HAVING CUT-POINTS IS NOT
A WHITNEY REVERSIBLE PROPERTY

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Abstract. We show that the property of having cut-points is not a Whitney reversible property. This answers in the negative a question posed by Illanes and Nadler.

1. Introduction

In this note, all spaces are separable metrizable spaces and maps are continuous. We denote the interval [0, 1] by I. A compact metric space is called a compactum, and continuum means a connected compactum. If X is a continuum, C(X) denotes the space of all subcontinua of X with the topology generated by the Hausdorff metric.

A topological property \( P \) is called a Whitney property provided that if a continuum \( X \) has property \( P \), so does \( \mu^{-1}(t) \) for each Whitney map (see p. 105 of [2]) \( \mu \) for \( C(X) \) and for each \( t \in [0, \mu(X)) \). A topological property \( P' \) is called a Whitney reversible property provided that whenever \( X \) is a continuum such that \( \mu^{-1}(t) \) has property \( P' \) for all Whitney maps \( \mu \) for \( C(X) \) and all \( t \in (0, \mu(X)) \), then \( X \) has property \( P' \). These properties have been studied by many authors (see [2]).

A point \( p \) in a continuum \( X \) is called a cut-point of \( X \) provided that \( X \setminus \{p\} \) is disconnected. In this paper we prove that the property of having cut-points is not a Whitney reversible property (it is known that the property of having cut-points is not a Whitney property; see Exercise 43.4 of [2]). This answers in the negative question 43.3 of [2] posed by Illanes and Nadler.

2. Main theorem

A map \( f : X \to Y \) between continua is called an atomic map if \( f^{-1}(f(A)) = A \) for each \( A \in C(X) \) such that \( f(A) \) is nondegenerate.

A subcontinuum \( T \) of a continuum \( X \) is terminal if every subcontinuum of \( X \) which intersects both \( T \) and its complement must contain \( T \). It is known that a map \( f \) of a continuum \( X \) onto a continuum \( Y \) is atomic if and only if every fiber of \( f \) is a terminal subcontinuum of \( X \).

The main aim of this paper is to prove Theorem 2.2. To prove this theorem, we need the next result, proved by Anderson [1].

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Lemma 2.1 (see Theorem of [4]). For each continuum Y, there exist a continuum X and a monotone open map \( f : X \to Y \) such that \( f^{-1}(y) \) is a nondegenerate terminal subcontinuum of X for each \( y \in Y \).

Theorem 2.2. There exists a continuum \( Z \) such that:

(A) \( Z \) does not have a cut-point, and

(B) \( \mu^{-1}(s) \) has a cut-point for each Whitney map \( \mu : C(Z) \to [0, \mu(Z)] \) and for each \( s \in (0, \mu(Z)) \).

Proof. By Lemma 2.1, there exist a continuum X and a monotone open map \( f : X \to I \) such that \( f^{-1}(y) \) is a nondegenerate terminal subcontinuum of X for each \( y \in I \). Let Z be the quotient space obtained from X by shrinking \( f^{-1}(1) \) to the point. Let \( p : X \to Z \) be the natural projection and \( q = f \circ p^{-1} : Z \to I \). Note that q is a monotone open map such that \( q^{-1}(1) \) is a one-point set and \( q^{-1}(y) \) is a nondegenerate terminal subcontinuum of Z for each \( y \in [0, 1) \) (hence q is atomic).

We show that Z has the required properties. At first we prove that Z does not have a cut-point. Let \( z \in Z \). If \( \{z\} = q^{-1}(1) \), then \( Z \setminus \{z\} = q^{-1}([0, 1)) \). Since q is monotone, \( q^{-1}((0, 1]) \) is connected. Hence in this case \( z \) is not a cut-point of Z. Assume that \( z \in q^{-1}(t) \) for some \( t \in [0, 1) \) and that \( z \) is a cut-point of Z. Then there exist nonempty open subsets \( O, H \subset Z \) such that \( Z \setminus \{z\} = O \cup H \) and \( O \cap H = \emptyset \). Then \( O \cup \{z\} \) and \( H \cup \{z\} \) are nondegenerate continua (see Proposition 6.3 of [3]). We may assume that \( q^{-1}(t) \cap H \neq \emptyset \). Since \( O \cup \{z\} \) is a nondegenerate continuum, there exists \( \{z_i\}_{i=1}^{\infty} \subset O \) such that \( \lim_{1 \to \infty} z_i = z \). For each \( i = 1, 2, ..., \) let \( t_i = q(z_i) \). Note that \( q^{-1}(t_i) \subset O \) for each \( i = 1, 2, ... \). Since q is an open map, \( \lim_{1 \to \infty} q^{-1}(t_i) = q^{-1}(t) \). Then \( q^{-1}(t) \subset O \cup \{z\} \). This is a contradiction because \( q^{-1}(t) \cap H \neq \emptyset \). So Z does not have a cut-point.

Next we prove that \( \mu^{-1}(s) \) has a cut-point for each Whitney map \( \mu : C(Z) \to [0, \mu(Z)] \) and for each \( s \in (0, \mu(Z)) \). Take \( a, b \in (0, 1) \) such that \( \mu(q^{-1}([a, b])) = s \) (this is possible because q is a monotone open map and \( q^{-1}(1) \) is a one-point set). Now we show that

1. \( \mu^{-1}(s) = \{ C \in \mu^{-1}(s) \mid C \subset q^{-1}([0, b]) \} \cup \{ C \in \mu^{-1}(s) \mid C \subset q^{-1}([a, 1]) \} \).

If not, there exists \( D \in \mu^{-1}(s) \) such that \( D \cap q^{-1}([0, a]) \neq \emptyset \neq D \cap q^{-1}([b, 1]) \). Then \( q(D) \) contains \([a, b] \) as a proper subcontinuum of \( q(D) \). Since q is atomic, \( D = q^{-1}(q(D)) \). So \( D \) contains \( q^{-1}([a, b]) \) as a proper subcontinuum of \( D \). This is a contradiction because \( D, q^{-1}([a, b]) \subset \mu^{-1}(s) \). Hence (1) holds.

It is easy to see that

2. \( \{ C \in \mu^{-1}(s) \mid C \subset q^{-1}([0, b]) \} \cap \{ C \in \mu^{-1}(s) \mid C \subset q^{-1}([a, 1]) \} = \{ q^{-1}([a, b]) \} \)

and

3. \( \{ C \in \mu^{-1}(s) \mid C \subset q^{-1}([0, b]) \} \) and \( \{ C \in \mu^{-1}(s) \mid C \subset q^{-1}([a, 1]) \} \)

are nondegenerate subcontinua of \( \mu^{-1}(s) \).

By (1), (2) and (3) we see that \( q^{-1}([a, b]) \) is a cut-point of \( \mu^{-1}(s) \).

By this result, we see that the property of having cut-points is not a Whitney reversible property.
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References


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