICT MULTIPLE ZETA VALUES AND BASIC HYPERGEOMETRIC SERIES

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Abstract. We consider the generating function for a q-analogue of non-strict multiple zeta values (or multiple zeta-star values) and prove an explicit formula for it in terms of a basic hypergeometric series ${}_3\phi_2$. By specializing the variables in the generating function, we reproduce the sum formula obtained by Ohno and Okuda and get some relations in the case of full height.

1. Introduction

In this paper we consider the generating function for a q-analogue of non-strict multiple zeta values and prove an explicit formula for it in terms of a basic hypergeometric series ${}_3\phi_2$.

First we recall the definition of the multiple zeta value (MZV). A multi-index $\mathbf{k}=(k_1,\ldots,k_n)$ $(k_i\in\mathbb{Z}_{>0})$ is called *admissible* if $k_1\geq 2$. The weight, the depth and the height of an index $\mathbf{k}=(k_1,\ldots,k_n)$ are defined by $\mathrm{wt}(\mathbf{k}):=\sum_{i=1}^{n}k_i, \mathrm{dep}(\mathbf{k}):=n$ and $\mathrm{ht}(\mathbf{k}):=\#\{i\,|\,k_i\geq 2\}$, respectively. For an admissible index $\mathbf{k}$, the MZV is defined by

$$\zeta(\mathbf{k}):=\sum_{m_1>\cdots>m_n>0}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

The non-strict multiple zeta value $\zeta^*(\mathbf{k})$ is defined by

$$\zeta^*(\mathbf{k}):=\sum_{m_1\geq\cdots\geq m_n\geq 1}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}},$$

which is also called a multiple zeta-star value (MZSV).

The subject of this article is the relations between MZVs or MZSVs and generalized hypergeometric series, and their q-analogue. The first result of this kind was obtained by Ohno and Zagier [9]. They considered a generating function for MZVs and found that it is explicitly written in terms of the value at $z=1$ of the hypergeometric series ${}_2F_1(\alpha,\beta,\gamma;z)$. Li refined Ohno-Zagier's formula by introducing generalized heights [6].

Aoki, Kombu and Ohno obtained an explicit formula for the generating function of MZSVs [1]. Denote by $I_0(k,n,s)$ the set of admissible indices of weight k, depth

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n and height s. Then Aoki-Kombu-Ohno's formula is equivalent to the following equality:

$$\begin{aligned}\sum_{k,n,s}\left(\sum_{\mathbf{k}\in I_0(k,n,s)}\zeta^*(\mathbf{k})\right)x^{k-n-s}y^{n-s}z^{s-1} \\ = \frac{1}{(1-x)(1-y)-z}\,{}_3F_2\left[\begin{matrix}1,\,1,\,1-x\\ 2-\alpha,\,2-\beta\end{matrix};1\right],\end{aligned}\tag{1.1}$$

where α and β are determined by

$$\alpha+\beta=x+y,\quad \alpha\beta=xy-z,$$

and ${}_3F_2$ is the generalized hypergeometric series

$${}_3F_2\left[\begin{matrix}\alpha_1,\,\alpha_2,\,\alpha_3\\ \beta_1,\,\beta_2\end{matrix};z\right]:=\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+n)\Gamma(\alpha_2+n)\Gamma(\alpha_3+n)}{n!\,\Gamma(\beta_1+n)\Gamma(\beta_2+n)}z^n.$$

The formula (1.1) is obtained from that in Remark 3.2 of [1] by using the Kummer-Thomae-Whipple formula

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\alpha_1,\,\alpha_2,\,\alpha_3\\ \beta_1,\,\beta_2\end{matrix};1\right]\\ &=\frac{\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_1)\Gamma(\beta_1+\beta_2-\alpha_2-\alpha_3)}{}_3F_2\left[\begin{matrix}\alpha_1,\,\beta_1-\alpha_2,\,\beta_1-\alpha_3\\ \beta_1,\,\beta_1+\beta_2-\alpha_2-\alpha_3\end{matrix};1\right].\end{aligned}$$

A refinement of (1.1) in the same direction as Li's result is obtained by Aoki, Ohno and Wakabayashi [2].

Now let us consider q-analogues. For an admissible index $\mathbf{k}=(k_1,\ldots,k_n)$, the q-analogues of MZV and MZSV are defined [10, 3, 4, 7] by

$$\zeta_q(\mathbf{k}):=\sum_{m_1>\cdots>m_n>0}\frac{q^{(k_1-1)m_1+\cdots+(k_n-1)m_n}}{[m_1]^{k_1}\cdots[m_n]^{k_n}},$$

$$\zeta_q^*(\mathbf{k}):=\sum_{m_1\geq\cdots\geq m_n\geq 1}\frac{q^{(k_1-1)m_1+\cdots+(k_n-1)m_n}}{[m_1]^{k_1}\cdots[m_n]^{k_n}},$$

where $0<q<1$ and $[n]$ is the q-integer $[n]:=(1-q^n)/(1-q)$. In this article we call the q-analogues $\zeta_q(\mathbf{k})$ and $\zeta_q^*(\mathbf{k})$, qMZV and qMZSV, respectively, for short. In [8], Okuda and the author proved a formula of Ohno-Zagier type for qMZVs. It is a generalization of Bradley's formula [3] for a generating function of qMZVs of type $\zeta_q(m+2,1,\ldots,1)$. See [3] for other linear relations among qMZVs. On the other hand, less is known about qMZSVs. Bradley studied a finite version of qMZSVs [4]. Ohno and Okuda obtained two kinds of sum formulas for qMZSVs [7].

The main result of this paper is a q-analogue of Aoki-Kombu-Ohno's formula (1.1). To write our formula, we need the basic hypergeometric series ${}_{r+1}\phi_r$ defined by

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{matrix};t\right]:=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_{r+1})_n}{(b_1)_n\cdots(b_r)_n(q)_n}t^n,$$

where $(x)_n=(x;q)_n:=\prod_{j=1}^{n}(1-q^{j-1}x)$.

Theorem 1.1 (Generating function of qMZSVs)**.**

$$\sum_{k,n,s}\left(\sum_{\mathbf{k}\in I_0(k,n,s)}\zeta_q^*(\mathbf{k})\right)x^{k-n-s}y^{n-s}z^{s-1} \tag{1.2}$$
$$= \frac{q}{(1-qx)(1-y)-qz}\,{}_3\phi_2\left[\begin{matrix} q,\, q,\, (1+(1-q)x)q \\ q^2/a,\, q^2/b \end{matrix};\frac{q}{1-(1-q)y}\right],$$

where a and b are determined by

$$a+b=\frac{2+(1-q)(x-y)+(1-q)^2(z-xy)}{1+(1-q)x},\quad ab=\frac{1-(1-q)y}{1+(1-q)x}.$$

The rest of the paper is organized as follows. We prove Theorem 1.1 in Section 2. In Section 3 we consider two specializations of the variables x, y and z in (1.2). First we set $z = xy$ to reproduce Ohno-Okuda's sum formula for qMZSVs. The second specialization is $y = 0$, which gives a formula for qMZSVs with full height; that is, $\mathrm{ht}(\mathbf{k}) = \mathrm{dep}(\mathbf{k})$. This is a q-analogue of Theorem 4.2 in [1].

2. Proof of Theorem 1.1

The proof is quite similar to that of Theorem 1 in [8]. We make use of the q-analogue of the multiple polylogarithms with equality:

$$\mathrm{Li}^*_{\mathbf{k}}(t) := \sum_{m_1\ge\cdots\ge m_n\ge 1}\frac{t^{m_1}}{[m_1]^{k_1}\cdots[m_n]^{k_n}}.$$

The right hand side converges if $|t| < 1$ for any index $\mathbf{k} = (k_1, \dots, k_n)$ $(k_i \in \mathbb{Z}_{>0})$. For an admissible index $\mathbf{k}$, the value $\mathrm{Li}^*_{\mathbf{k}}(q)$ is related to qMZSVs as follows:

$$\mathrm{Li}^*_{\mathbf{k}}(q) = \sum_{a_1=2}^{k_1}\sum_{a_2=1}^{k_2}\cdots\sum_{a_n=1}^{k_n}\binom{k_1-2}{a_1-2}\left\{\prod_{j=2}^{n}\binom{k_j-1}{a_j-1}\right\} \tag{2.1}$$
$$\times (1-q)^{\sum_{j=1}^{n}(k_j-a_j)}\zeta_q^*(a_1,\dots,a_n).$$

Denote by $I(k, n, s)$ the set of indices of weight k, depth n and height s, and by $I_0(k, n, s)$ the subset consisting of admissible indices. Set

$$G(k,n,s;t) := \sum_{\mathbf{k}\in I(k,n,s)}\mathrm{Li}^*_{\mathbf{k}}(t),\quad G_0(k,n,s;t) := \sum_{\mathbf{k}\in I_0(k,n,s)}\mathrm{Li}^*_{\mathbf{k}}(t).$$

By definition we set $G(0, 0, 0; t) = 1$ and $G(k, n, s; t) = 0$ unless $k \ge n + s$ and $n \ge s \ge 0$. Now introduce the two generating functions

$$\Phi(t) := \sum_{k,n,s\ge 0} G(k,n,s;t)\, u^{k-n-s}v^{n-s}w^{s},$$
$$\Phi_0(t) := \sum_{k,n,s\ge 0} G_0(k,n,s;t)u^{k-n-s}v^{n-s}w^{s-1}.$$

From (2.1) we see that

$$\Phi_0(q) = \frac{1}{1-(1-q)u}\sum_{k,n,s}\left(\sum_{\mathbf{k}\in I_0(k,n,s)}\zeta_q^*(\mathbf{k})\right)x^{k-n-s}y^{n-s}z^{s-1}, \tag{2.2}$$

where x, y, z are determined by

$$x = \frac{u}{1-(1-q)u}, \quad y = \frac{v+(1-q)(w-uv)}{1-(1-q)u}, \quad z = \frac{w}{(1-(1-q)u)^2}.$$

The q-difference operator $\mathcal{D}_q$ is defined by

$$(\mathcal{D}_q f)(t) := \frac{f(t)-f(qt)}{(1-q)t}.$$

From the recurrence relations

$$\mathcal{D}_q \mathrm{Li}^*_{\mathbf{k}}(t) = \begin{cases} \dfrac{1}{t}\mathrm{Li}^*_{k_1-1,k_2,\ldots,k_n}(t) & (k_1>1), \\ \dfrac{1}{t(1-t)}\mathrm{Li}^*_{k_2,\ldots,k_n}(t) & (k_1=1, n>1), \\ \dfrac{1}{1-t} & (\mathbf{k}=(1)) \end{cases}$$

we obtain the following difference equations by the same calculation as in [1]:

$$\mathcal{D}_q\Phi_0 = \frac{1}{vt}\left(\Phi - 1 - w\Phi_0\right) + \frac{u}{t}\Phi_0,$$
$$\mathcal{D}_q\left(\Phi - \Phi_0\right) = \frac{v}{t(1-t)}(\Phi-1) + \frac{v}{1-t}.$$

Eliminate Φ from the two equations. By using the formula $\mathcal{D}_q(tf(t)) = qt\cdot\mathcal{D}_q f(t) + f(t)$, we find

$$qt^2(1-t)\mathcal{D}_q^2\Phi_0 + t((1-u)(1-t)-v)\mathcal{D}_q\Phi_0 + (uv-w)\Phi_0 = t. \tag{2.3}$$

Let us solve (2.3). Assume that $|u|, |v|$ and $|w|$ are small enough. Then $\Phi_0(t)$ is regular at $t=0$ and satisfies $\Phi_0(0)=0$. Set $\Phi_0(t) = \sum_{n=1}^{\infty} c_n t^n$ and substitute it into (2.3). We see that

$$c_1 = \frac{1}{(1-u)(1-v)-w},$$
$$c_{n+1} = \frac{[n](1-u+q[n-1])}{q[n+1][n]+(1-u-v)[n+1]+uv-w}\, c_n \quad (n=1,2,\ldots). \tag{2.4}$$

Now introduce the two variables a and b determined by

$$a+b = 2-(1-q)(u+v), \quad ab = 1-(1-q)(u+v)+(1-q)^2(uv-w).$$

Then the coefficient in (2.4) is factored as

$$c_{n+1} = \frac{(1-q^n)(1-\frac{q^n}{1-(1-q)u})}{(1-q^{n+1}/a)(1-q^{n+1}/b)} \cdot \frac{1-(1-q)u}{ab}\, c_n.$$

Thus we obtain

$$\Phi_0(t) = \frac{t}{(1-u)(1-v)-w}\, {}_3\phi_2\left[\begin{matrix} q,\ q,\ \frac{q}{1-(1-q)u} \\ q^2/a,\ q^2/b \end{matrix}; \frac{1-(1-q)u}{ab}\, t\right].$$

Set $t=q$ and compare it with (2.2). Expressing u, v and w in terms of x, y and z, we finally get (1.2). This completes the proof of Theorem 1.1.

3. Specialization of parameters

Let us consider two specializations of (1.2) at (i) $z = xy$ and (ii) $y = 0$. Before proceeding we rewrite the right hand side of (1.2) by using the q-analogue of the Kummer-Thomae-Whipple formula (see [5], Eq. (3.2.7)):

$$ {}_3\phi_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} ; \frac{b_1 b_2}{a_1 a_2 a_3}\right] = \frac{(b_2/a_1)_\infty (b_1 b_2/a_2 a_3)_\infty}{(b_2)_\infty (b_1 b_2/a_1 a_2 a_3)_\infty} {}_3\phi_2\left[\begin{matrix} a_1, b_1/a_2, b_1/a_3 \\ b_1, b_1 b_2/a_2 a_3 \end{matrix} ; \frac{b_2}{a_1}\right]. $$

We can apply this equality to ${}_3\phi_2$ in (1.2) because $1-(1-q)y = ab(1+(1-q)x)$. Then we see that

$$ \begin{aligned} & {}_3\phi_2\left[\begin{matrix} q,\, q,\, (1+(1-q)x)q \\ q^2/a,\, q^2/b \end{matrix} ; \frac{q}{1-(1-q)y}\right] \\ &= \frac{(q/b)_\infty}{(q^2/b)_\infty} \frac{(\frac{q^2}{1-(1-q)y})_\infty}{(\frac{q}{1-(1-q)y})_\infty} {}_3\phi_2\left[\begin{matrix} q,\, q/a,\, \frac{q}{a(1-(1-q)x)} \\ q^2/a,\, \frac{q^2}{1-(1-q)y} \end{matrix} ; q/b\right] \\ &= \left(1-\frac{q}{a}\right)\left(1-\frac{q}{b}\right) \sum_{n=0}^{\infty} \frac{\left(\dfrac{q}{a(1+(1-q)x)}\right)_n}{\left(\dfrac{q}{1-(1-q)y}\right)_{n+1}} \frac{\left(\dfrac{q}{b}\right)^n}{1-\dfrac{q^{n+1}}{a}}. \end{aligned} $$

In the following we specialize the variables x, y and z in the equality obtained by rewriting the right hand side of (1.2) as above.

3.1. The case of $z = xy$. We can take $a = \frac{1-(1-q)y}{1+(1-q)x}$ and $b = 1$. Then we reproduce the following formula obtained by Ohno and Okuda [7]:

$$ \sum_{k,n} \left(\sum_{\mathbf{k} \in I_0(k,n)} \zeta_q^*(\mathbf{k}) \right) x^{k-n-1} y^{n-1} = \sum_{n=1}^{\infty} \frac{q^n(1-(1-q)y)}{([n]-y)([n]-(q^n x + y))}, $$

where $I_0(k,n)$ is the set of admissible indices of weight k and depth n. It implies the sum formula for qMZSVs:

$$ \sum_{\mathbf{k} \in I_0(k,n)} \zeta_q^*(\mathbf{k}) = \frac{1}{k-1}\binom{k-1}{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (k-1-l)(1-q)^l \zeta_q(k-l). $$

3.2. The case of $y = 0$. The right hand side of (1.2) becomes

$$ -\frac{1}{z}\left(1 - {}_2\phi_1\left[\begin{matrix} 1/a,\, b \\ q/a \end{matrix} ; q/b\right]\right). $$

Now using Heine's summation formula

$$ {}_2\phi_1\left[\begin{matrix} a_1,\, a_2 \\ b_1 \end{matrix} ; \frac{b_1}{a_1 a_2}\right] = \frac{(b_1/a_1)_\infty (b_1/a_2)_\infty}{(b_1)_\infty (b_1/a_1 a_2)_\infty}, $$

we obtain

$$ {}_2\phi_1\left[\begin{matrix} 1/a,\, b \\ q/a \end{matrix} ; q/b\right] = \frac{(q)_\infty (q/ab)_\infty}{(q/a)_\infty (q/b)_\infty} = \prod_{n=1}^{\infty} \frac{1-\frac{q^n}{[n]}x}{\left(1-\frac{q^n}{[n]}s\right)\left(1-\frac{q^n}{[n]}t\right)}, $$

where s and t are determined by

$$ s + t = x + (1-q)z, \quad st = -z. \tag{3.1} $$

From the expansion

$$\log \prod_{n=1}^{\infty}\left(1-\frac{q^n}{[n]}x\right)=\frac{1}{q-1}\log\left(1+x(1-q)\right)\sum_{n=1}^{\infty}\frac{q^n}{[n]} -\sum_{n=2}^{\infty}\zeta_q(n)\sum_{m=0}^{\infty}\frac{(q-1)^m}{m+n}x^{m+n},$$

we get the following formula.

Theorem 3.1 (Generating function of qMZSVs with full height)**.**

$$\sum_{k,n}\left(\sum_{\mathbf{k}\in I_0(k,n,n)}\zeta_q^*(\mathbf{k})\right)x^{k-n-s}z^{n-1} = -\frac{1}{z}\left\{1-\exp\left(\sum_{n=2}^{\infty}\zeta_q(n)\sum_{m=0}^{\infty}\frac{(q-1)^m}{m+n}(s^{m+n}+t^{m+n}-x^{m+n})\right)\right\},$$

where s and t are determined by (3.1).

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