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A FUNCTIONAL-ANALYTICAL APPROACH TO THE ASYMPTOTICS OF RECURSIONS

CHRISTIAN PÖTZSCHE

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Abstract. We propose a functional-analytical method to investigate the long-term behavior of recursions (difference equations). It is based on a formulation of given (implicit) recursions as abstract operator equations in sequence spaces. Solving them using appropriate tools from nonlinear analysis yields quantitative convergence results and equips us with a method to verify summable or subexponential decay.

1. Introduction

The classical and generally applicable tools in the stability theory of discrete dynamical systems (or in synonymic terminology, recursions or difference equations) are typically based on estimates using various Gronwall-type inequalities or Lyapunov functions (cf., e.g., [Aga00, Chapter 5]). In particular in the autonomous (time-invariant) case, where the law of evolution given by the right-hand side does not depend on time, asymptotic stability of solutions generically goes hand in hand with exponential decay. It is not surprising that one encounters more complex behavior in the general setting of nonautonomous (time-variant) recursions. Here, subexponential decay can occur in the sense that solutions are, for example, only summable. Accordingly, in order to deal with such problems, more flexible convergence notions have been developed using weighted norms (e.g., [Pin98]) or criteria for ℓ^p-summability of solutions (see [Gor71] for a method using Lyapunov functions).

In this paper we suggest another approach to stability or attractivity problems for difference equations. It is based on an abstract formulation of a recursion (say in $\mathbb{K}^d$) incorporating initial conditions, as an operator equation in the infinite-dimensional space $\ell(\mathbb{K}^d)$ of all sequences in $\mathbb{K}^d$. The corresponding operator is composed of a trivial embedding map to include initial conditions, as well as a right shift and a substitution (Nemytskii) operator. Choosing an appropriate normed subspace of $\ell(\mathbb{K}^d)$ to capture the specific kind of decay, and appropriate techniques from nonlinear analysis to solve such problems, endows us with quantitative convergence results.

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The advantage of such a reformulation is that proofs become conceptually clear and transparent. Indeed, one only has to show that certain mappings on sequence spaces are well-defined and satisfy a structural assumption guaranteeing that e.g. fixed point theorems can be employed, such as being contractive, completely continuous or condensing. Results of this kind appear to be well-documented in the literature (cf. [AZ90, DG05] or for the linear case [Mad70, Wil84]). Moreover, the proofs follow the same lines for various convergence notions, since only the space setting needs to be modified. Beyond convergence, one also obtains the existence of solutions for implicit problems.

Although the above idea is contiguous, so far it seems rarely analyzed in the literature. The only related reference we are aware of is the work of Siafarikas and his collaborators (see, for example, [PS05]), which is essentially restricted to ℓ^2-convergence. The more flexible situation of arbitrary sequence spaces has been considered in [EP07], a paper which focuses on fixed point results (of Banach, Krasnoselski-, Reinermann- and Goebel-Kirk-type) to solve the resulting sequential operator equations.

However, fixed point methods have the disadvantage that the domain and range of the mappings under consideration need to coincide. The paper at hand circumvents this problem by using a result (Theorem 2.1) which easily follows from Darbo's or Sadovskiĭ's fixed point theorem (cf. [Dar55] or [Sad67], respectively). Having this tool available we apply our methodology to implicit recursions of the form

$$x_{k+1} = f_k(x_k, x_{k+1}) + g_k(x_k, x_{k+1}), \tag{1.1}$$

where the mapping f_k is assumed to yield an adequate admissibility property. This means that for each perturbation sequence $\psi = (\psi_k)$ from a sequence space Y there exists a unique solution $\phi = (\phi_k)$ for the initial value problem

$$x_{k+1} = f_k(x_k, x_{k+1}) + \psi_{k+1}, \quad x_0 = \psi_0 \tag{1.2}$$

in a sequence space X. Such admissibility results under dichotomy assumptions, among others, can be found in [CS67, NP97, Pin98, Sas06]. Then, in order to obtain convergence results for (1.1), it will be sufficient to show that the substitution operator induced by g_k will have a specific compactness property between X and Y. Examples of admissible spaces for semi-linear recursions close our considerations.

2. Preliminaries and a Darbo-type result

We define the discrete interval $\mathbb{N}_0 := \{k \in \mathbb{Z} : k \geq 0\}$ of nonnegative integers. As usual, $\mathbb{K}$ denotes the field of real or complex numbers and $\mathbb{K}^d$ the d-dimensional Euclidean (or Hermitian) space, equipped with norm $|\cdot|$.

Let X be a Banach space with norm $\|\cdot\|_X$ and corresponding family $\mathcal{B}(X)$ of bounded subsets. The closed ball in X with radius ρ centered at 0 is denoted by $\bar{B}_\rho(X)$. We briefly write $\ell(\mathbb{K}^d)$ for the linear space of sequences $\phi = (\phi_k)_{k\geq 0}$ in $\mathbb{K}^d$.

The *Hausdorff measure of noncompactness* $\chi_X : \mathcal{B}(X) \to [0, \infty)$ is defined as

$$\chi_X(B) := \inf \{\epsilon > 0 : B \text{ has a finite } \epsilon\text{-net in } X\}$$

(cf. [ADL97, pp. 20ff], which will be our standard reference concerning χ_X). Since it is possible to determine χ_X explicitly for certain spaces X (see below), we will use this particular measure of noncompactness exclusively throughout the paper.

Well-known properties of χ_X needed here include translation invariance $\chi_X(x+B) = \chi_X(B)$ for every $B \in \mathcal{B}(X)$ and $x \in X$. Let χ_Y denote the Hausdorff measure of noncompactness on a further Banach space Y. Then a mapping $\mathcal{T} : X \to Y$ is called *condensing* (see, e.g., [ADL97, p. 38, Definition 5.1(b)]) if it is continuous and one has

$$\chi_X(B) > 0 \quad \Rightarrow \quad \chi_Y(\mathcal{T}(B)) < \chi_X(B) \quad \text{for all } B \in \mathcal{B}(X). \tag{2.1}$$

In concrete applications it is frequently difficult to verify the above implication, and one needs sufficient conditions for (2.1) to hold. For example, a continuous mapping $\mathcal{T}$ is condensing if it is *completely continuous* (one has $\chi_Y(\mathcal{T}(B)) = 0$ for $B \in \mathcal{B}(X)$), or a *set-contraction*, where we have a $\lambda \in [0,1)$ such that

$$\chi_Y(\mathcal{T}(B)) \le \lambda \chi_X(B) \quad \text{for all } B \in \mathcal{B}(X).$$

Also contractive mappings are set-contractions and more general, if $\mathcal{T} : X \to Y$ satisfies a global Lipschitz condition with constant $\operatorname{lip} \mathcal{T}$, then it follows that

$$\chi_Y(\mathcal{T}(B)) \le \operatorname{lip} \mathcal{T} \chi_X(B) \quad \text{for all } B \in \mathcal{B}(X).$$

After these preparations we arrive at

Theorem 2.1. *Let $\mathcal{A} : D(\mathcal{A}) \subseteq X \to Y$ and $\mathcal{G} : X \to Y$ be mappings satisfying:*

(i) *there exist reals $\alpha, \beta, \gamma \ge 0$ such that $\|\mathcal{G}(x)\|_Y \le \alpha + \beta \|x\|_X^\gamma$ for all $x \in X$,*
(ii) *$\mathcal{A}^{-1} : Y \to X$ exists and is globally Lipschitz with $\mathcal{A}^{-1}(0) = 0$,*
(iii) *$\mathcal{A}^{-1} \circ \mathcal{G} : X \to X$ is a set-contraction,*
(iv) *the inequality $(\alpha + \beta\rho^\gamma) \operatorname{lip} \mathcal{A}^{-1} \le \rho$ admits a solution $\rho_0 > 0$.*

Then the set $\{x \in X : \mathcal{A}(x) = \mathcal{G}(x)\} \subseteq \bar{B}_{\rho_0}(X)$ is nonempty and compact.

Remark 2.2. (1) Instead of being a set-contraction, in order to ensure that there exists a solution of the coincidence equation $\mathcal{A}(x) = \mathcal{G}(x)$ in $\bar{B}_{\rho_0}(X)$, one can assume that the composition $\mathcal{A}^{-1} \circ \mathcal{G} : X \to X$ is condensing. For this generalization, Sadovskiĭ's theorem (as stated in [ADL97, p. 40, Theorem 5.4]) has to be used in the subsequent proof.

(2) It is elementary to derive criteria sufficient for hypothesis (iv). Examples are given in the following three cases:

- $\gamma < 1$,
- $\gamma = 1$ and $\beta \operatorname{lip} \mathcal{A}^{-1} < 1$,
- $\gamma > 1$, $\beta > 0$ and $\left(\beta\gamma\right)^{\frac{1}{1-\gamma}} \left[\alpha + \beta\left(\beta\gamma\right)^{\frac{\gamma}{1-\gamma}}\right] \operatorname{lip} \mathcal{A}^{-1} \le 1$.

Proof of Theorem 2.1. By assumption the composition $\mathcal{A}^{-1} \circ \mathcal{G} : X \to X$ satisfies

$$\left\|\mathcal{A}^{-1}(\mathcal{G}(x))\right\|_X \le \operatorname{lip} \mathcal{A}^{-1} \|\mathcal{G}(x)\|_Y \le \operatorname{lip} \mathcal{A}^{-1}(\alpha + \beta\rho_0^\gamma) \le \rho_0 \quad \text{for all } x \in \bar{B}_{\rho_0}(X)$$

and therefore $\mathcal{A}^{-1} \circ \mathcal{G}$ maps the bounded, closed and convex set $\bar{B}_{\rho_0}(X)$ into itself. Thus, thanks to assumption (iii), we can apply Darbo's theorem (cf. [BG80, p. 17, Theorem 5.1]) to verify that the set of fixed points for $\mathcal{A}^{-1} \circ \mathcal{G}$ is nonempty compact. Obviously, solutions of $\mathcal{A}(x) = \mathcal{G}(x)$ are fixed points of $\mathcal{A}^{-1} \circ \mathcal{G}$ and vice versa. $\square$

Our goal is to deduce attractivity properties for forward solutions of (1.1) from Theorem 2.1. Thereto, the following complete normed subspaces of $\ell(\mathbb{K}^d)$ are crucial:

$$\ell^\infty(\mathbb{K}^d) := \left\{ \phi = (\phi_k)_{k\geq 0} : \sup_{k\geq 0} |\phi_k| < \infty \right\}, \qquad \|\phi\|_{\ell^\infty} := \sup_{k\geq 0} |\phi_k|,$$
$$\ell_0(\mathbb{K}^d) := \left\{ \phi = (\phi_k)_{k\geq 0} : \lim_{k\to\infty} \phi_k = 0 \right\}, \qquad \|\phi\|_{\ell_0} := \sup_{k\geq 0} |\phi_k|,$$
$$\ell^p(\mathbb{K}^d) := \left\{ \phi = (\phi_k)_{k\geq 0} : \sum_{k\geq 0} |\phi_k|^p < \infty \right\}, \qquad \|\phi\|_{\ell^p} := \sqrt[p]{\sum_{k\geq 0} |\phi_k|^p}$$

with a real number $p \geq 1$. For Banach spaces possessing a Schauder basis, thus in particular $\ell_0(\mathbb{K}^d)$ and $\ell^p(\mathbb{K}^d)$, the quantity $\chi_X(B) \geq 0$ can be computed explicitly (cf., for instance, [ADL97, pp. 35ff] in connection with [BG80, p. 23]) as follows:

$$\chi_{\ell_0}(B) = \lim_{n\to\infty} \sup_{x\in B} \sup_{k>n} |x_k| \quad \text{for all } B \in \mathcal{B}(\ell_0(\mathbb{K}^d)),$$
$$\chi_{\ell^p}(B) = \lim_{n\to\infty} \sup_{x\in B} \sqrt[p]{\sum_{k>n} |x_k|^p} \quad \text{for all } B \in \mathcal{B}(\ell^p(\mathbb{K}^d)).$$

3. Asymptotic behavior of recursions

We consider the discrete implicit initial value problem (IVP for short)

$$x_{k+1} = f_k(x_k, x_{k+1}), \tag{3.1}$$
$$x_0 = \xi \tag{3.2}$$

for initial values $\xi \in \mathbb{K}^d$ and a right-hand side $f_k : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^d$, $k \in \mathbb{N}_0$. For simplicity reasons we suppose that f_k is globally defined on the whole space $\mathbb{K}^d \times \mathbb{K}^d$ and satisfies global assumptions (see below) throughout. This seemingly severe restriction has to be related to the goals of our approach: On the one hand, under global assumptions which are rarely satisfied in applications, we can deduce global convergence results. On the other hand, though, in stability theory one frequently encounters the situation where a small set (typically a neighborhood of 0) is forward invariant under the dynamics of (3.1). Then an appropriate modification of f_k outside such an invariant set using retraction mappings allows us to deduce local convergence criteria from our global results; we demonstrate this standard procedure in Example 4.4 below.

In order to embed (3.1), (3.2) into our functional-analytical framework we introduce

- the linear *embedding operator* $\mathcal{E} : \mathbb{K}^d \to \ell(\mathbb{K}^d)$, $\mathcal{E}\xi := (\xi, 0, 0, \dots)$,
- the linear *right shift operator* $\mathcal{S} : \ell(\mathbb{K}^d) \to \ell(\mathbb{K}^d)$, $\mathcal{S}\phi := (0, \phi_0, \phi_1, \dots)$,
- the *substitution operator* $\mathcal{F}_f : \ell(\mathbb{K}^d) \to \ell(\mathbb{K}^d)$, $\mathcal{F}_f(\phi) := \big(f_k(\phi_k, \phi_{k+1})\big)_{k\geq 0}$,
- $\mathcal{G}_f : \ell(\mathbb{K}^d) \times \mathbb{K}^d \to \ell(\mathbb{K}^d)$, given by

$$\mathcal{G}_f(\phi, \xi) := \mathcal{E}\xi + \mathcal{S}\mathcal{F}_f(\phi). \tag{3.3}$$

The operator $\mathcal{G}_f$ depends linearly on the right-hand side f_k and, if $f_k : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^d$ is a linear mapping, then $\mathcal{G}_f$ becomes affine linear. Our fundamental device to reformulate the IVP (3.1), (3.2) as an operator equation in $\ell(\mathbb{K}^d)$ is given in

Theorem 3.1. *Let $\xi \in \mathbb{K}^d$ and $\phi \in \ell(\mathbb{K}^d)$ be a sequence in $\mathbb{K}^d$. Then ϕ is a solution of the IVP* (3.1), (3.2) *if and only if ϕ solves the fixed point equation*

$$\phi = \mathcal{G}_f(\phi, \xi). \tag{3.4}$$

For an explicit recursion (3.1), *a fixed-point of the mapping $\mathcal{G}_f(\cdot, \xi)$ is unique.*

Proof. We refer to [EP07, Theorem 3.3] for the easy proof. □

To get a flavor of our methodology, we give a first application of the interplay between Theorems 2.1 and 3.1. Its proof is based on the continuous embedding $\ell^p(\mathbb{K}^d) \hookrightarrow \ell^q(\mathbb{K}^d)$ for $p \leq q$ and can be adapted to other pairs of continuously embedded sequence spaces.

Proposition 3.2. *Let $p \geq 1$ and $f_k : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^d$ be continuous satisfying:*

(i) *there exist sequences $a \in \ell^p(\mathbb{R})$, $b, c \in \ell_0(\mathbb{R})$ and reals $\gamma \geq 1$ such that*

$$|f_k(x, y)| \leq a_k + \max\left\{b_k |x|^\gamma, c_k |y|^\gamma\right\} \quad \text{for all } k \in \mathbb{N}_0,\ x, y \in \mathbb{K}^d,$$

(ii) *the inequality $|\xi| + \|a\|_{\ell^p} + (\|b\|_{\ell_0} + \|c\|_{\ell_0})\, \rho^\gamma \leq \rho$ admits a solution $\rho_0 > 0$.*

Then there exists a solution $\phi \in \bar{B}_{\rho_0}(\ell^p(\mathbb{K}^d))$ of the IVP (3.1), (3.2).

Proof. Let $\xi \in \mathbb{K}^d$ be fixed and define $q := \gamma p \geq p$. Let us introduce the linear operator $\mathcal{A} : D(\mathcal{A}) \subseteq \ell^q(\mathbb{K}^d) \to \ell^p(\mathbb{K}^d)$, $\mathcal{A}\phi := \phi$ with $D(\mathcal{A}) := \ell^p(\mathbb{K}^d)$. Owing to the continuous embedding $\ell^p(\mathbb{K}^d) \hookrightarrow \ell^q(\mathbb{K}^d)$ we know that the inverse $\mathcal{A}^{-1} : \ell^p(\mathbb{K}^d) \to \ell^q(\mathbb{K}^d)$ exists as a continuous mapping with norm 1.

On the other hand, from our preliminaries in [EP07, Lemma 5.2, 5.3] we obtain that $\mathcal{G}_f(\cdot, \xi) : \ell^q(\mathbb{K}^d) \to \ell^p(\mathbb{K}^d)$ is completely continuous with

$$\|\mathcal{G}_f(\phi, \xi)\|_{\ell^p} \leq |\xi| + \|a\|_{\ell^p} + (\|b\|_{\ell_0} + \|c\|_{\ell_0})\, \|\phi\|_{\ell^q}^\gamma \quad \text{for all } \phi \in \ell^q(\mathbb{K}^d),$$

and by Theorem 2.1 with $X = \ell^q(\mathbb{K}^d)$ and $Y = \ell^p(\mathbb{K}^d)$ there exists a $\phi \in \bar{B}_{\rho_0}(\ell^p(\mathbb{K}^d))$ such that $\phi = \mathcal{A}\phi = \mathcal{G}_f(\phi, \xi)$. The assertion follows from Theorem 3.1. □

A crucial terminology for our investigations is the notion of *admissibility*, which in the linear case dates back to [CS67]. Thereto, the admissibility of subspaces $X, Y \subseteq \ell(\mathbb{K}^d)$ means that inputs $\psi \in Y$ of the form (1.2) into (3.1) produce unique outputs in X. To be more precise, we say that the difference equation (3.1) is (X, Y)-*admissible* if for each perturbation $\psi \in Y$ there exists a unique forward solution $\mathcal{J}(\psi) \in X$ of the IVP (1.2) and such that the *admissibility map* $\mathcal{J} : Y \to X$ satisfies $\operatorname{lip} \mathcal{J} < \infty$.

Theorem 3.3. *Let $\xi \in \mathbb{K}^d$, $X, Y \subseteq \ell(\mathbb{K}^d)$ be normed sequence subspaces, suppose that* (3.1) *is (X, Y)-admissible with admissibility map $\mathcal{J}$ and*

$$x = f_k(0, x) \quad \Leftrightarrow \quad x = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{3.5}$$

If the mappings $g_k : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^d$ satisfy

(i) *there exist reals $\alpha, \beta, \gamma \geq 0$ such that*

$$\|\mathcal{G}_g(\phi, \xi)\|_Y \leq \alpha + \beta\, \|\phi\|_X^\gamma \quad \text{for all } \phi \in X, \tag{3.6}$$

(ii) *$\mathcal{F}_g : X \to Y$ is continuous and there exists a $\lambda \geq 0$ such that*

$$\chi_Y(\mathcal{F}_g(B)) \leq \lambda \chi_X(B) \quad \text{for all } B \in \mathcal{B}(X),$$

(iii) *$\lambda \operatorname{lip} \mathcal{J} < 1$ and the inequality $(\alpha + \beta\rho^\gamma) \operatorname{lip} \mathcal{J} \leq \rho$ admits a solution $\rho_0 > 0$,* *then the IVP*

$$x_{k+1} = f_k(x_k, x_{k+1}) + g_k(x_k, x_{k+1}), \quad x_0 = \xi \tag{3.7}$$

possesses a solution in $\bar{B}_{\rho_0}(X)$.

Proof. Suppose $\xi \in \mathbb{K}^d$. Thanks to our basic Theorem 3.1 we know that every solution ϕ of the IVP (3.7) satisfies the fixed point equation $\phi = \mathcal{E}\xi + \mathcal{S}\mathcal{F}_f(\phi) + \mathcal{S}\mathcal{F}_g(\phi)$ (cf. (3.3)) or equivalently the nonlinear equation

$$\mathcal{A}(\phi) = \mathcal{G}_g(\phi, \xi) \tag{3.8}$$

with the operator $\mathcal{A} : \ell(\mathbb{K}^d) \to \ell(\mathbb{K}^d)$, $\mathcal{A}(\phi) := \phi - \mathcal{S}\mathcal{F}_f(\phi)$. In order to utilize Theorem 2.1 we choose $\psi \in Y$ and remark that $\mathcal{A}(\phi) = \psi$ has the explicit formulation

$$\phi_0 = \psi_0, \qquad \phi_{k+1} - f_k(\phi_k, \phi_{k+1}) = \psi_{k+1} \quad \text{for all } k \in \mathbb{N}_0.$$

Hence, due to the assumed (X, Y)-admissibility of (3.1) we know that there exists a unique sequence $\phi \in X$ such that $\mathcal{A}(\phi) = \psi$ holds, i.e., the admissibility map $\mathcal{J} : Y \to X$ is the inverse of $\mathcal{A}$. In addition, from (3.5) we get $\mathcal{J}(0) = 0$. The translation invariance of χ_Y implies $\chi_Y(\mathcal{G}_g(B, \xi)) = \chi_Y(\mathcal{F}_g(B))$ and our assumptions yield

$$\chi_X\left(\mathcal{A}^{-1}(\mathcal{G}_g(B, \xi))\right) \leq \operatorname{lip} \mathcal{A}^{-1} \chi_Y(\mathcal{G}_g(B, \xi)) = \operatorname{lip} \mathcal{J} \chi_Y(\mathcal{F}_g(B)) \leq \lambda \operatorname{lip} \mathcal{J} \chi_X(B)$$

for all $B \in \mathcal{B}(X)$. Consequently, $\mathcal{A}^{-1} \circ \mathcal{G}_g(\cdot, \xi)$ is a set-contraction. Then Theorem 2.1 guarantees that (3.8) has a solution in $\bar{B}_{\rho_0}(X)$, which by Theorem 3.1 is our claim. □

4. Admissibility and applications

We continue this paper with three explicit applications of Theorem 3.3. A particularly interesting special case of (3.1) are linear difference equations of the form

$$x_{k+1} = A_k x_k + B_k x_{k+1} \tag{4.1}$$

with matrices $A_k, B_k \in \mathbb{K}^{d \times d}$ such that $I - B_k$ is invertible for every $k \in \mathbb{N}_0$. Then the *transition operator* $\Phi(k, n) \in \mathbb{K}^{d \times d}$ of (4.1) is given by

$$\Phi(k, n) := \begin{cases} I & \text{for } k = n, \\ (I - B_{k-1})^{-1} A_{k-1} \cdots (I - B_n)^{-1} A_n & \text{for } k > n. \end{cases}$$

We will see that admissibility properties for (4.1) can be derived using fairly classical operator-theoretical tools from, for example, [Mad70, Wil84]. A different approach to the admissibility of linear difference equations can be found in [CS67, NP97, Sas06] and related results under semi-linear perturbations are considered in [Pin98].

Lemma 4.1 ((ℓ^p, ℓ^p)-admissibility). *Let $p, q > 1$ be reals with $\frac{1}{p} + \frac{1}{q} = 1$. If we have*

$$\begin{aligned} \nu_1 &:= \sup_{k \geq 0} |\Phi(k, 0)| + \sup_{k \geq 1} \sum_{n=1}^{k} \left|\Phi(k, n)(I - B_{n-1})^{-1}\right| < \infty, \\ \nu_2 &:= \max\left\{ \sum_{k=0}^{\infty} |\Phi(k, 0)|, \sup_{n \geq 1} \sum_{k=n}^{\infty} \left|\Phi(k, n)(I - B_{n-1})^{-1}\right| \right\} < \infty, \end{aligned} \tag{4.2}$$

then the linear difference equation (4.1) *is* $(\ell^p(\mathbb{K}^d), \ell^p(\mathbb{K}^d))$*-admissible with linear admissibility map* $\mathcal{J}$ *satisfying*

$$\operatorname{lip} \mathcal{J} \leq \sqrt[q]{\nu_1} \sqrt[p]{\nu_2}. \tag{4.3}$$

Example 4.2. Let $\theta < 1$ be given. For constant matrices $A, B \in \mathbb{K}^{d\times d}$ the admissibility conditions (4.2) hold if the eigenvalues $\lambda \in \mathbb{C}$ of $(I-B)^{-1}A \in \mathbb{K}^{d\times d}$ satisfy $|\lambda| \leq \theta$ and such that eigenvalues with modulus $|\lambda| = \theta$ are semi-simple. Indeed, we have

$$\nu_1 \leq 1 + \tfrac{1}{1-\theta}\left|(I-B)^{-1}\right|, \qquad \nu_2 \leq \tfrac{1}{1-\theta}\max\left\{1, \left|(I-B)^{-1}\right|\right\}.$$

Proof of Lemma 4.1. Let $\psi \in \ell^p(\mathbb{K}^d)$. It is not difficult to see that the sequence

$$\phi_k := \Phi(k,0)\psi_0 + \sum_{n=0}^{k-1}\Phi(k,n+1)(I-B_n)^{-1}\psi_{n+1} \quad \text{for all } k \in \mathbb{N}_0 \tag{4.4}$$

is the unique forward solution of the perturbed linear difference equation

$$x_{k+1} = A_k x_k + B_k x_{k+1} + \psi_{k+1} \tag{4.5}$$

satisfying $\phi_0 = \psi_0$. In order to deduce the inclusion $\phi \in \ell^p(\mathbb{K}^d)$, we interpret (4.4) formally as rows of a linear equation $\phi = \mathcal{J}\psi$, where the infinite matrix $\mathcal{J}$ is lower triangular. Adopting the well-definedness condition for matrix operators $\mathcal{J}$ from ℓ^p to ℓ^p given in [Mad70, p. 174, Theorem 9], we arrive at $\|\phi\|_{\ell^p} \leq \sqrt[q]{\nu_1}\sqrt[p]{\nu_2}\|\psi\|_{\ell^p}$. Thus, the linear mapping $\mathcal{J}(\psi) = \phi$ is our desired admissibility map satisfying (4.3). $\square$

Proposition 4.3. *Let $\xi \in \mathbb{K}^d$ and $p, q > 1$ be reals with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose the estimates* (4.2) *hold and that the continuous mappings $g_k : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^d$ satisfy:*

(i) *there exist sequences $a \in \ell^p(\mathbb{R})$, $b, c \in \ell^\infty(\mathbb{R})$ such that*

$$|g_k(x,y)| \leq a_k + \max\{b_k|x|, c_k|y|\} \quad \textit{for all } k \in \mathbb{N}_0,\ x, y \in \mathbb{K}^d,$$

(ii) $\sqrt[q]{\nu_1}\sqrt[p]{\nu_2}(\|b\|_{\ell^\infty} + \|c\|_{\ell^\infty}) < 1$.

Then there exists a solution $\phi \in \bar{B}_{\rho_0}(\ell^p(\mathbb{K}^d))$ of the semi-linear IVP

$$x_{k+1} = A_k x_k + B_k x_{k+1} + g_k(x_k, x_{k+1}), \quad x_0 = \xi, \tag{4.6}$$

where we have

$$\rho_0 := \frac{|\xi| + \sqrt[q]{\nu_1}\sqrt[p]{\nu_2}\|a\|_{\ell^p}}{1 - \sqrt[q]{\nu_1}\sqrt[p]{\nu_2}(\|b\|_{\ell^\infty} + \|c\|_{\ell^\infty})}.$$

Proof. Let $\xi \in \mathbb{K}^d$ be given and successively check the assumptions of Theorem 3.3 with Banach spaces $X = \ell^p(\mathbb{K}^d)$, $Y = \ell^p(\mathbb{K}^d)$. Above all, Lemma 4.1 guarantees that (4.1) is $(\ell^p(\mathbb{K}^d), \ell^p(\mathbb{K}^d))$-admissible with an admissibility map $\mathcal{J}$ satisfying (4.3). From the invertibility of $I - B_k$ we obtain (3.5). Moreover, [DG05, Theorems 1.1, 1.3] implies that $\mathcal{G}_g : \ell^p(\mathbb{K}^d) \times \mathbb{K}^d \to \ell^p(\mathbb{K}^d)$ is well-defined and continuous in the first argument. Due to [EP07, Lemma 5.2] we see that

$$\|\mathcal{G}_f(\phi,\xi)\|_{\ell^p} \leq |\xi| + \|a\|_{\ell^p} + (\|b\|_{\ell^\infty} + \|c\|_{\ell^\infty})\|\phi\|_{\ell^p} \quad \text{for all } \phi \in \ell^p(\mathbb{K}^d)$$

and this yields (3.6) with $\alpha = |\xi| + \|a\|_{\ell^p}$, $\beta = \|b\|_{\ell^\infty} + \|c\|_{\ell^\infty}$ and $\gamma = 1$. Now let $B \subseteq \ell^p(\mathbb{K}^d)$ be a bounded set and choose $\phi \in B$. Due to the translation invariance of the measure of noncompactness χ_{ℓ^p} we can suppose $g_k(0,0) \equiv 0$ in the estimate

$$\begin{aligned}
&\sqrt[p]{\sum_{k>n} |g_k(\phi_k, \phi_{k+1})|^p} \leq \sqrt[p]{\sum_{k>n} \max\{b_k |\phi_k|, c_k |\phi_{k+1}|\}^p} \\
&\leq \sqrt[p]{\sum_{k>n} (b_k |\phi_k| + c_k |\phi_{k+1}|)^p} \leq (\|b\|_{\ell^\infty} + \|c\|_{\ell^\infty}) \sqrt[p]{\sum_{k>n} |\phi_k|^p} \\
&\leq (\|b\|_{\ell^\infty} + \|c\|_{\ell^\infty}) \sup_{\phi \in B} \sqrt[p]{\sum_{k>n} |\phi_k|^p} \quad \text{for all } n \in \mathbb{N}_0;
\end{aligned}$$

passing to the supremum over all $\phi \in B$ and taking the limit $n \to \infty$ leads to the inequality $\chi_{\ell^p}(\mathcal{F}_g(B)) \leq (\|b\|_{\ell^\infty} + \|c\|_{\ell^\infty}) \chi_{\ell^p}(B)$. Hence, our assumptions are sufficient for the hypotheses of Theorem 3.3. $\square$

The following illustrative example demonstrates that Proposition 4.3, as well as our method in general, are easily applicable to implicit problems.

Example 4.4. Let us consider the nonautonomous implicit complex recursion

$$z_{k+1} = \alpha z_k + \gamma_k |z_k| |z_{k+1}| + \delta_k, \tag{4.7}$$

where $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, $(\gamma_k)_{k\geq 0} \in \ell^\infty(\mathbb{C})$ and $(\delta_k)_{k\geq 0} \in \ell^p(\mathbb{C})$ for some $p > 1$. We are using the radial retraction $R_r : \mathbb{C} \to \mathbb{C}$, $r > 0$, given by

$$R_r(z) := \begin{cases} z, & |z| \leq r \\ \frac{r}{|z|} z, & |z| > r \end{cases}$$

to prepare the nonlinear equation (4.7) as follows:

$$z_{k+1} = \alpha z_k + \gamma_k |R_r(z_k)| |R_r(z_{k+1})| + \delta_k. \tag{4.8}$$

Note that recursion (4.8) satisfies the assumptions of Proposition 4.3 with $A_k = \alpha$, $B_k = 0$, $g_k(z, w) = \gamma_k |R_r(z)| |R_r(w)| + \delta_k$ and constants $\nu_1 \leq \frac{2-|\alpha|}{1-|\alpha|}$, $\nu_2 \leq \frac{1}{1-|\alpha|}$, $a_k = |\delta_k|$, $b_k = c_k = r |\gamma_k|$, provided $r > 0$ is chosen so small that

$$2 \sqrt[q]{\nu_1} \sqrt[p]{\nu_2} \sup_{k\geq 0} |\gamma_k| r < 1.$$

Hence, referring to Proposition 4.3 there exists a solution $\phi \in \bar{B}_{\rho_0}(\ell^p(\mathbb{C}))$ of (4.8) starting in $\zeta \in \mathbb{C}$, which in particular satisfies

$$|\phi_k| \leq \sqrt[p]{\rho_0} = \sqrt[p]{\frac{|\zeta| + \sqrt[q]{\nu_1} \sqrt[p]{\nu_2} \|a\|_{\ell^p}}{1 - 2 \sqrt[q]{\nu_1} \sqrt[p]{\nu_2} \|\gamma\|_{\ell^\infty} r}} \quad \text{for all } k \in \mathbb{N}_0.$$

With sufficiently small ℓ^p-norm of a and initial values ζ close of 0, we have $|\phi_k| \leq r$ for all $k \in \mathbb{N}_0$. Consequently, for such values of ζ, the solution ϕ stays in the ball $\bar{B}_r(\mathbb{C})$ and due to the fact that (4.7) and (4.8) coincide on $\bar{B}_r(\mathbb{C})$, the sequence $\phi \in \bar{B}_{\rho_0}(\ell^p(\mathbb{C}))$ is also a solution of the original equation (4.7) starting in ζ.

Lemma 4.5 ((ℓ_0, ℓ_0)-admissibility)**.** *If we have*

$$\lim_{k\to\infty} \Phi(k, n) = 0 \quad \textit{for all } n \geq 0,$$

$$\nu := \sup_{k\geq 0} |\Phi(k, 0)| + \sup_{k\geq 1} \sum_{n=1}^{k} \left|\Phi(k, n)(I - B_{n-1})^{-1}\right| < \infty, \tag{4.9}$$

then the linear difference equation (4.1) *is* $(\ell_0(\mathbb{K}^d), \ell_0(\mathbb{K}^d))$*-admissible with linear admissibility map* $\mathcal{J}$ *satisfying*

$$\operatorname{lip} \mathcal{J} \leq \nu. \tag{4.10}$$

Example 4.6. Under the assumptions of Example 4.2 the admissibility conditions (4.9) are satisfied with $\nu \leq 1 + \frac{1}{1-\theta} \left|(I-B)^{-1}\right|$.

Proof of Lemma 4.5. For a given inhomogeneity $\psi \in \ell_0(\mathbb{K}^d)$ we again define the uniquely determined forward solution ϕ of (4.5) satisfying $\phi_0 = \psi_0$ by (4.4). Arguing as in the proof of Lemma 4.1 we deduce from [Mad70, p. 163, Theorem 1] that $\phi \in \ell_0(\mathbb{K}^d)$ with $\|\phi\|_{\ell_0} \leq \nu \|\psi\|_{\ell_0}$. Thus, the claim follows with admissibility map $\mathcal{J}(\psi) := \phi$. □

Proposition 4.7. *Let* $\xi \in \mathbb{K}^d$*. Suppose the relations* (4.9) *hold and that the mappings* $g_k : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^d$ *satisfy:*

(i) *for each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for every* $k \in \mathbb{N}_0$ *and* $x, \bar{x}, y, \bar{y} \in \mathbb{K}^d$ *with* $|x - \bar{x}|, |y - \bar{y}| < \delta$ *one has*

$$|g_k(x, y) - g_k(\bar{x}, \bar{y})| < \epsilon,$$

(ii) *there exist sequences* $a \in \ell_0(\mathbb{R})$ *and* $b, c \in \ell^\infty(\mathbb{R})$ *such that*

$$|g_k(x, y)| \leq a_k + \max\{b_k |x|, c_k |y|\} \quad \textit{for all } k \in \mathbb{N}_0,\ x, y \in \mathbb{K}^d,$$

(iii) $\nu \max\{\|b\|_{\ell^\infty}, \|c\|_{\ell^\infty}\} < 1$.

Then there exists a solution $\phi \in \bar{B}_{\rho_0}(\ell_0(\mathbb{K}^d))$ *of the IVP* (4.6)*, where we have*

$$\rho_0 := \frac{|\xi| + \nu \|a\|_{\ell_0}}{1 - \nu \max\{\|b\|_{\ell^\infty}, \|c\|_{\ell^\infty}\}}.$$

Proof. For given initial value $\xi \in \mathbb{K}^d$, let us check the assumptions of Theorem 3.3 with spaces $X = Y = \ell_0(\mathbb{K}^d)$. By Lemma 4.5 the linear difference equation (4.1) is $(\ell_0(\mathbb{K}^d), \ell_0(\mathbb{K}^d))$-admissible and (4.10) holds. In addition, by [EP07, Lemma 5.5] the mapping $\mathcal{G}_g : \ell_0(\mathbb{K}^d) \times \mathbb{K}^d \to \ell_0(\mathbb{K}^d)$ is well-defined with

$$\|\mathcal{G}_g(\phi, \xi)\|_{\ell_0} \leq |\xi| + \|a\|_{\ell_0} + \max\{\|b\|_{\ell^\infty}, \|c\|_{\ell^\infty}\} \|\phi\|_{\ell_0} \quad \text{for all } \phi \in \ell_0(\mathbb{K}^d)$$

and therefore (3.6) holds with $\alpha = |\xi| + \|a\|_{\ell_0}$, $\beta = \max\{\|b\|_{\ell^\infty}, \|c\|_{\ell^\infty}\}$ and $\gamma = 1$. We leave it to the interested reader to show that our assumption (i) implies the continuity of $\mathcal{F}_g : \ell_0(\mathbb{K}^d) \to \ell_0(\mathbb{K}^d)$. Let $B \subseteq \ell_0(\mathbb{K}^d)$ be a bounded set and pick $\phi \in B$. The translation invariance of χ_{ℓ_0} allows us to suppose $g_k(0,0) \equiv 0$ in the estimate

$$\sup_{k>n} |\mathcal{F}_g(\phi)_k| \leq \sup_{k>n} \max\{b_k |\phi_k|, c_k |\phi_{k+1}|\} \leq \max\{\|b\|_{\ell^\infty}, \|c\|_{\ell^\infty}\} \sup_{\phi \in B} \sup_{k>n} \|\phi\|_{\ell_0}$$

for all $n \in \mathbb{N}_0$. We take the least upper bound for $\phi \in B$ and pass to the limit $n \to \infty$ to arrive at $\chi_{\ell_0}(\mathcal{F}_g(B)) \leq \max\{\|b\|_{\ell^\infty}, \|c\|_{\ell^\infty}\} \chi_{\ell_0}(B)$ for $B \in \mathcal{B}(\ell_0(\mathbb{K}^d))$. Having this available, our assertion follows from Theorem 3.3. □

Example 4.8. We consider a linear inhomogeneous difference equation

$$x_{k+1} = (A + C_k)x_k + (B + D_k)x_{k+1} + d_k \tag{4.11}$$

with matrices $A, B, C_k, D_k \in \mathbb{K}^{d \times d}$ such that $I - B$ is invertible for $k \in \mathbb{N}_0$ and A, B satisfy the stability assumptions stated in Example 4.2; moreover, suppose

$d \in \ell_0(\mathbb{K}^d)$. Then the above Proposition 4.7 can be applied with $g_k(x,y) = C_k x + D_k y$, $a_k = |d_k|$, $b_k = 2|C_k|$, $c_k = 2|D_k|$ and $\nu \le 1 + \frac{1}{1-\theta}\left|(I-B)^{-1}\right|$, provided the perturbations fulfill

$$2\nu \sup_{k\ge 0} \max\{|C_k|, |D_k|\} < 1.$$

Hence, by Proposition 4.7 for every initial value $\xi \in \mathbb{K}^d$ there exists a forward solution of (4.11) decaying to 0. In particular, the linear system (4.11) is asymptotically stable.

Lemma 4.9 ((ℓ^p, ℓ^1)-admissibility)**.** *Let $p \ge 1$ be a real number. If we have*

$$\nu := \max\left\{\sum_{k=0}^{\infty} |\Phi(k,0)|^p, \sup_{k\ge 1}\sum_{n=1}^{k}\left|\Phi(k,n)(I-B_{n-1})^{-1}\right|^p\right\} < \infty, \tag{4.12}$$

then the linear difference equation (4.1) *is $(\ell^p(\mathbb{K}^d), \ell^1(\mathbb{K}^d))$-admissible with linear admissibility map $\mathcal{J}$ satisfying*

$$\operatorname{lip}\mathcal{J} \le \sqrt[p]{\nu}. \tag{4.13}$$

Example 4.10. In the situation of Example 4.2 the admissibility conditions (4.12) hold with $\nu \le \frac{1}{1-\theta^p}\max\left\{1, \left|(I-B)^{-1}\right|^p\right\}$.

Proof of Lemma 4.9. Let $\psi \in \ell^1(\mathbb{K}^d)$. Analogous to the proof of Lemma 4.1 we define the unique forward solution ϕ of (4.5) satisfying $\phi_0 = \psi_0$ by (4.4). Then [Mad70, p. 167, Theorem 5] yields $\phi \in \ell^p(\mathbb{K}^d)$ and we set $\mathcal{J}(\psi) := \phi$ as admissibility map. □

Proposition 4.11. *Let $\xi \in \mathbb{K}^d$ and $p \ge 1$ be a real number. Suppose the estimate* (4.12) *holds and that the continuous mappings $g_k : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^d$ satisfy:*

(i) *there exist sequences $a, b, c \in \ell^1(\mathbb{R})$ such that*

$$|g_k(x,y)| \le a_k + \max\{b_k |x|^p, c_k |y|^p\} \quad \text{for all } k \in \mathbb{N}_0,\ x, y \in \mathbb{K}^d,$$

(ii) *the inequality $\sqrt[p]{\nu}\left[|\xi| + \|a\|_{\ell^1} + (\|b\|_{\ell^1} + \|c\|_{\ell^1})\right]\rho^p \le \rho$ admits a solution $\rho_0 > 0$.*

Then there exists a solution $\phi \in \bar{B}_{\rho_0}(\ell_0(\mathbb{K}^d))$ of the IVP (4.6).

Proof. For given initial value $\xi \in \mathbb{K}^d$ we verify the assumptions of Theorem 3.3 with sequence spaces $X = \ell^p(\mathbb{K}^d)$ and $Y = \ell^1(\mathbb{K}^d)$. Having Lemma 4.9 at hand, we find that the linear difference equation (4.1) is $(\ell^p(\mathbb{K}^d), \ell^1(\mathbb{K}^d))$-admissible and (4.13) holds. In addition, the mapping $\mathcal{G}_g : \ell^p(\mathbb{K}^d) \times \mathbb{K}^d \to \ell^1(\mathbb{K}^d)$ is well-defined, since we have

$$\|\mathcal{G}_g(\phi,\xi)\|_{\ell^1} = |\xi| + \sum_{k\ge 0} |g_k(\phi_k, \phi_{k+1})| \le |\xi| + \|a\|_{\ell^1} + (\|b\|_{\ell^1} + \|c\|_{\ell^1})\,\|\phi\|_{\ell^p}^p$$

for all $\phi \in \ell^p(\mathbb{K}^d)$ and thus (3.6) holds with $\alpha = |\xi| + \|a\|_{\ell^1}$, $\beta = \|b\|_{\ell^1} + \|c\|_{\ell^1}$, $\gamma = p$. It is not difficult to show that $\mathcal{F}_g : \ell^p(\mathbb{K}^d) \to \ell^1(\mathbb{K}^d)$ is continuous (we again refer to [DG05, Theorems 1.1, 1.3]). Now let $B \subseteq \ell^p(\mathbb{K}^d)$ be bounded and choose $\phi \in B$. As above, the translation invariance of χ_{ℓ^1} allows us to suppose $g_k(0,0) \equiv 0$ in the estimate

$$\sum_{k>n} |\mathcal{F}_g(\phi)_k| \le \sum_{k>n} \max\{b_k |\phi_k|^p, c_k |\phi_{k+1}|^p\} \le \sum_{k>n} b_k \sup_{\phi\in B}\|\phi\|_{\ell^p}^p + \sum_{k>n} c_k \sup_{\phi\in B}\|\phi\|_{\ell^p}^p$$

for all $n \geq 0$. We take the least upper bound over $\phi \in B$ and pass to the limit $n \to \infty$ to arrive at $\chi_{\ell^1}(\mathcal{F}_g(B)) = 0$ for $B \in \mathcal{B}(\ell_0(\mathbb{K}^d))$; i.e., the mapping $\mathcal{F}_g : \ell^p(\mathbb{K}^d) \to \ell^1(\mathbb{K}^d)$ is completely continuous. Then our assertion follows from Theorem 3.3. $\square$

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Technische Universität München, Zentrum Mathematik, Boltzmannstrasse 3, D-85748 Garching, Germany

E-mail address: christian.poetzsche@ma.tum.de