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THE COMPLEX VOLUMES OF TWIST KNOTS

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Abstract. For a given hyperbolic knot, the third author defined a function whose imaginary part gives the hyperbolic volume of the knot complement. We show that, for a twist knot, the function actually gives the complex volume of the knot complement using Zickert's and Neumann's theory of the extended Bloch groups and the complex volumes.

1. Introduction

In [1], Kashaev defined an invariant of a link and conjectured the following:

$$\mathrm{vol}(L) = 2\pi \lim_{N\to\infty} \frac{\log |\langle L\rangle_N|}{N},$$

where L is a hyperbolic link, $\mathrm{vol}(L)$ is the hyperbolic volume of the link complement, $\langle L\rangle_N$ is the Kashaev invariant of L, and N is the positive-integer variable. Afterwards, in [3], the complexified volume conjecture was proposed as follows and confirmed in some cases with numerical computations:

$$i(\mathrm{vol}(L) + i\,\mathrm{cs}(L)) \equiv 2\pi \lim_{N\to\infty} \frac{\log\langle L\rangle_N}{N} \pmod{\pi^2},$$

where $\mathrm{cs}(L)$ is the Cheeger-Chern-Simons invariant of the hyperbolic link complement. We call $\mathrm{vol}(L) + i\,\mathrm{cs}(L)$ *the complex volume* of L.

On the other hand, the third author developed a special ideal triangulation of a hyperbolic knot complement in [6] and showed that this triangulation has a close relation with the Kashaev invariant. He also defined a function V from this triangulation and showed that, after evaluating V with the geometric solution of the hyperbolicity equations, the imaginary part of V becomes the volume of the knot complement.

Therefore, it is natural to expect that, after evaluating, the real part of V becomes the Cheeger-Chern-Simons invariant of the knot complement modulo π^2. In this article, we show such a coincidence for the twist knots using Zickert's and Neumann's theory of the extended Bloch groups and the complex volumes in [7] and [4].

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2. Preliminaries

For a positive integer n, let T_n be the twist knot with $n+4$ crossings as in Figure 1. We label some of the crossings with integers $0, 1, \ldots, n+1$ as in Figure 1. Also consider a (1,1)-tangle diagram whose closure is T_n. Following the method in [6], we assign a complex variable z_k to the edge of T_n that connects the over-crossing point of the kth crossing and the under-crossing point of the $(k+1)$th crossing. (See Figure 1. The arcs around the crossing points represent the surviving tetrahedra after the collapsing process, which will be briefly mentioned after Figure 2.)

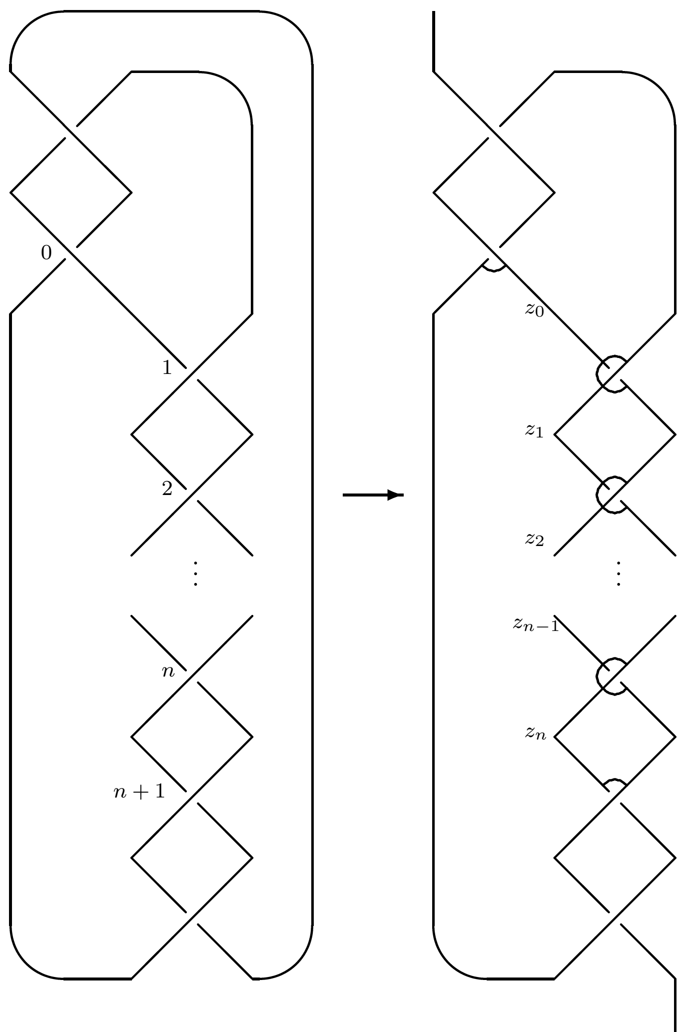

Figure 1

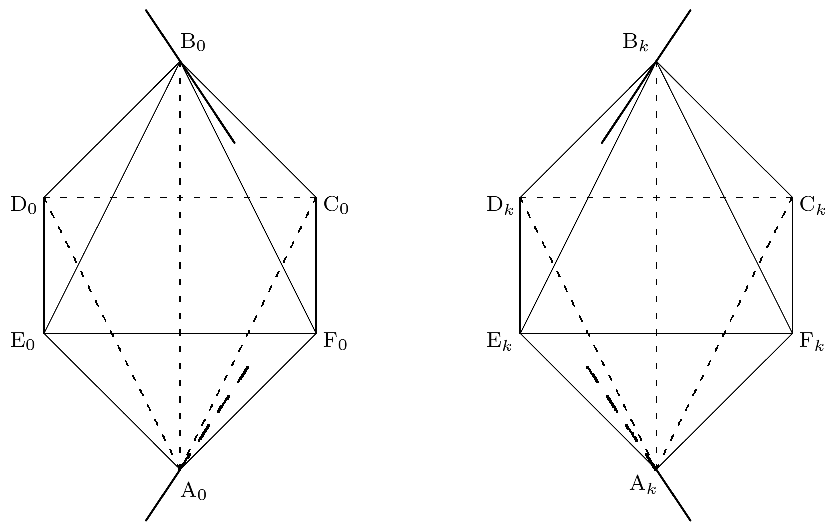

FIGURE 2

Let $I = \{1, 2, \ldots, n\}$. We put an octahedron $A_kB_kC_kD_kE_kF_k$ at the kth crossing of the tangle diagram for $k \in I \cup \{0, n+1\}$ so that A_kB_k is perpendicular to the plane; A_k and B_k touch the under-crossing point and the over-crossing point respectively; and C_k, D_k, E_k, F_k are to be the northeast, northwest, southwest, and southeast respectively. (See Figure 2.) Note that the octahedron $A_kB_kC_kD_kE_kF_k$ splits into 4 tetrahedra: in our case, $A_kB_kC_kD_k$, $A_kB_kD_kE_k$, $A_kB_kE_kF_k$, and $A_kB_kF_kC_k$.

We identify the edges A_0D_0 with A_0F_0, B_0C_0 with B_0E_0, A_kC_k with A_kE_k, and B_kD_k with B_kF_k respectively for $k \in I \cup \{n+1\}$. To obtain the ideal triangulation of the knot complement, we collapse some of the tetrahedra and glue the surviving tetrahedra together following the knot diagram. (For details, see [2] or [6].) In our case, we can obtain the ideal triangulation of the knot complement by gluing the $3n+2$ tetrahedra $A_0B_0E_0F_0$, $A_kB_kC_kD_k$, $A_kB_kD_kE_k$, $A_kB_kE_kF_k$, and $A_{n+1}B_{n+1}C_{n+1}D_{n+1}$ for $k \in I$.

To apply the method in [6] to this triangulation, we parameterize each ideal tetrahedra with complex numbers as follows. Assign z_0 to the edge A_0B_0 of $A_0B_0E_0F_0$, z_{k-1} to the edge A_kB_k of $A_kB_kC_kD_k$, $\frac{z_k}{z_{k-1}}$ to the edge A_kB_k of $A_kB_kD_kE_k$, $\frac{1}{z_k}$ to the edge A_kB_k of $A_kB_kE_kF_k$, and z_n to the edge $A_{n+1}B_{n+1}$ of $A_{n+1}B_{n+1}C_{n+1}D_{n+1}$ respectively for $k \in I$. Then, from [5] or [6], the hyperbolicity equations of the knot complement become

$$\begin{aligned} 1 - \frac{z_0}{z_1} &= (1 - z_0)(1 - \frac{1}{z_0}), \\ (1 - \frac{z_k}{z_{k+1}})(1 - \frac{1}{z_k}) &= (1 - z_k)(1 - \frac{z_{k-1}}{z_k}) \quad \text{for } k = 1, 2, \ldots, n-1, \\ 1 - \frac{1}{z_n} &= (1 - z_n)(1 - \frac{z_{n-1}}{z_n}). \end{aligned} \tag{2.1}$$

Note that the solution of these equations which induces the complete hyperbolic structure of the knot complement exists uniquely by virtue of the Lemma 2.3 of [6]. In this article, the unique solution is called *the geometric solution.*

Now let r_k be the even integer satisfying the following equations for $k \in I \cup \{0\}$:

$$\begin{aligned} r_0 \pi i &= \log(1 - \frac{z_0}{z_1}) - \log(1 - z_0) - \log(1 - \frac{1}{z_0}), \\ r_k \pi i &= \log(1 - \frac{z_k}{z_{k+1}}) + \log(1 - \frac{1}{z_k}) - \log(1 - z_k) - \log(1 - \frac{z_{k-1}}{z_k}) \\ &\qquad \text{for } k = 1, 2, \ldots, n-1, \\ r_n \pi i &= \log(1 - \frac{1}{z_n}) - \log(1 - z_n) - \log(1 - \frac{z_{n-1}}{z_n}). \end{aligned} \tag{2.2}$$

Then we can determine the V function, following [6], as

$$\begin{aligned} V(z_0, z_1, \ldots, z_n) = &- \sum_{k \in I \cup \{0\}} r_k \pi i \log z_k + \left(\frac{\pi^2}{6} - \mathrm{Li}_2(\frac{1}{z_0}) \right) \\ &+ \sum_{k \in I} \left\{ \left(\mathrm{Li}_2(z_{k-1}) - \frac{\pi^2}{6} \right) + \left(\frac{\pi^2}{6} - \mathrm{Li}_2(\frac{z_{k-1}}{z_k}) \right) + \left(\mathrm{Li}_2(\frac{1}{z_k}) - \frac{\pi^2}{6} \right) \right\} \\ &+ \left(\mathrm{Li}_2(z_n) - \frac{\pi^2}{6} \right). \end{aligned} \tag{2.3}$$

With these notations, we state the main theorem of this article.

Theorem 2.1. *Let $z_0, z_1, \ldots, z_n$ be the geometric solution of* (2.1)*, which induces the complete hyperbolic structure of the T_n complement. Then*

$$V(z_0, z_1, \ldots, z_n) \equiv i(\mathrm{vol}(T_n) + i\,\mathrm{cs}(T_n)) \pmod{\pi^2}.$$

We will prove this theorem in Section 3.

The main technique of the proof is Zickert's article [7]. To apply it, we need the vertex ordering of each tetrahedron. We equip the vertex ordering according to the order of the vertices as follows: $\mathrm{B}_0\mathrm{A}_0\mathrm{F}_0\mathrm{E}_0$, $\mathrm{A}_1\mathrm{B}_1\mathrm{D}_1\mathrm{C}_1$, $\mathrm{A}_k\mathrm{B}_k\mathrm{C}_k\mathrm{D}_k$ for $k = 2, 3, \ldots, n+1$, $\mathrm{A}_k\mathrm{B}_k\mathrm{E}_k\mathrm{D}_k$ for $k \in I$, and $\mathrm{A}_k\mathrm{B}_k\mathrm{E}_k\mathrm{F}_k$ for $k \in I$. For example, the vertices A_k, B_k, E_k, D_k of the tetrahedron $\mathrm{A}_k\mathrm{B}_k\mathrm{E}_k\mathrm{D}_k$ have the order 0, 1, 2, 3 respectively.

Note that the vertex ordering of each tetrahedron induces orientations of edges and the tetrahedron. The induced orientation can be different from the original orientation induced by the knot complement. In our case, such a situation arises for $\mathrm{A}_1\mathrm{B}_1\mathrm{D}_1\mathrm{C}_1$ and $\mathrm{A}_k\mathrm{B}_k\mathrm{E}_k\mathrm{D}_k$ for $k \in I$.

To apply the main technique, we need the following lemma, which is induced directly from the gluing pattern of the triangulation.

Lemma 2.2. *The ideal triangulation of the T_n complement given as above is in concordance with the ordering of each edge which is induced by the vertex ordering defined above. In other words, if two edges are glued together by the triangulation, then the orientations of two edges induced by each vertex ordering coincide.*

Now we will determine the element of the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ corresponding to each tetrahedron using the method of [7]. At first, let $J = \{1, 2, 3\}$

and define a sign function $\sigma_k^{(j)}$ as

$$\sigma_k^{(j)} = \begin{cases} -1 & \text{if } j=2 \text{ or } (k,j)=(1,1), \\ 1 & \text{otherwise,} \end{cases} \tag{2.4}$$

for $k \in I \cup \{0, n+1\}$ and $j \in J$ except $(k,j) = (0,1), (0,2), (n+1,2), (n+1,3)$. Also let

$$\begin{gathered} \sigma_0^{(3)}[z_0^{(3)}; p_0^{(3)}, q_0^{(3)}],\ \sigma_1^{(1)}[z_1^{(1)}; p_1^{(1)}, q_1^{(1)}],\ \sigma_k^{(1)}[z_k^{(1)}; p_k^{(1)}, q_k^{(1)}] \text{ for } k \neq 1, \\ \sigma_k^{(2)}[z_k^{(2)}; p_k^{(2)}, q_k^{(2)}],\ \sigma_k^{(3)}[z_k^{(3)}; p_k^{(3)}, q_k^{(3)}], \text{ and } \sigma_{n+1}^{(1)}[z_{n+1}^{(1)}; p_{n+1}^{(1)}, q_{n+1}^{(1)}] \end{gathered} \tag{2.5}$$

be the elements of $\widehat{\mathcal{B}}(\mathbb{C})$ corresponding to the tetrahedra $\mathrm{B}_0\mathrm{A}_0\mathrm{F}_0\mathrm{E}_0$, $\mathrm{A}_1\mathrm{B}_1\mathrm{D}_1\mathrm{C}_1$, $\mathrm{A}_k\mathrm{B}_k\mathrm{C}_k\mathrm{D}_k$, $\mathrm{A}_k\mathrm{B}_k\mathrm{E}_k\mathrm{D}_k$, $\mathrm{A}_k\mathrm{B}_k\mathrm{E}_k\mathrm{F}_k$, and $\mathrm{A}_{n+1}\mathrm{B}_{n+1}\mathrm{C}_{n+1}\mathrm{D}_{n+1}$ for $k \in I$ respectively. Then, by the definition above and the parameters of the ideal tetrahedra,

$$\begin{gathered} z_0^{(3)} = z_0,\ z_1^{(1)} = \frac{1}{z_0},\ z_k^{(1)} = z_{k-1} \text{ for } k \neq 1, \\ z_k^{(2)} = \frac{z_{k-1}}{z_k},\ z_k^{(3)} = \frac{1}{z_k}, \text{ and } z_{n+1}^{(1)} = z_n, \end{gathered} \tag{2.6}$$

for $k \in I$. To determine the integers $p_k^{(j)}$ and $q_k^{(j)}$ for $k \in I \cup \{0, n+1\}$ and $j \in J$, we use the following definition from (3.5) of [7]. For a tetrahedron with a vertex ordering,

$$p\pi i + \log z = \log c(g_{03}) + \log c(g_{12}) - \log c(g_{02}) - \log c(g_{13}), \tag{2.7}$$

$$q\pi i - \log(1-z) = \log c(g_{02}) + \log c(g_{13}) - \log c(g_{01}) - \log c(g_{23}), \tag{2.8}$$

where $[z; p, q]$ is the element of $\widehat{\mathcal{B}}(\mathbb{C})$ corresponding to the ideal tetrahedron and $c(g_{ab})$ is the complex number assigned to the edge connecting the ath and bth vertex.

Zickert gave an explicit method for calculating the complex numbers in [7], but, in this article, we do not need the exact numbers. What we need is only the fact that if two edges are identified by the gluing pattern of the ideal triangulation, then the numbers assigned to these identified edges are the same. Using this fact, the following lemma is directly deduced from the gluing pattern of the triangulation.

Lemma 2.3. *Let α_k, β_k, γ_k, γ_{k+1}, m_k, and δ be the complex numbers assigned to the edges $\mathrm{B}_k\mathrm{E}_k$, $\mathrm{B}_k\mathrm{F}_k$, $\mathrm{C}_k\mathrm{D}_k$, $\mathrm{E}_k\mathrm{F}_k$, $\mathrm{A}_k\mathrm{B}_k$, and $\mathrm{D}_k\mathrm{E}_k$ respectively for $k \in I$. Also let α_0 and β_0 be the complex numbers assigned to the edges $\mathrm{B}_0\mathrm{F}_0$ and $\mathrm{B}_0\mathrm{E}_0$ respectively. Then we can determine all the assigned complex numbers of each edge using α_k, β_k, γ_k, m_k, δ, α_0, β_0, and γ_{n+1} for $k \in I$ as in Figure* 3.

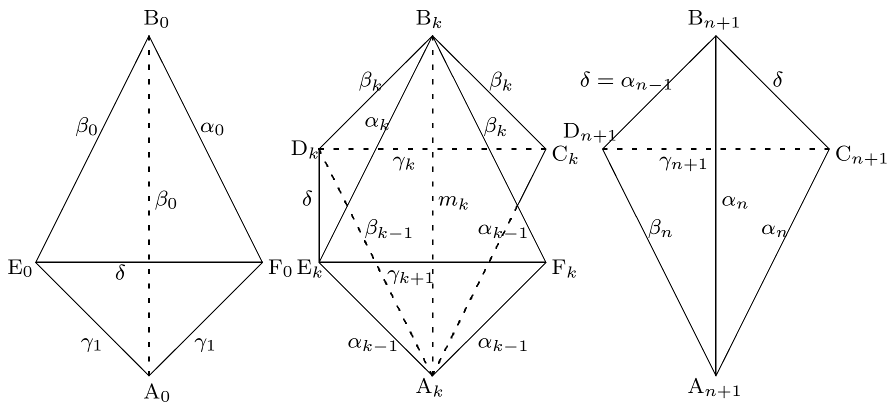

FIGURE 3

The following lemma is a direct consequence of (2.5), Lemma 2.2, and [4] and [7].

Lemma 2.4. *Let* $z_0, z_1, \ldots, z_n$ *be the geometric solution of* (2.1) *and let* $\sigma_k^{(j)}[z_k^{(j)}; p_k^{(j)}, q_k^{(j)}]$ *be the element of the extended Bloch group* $\widehat{\mathcal{B}}(\mathbb{C})$ *defined in* (2.4), (2.5), *and* (2.6). *Then*

$$i(\mathrm{vol}(T_n) + i\,\mathrm{cs}(T_n)) \equiv \sum_{\substack{k \in I \cup \{0, n+1\} \\ j \in J}} \sigma_k^{(j)} \widehat{L}[z_k^{(j)}; p_k^{(j)}, q_k^{(j)}] \pmod{\pi^2}, \tag{2.9}$$

where $\widehat{L}[z; p, q] = \mathrm{Li}_2(z) + \frac{1}{2}\log z \log(1-z) + \frac{\pi i}{2}(q \log z + p \log(1-z)) - \frac{\pi^2}{6}$ *is the extended dilogarithm function defined in* $\widehat{\mathcal{B}}(\mathbb{C})$.

To prove the main theorem, we prove the coincidence of the right side of (2.3) and the right side of (2.9) in the next section.

3. PROOF OF THE THEOREM 2.1

To simplify the calculation, we define $\hat{p}_k^{(j)}$ by

$$\hat{p}_k^{(j)} \pi i = p_k^{(j)} \pi i + \log z_k^{(j)}$$

for $k \in I \cup \{0, n+1\}$ and $j \in J$. Then, the right side of (2.9) becomes

$$\begin{aligned}
&\sum_{\substack{k \in I \cup \{0, n+1\} \\ j \in J}} \sigma_k^{(j)} \left\{ \mathrm{Li}_2(z_k^{(j)}) - \frac{\pi^2}{6} \right\} \\
&\qquad + \frac{\pi i}{2} \sum_{\substack{k \in I \cup \{0, n+1\} \\ j \in J}} \sigma_k^{(j)} \{ q_k^{(j)} \log z_k^{(j)} + \hat{p}_k^{(j)} \log(1 - z_k^{(j)}) \} \\
&= V(z_0, z_1, \ldots, z_n) + \sum_{k \in I \cup \{0\}} r_k \pi i \log z_k \\
&\qquad + \frac{\pi i}{2} \sum_{\substack{k \in I \cup \{0, n+1\} \\ j \in J}} \sigma_k^{(j)} \{ q_k^{(j)} \log z_k^{(j)} + \hat{p}_k^{(j)} \log(1 - z_k^{(j)}) \}.
\end{aligned}$$

Therefore, it is enough to show that

$$\sum_{\substack{k\in I\cup\{0,n+1\}\\ j\in J}} \sigma_k^{(j)}\pi i\{q_k^{(j)}\log z_k^{(j)}+\hat{p}_k^{(j)}\log(1-z_k^{(j)})\} \tag{3.1}$$
$$+2\left\{\sum_{k\in I\cup\{0\}} r_k\pi i\log z_k\right\}\equiv 0 \pmod{2\pi^2}.$$

Lemma 3.1.

$$\begin{aligned}
r_0\pi i &= (q_1^{(2)}-q_0^{(3)}-q_1^{(1)})\pi i,\\
r_k\pi i &= (q_k^{(3)}-q_k^{(2)}+q_{k+1}^{(2)}-q_{k+1}^{(1)})\pi i\\
&\qquad \textit{for } k=1,2,\ldots,n-1,\\
r_n\pi i &= (q_n^{(3)}-q_{n+1}^{(1)}-q_n^{(2)})\pi i.
\end{aligned}$$

Proof. From Lemma 2.3 and (2.8), we obtain

$$\begin{aligned}
q_1^{(1)}\pi i &= \log(1-\frac{1}{z_0})+\log\beta_0+\log\beta_1-\log\gamma_1-\log m_1,\\
q_{k+1}^{(1)}\pi i &= \log(1-z_k)+\log\alpha_k+\log\beta_{k+1}-\log\gamma_{k+1}-\log m_{k+1}\\
&\quad \text{for } k=1,2,\ldots,n-1,\\
q_{n+1}^{(1)}\pi i &= \log(1-z_n)+\log\alpha_{n-1}-\log\gamma_{n+1},\\
q_k^{(2)}\pi i &= \log(1-\frac{z_{k-1}}{z_k})+\log\alpha_{k-1}+\log\beta_k-\log\delta-\log m_k \quad \text{for } k\in I,\\
q_0^{(3)}\pi i &= \log(1-z_0)+\log\alpha_0+\log\gamma_1-\log\beta_0-\log\delta,\\
q_k^{(3)}\pi i &= \log(1-\frac{1}{z_k})+\log\alpha_{k-1}+\log\beta_k-\log\gamma_{k+1}-\log m_k \quad \text{for } k\in I.
\end{aligned}$$

Applying these equations and $\delta=\alpha_{n-1}$ to (2.2), we obtain the result. □

Lemma 3.2.

$$\begin{aligned}
\sum_{\substack{k\in I\cup\{0,n+1\}\\ j\in J}} \sigma_k^{(j)}\hat{p}_k^{(j)}\pi i\log(1-z_k^{(j)}) &\equiv \sum_{\substack{k\in I\cup\{0,n+1\}\\ j\in J}} \sigma_k^{(j)}q_k^{(j)}\pi i\log z_k^{(j)}\\
&\equiv -\sum_{k\in I\cup\{0\}} r_k\pi i\log z_k \pmod{2\pi^2}.
\end{aligned}$$

Proof. From (2.7) and Lemma 2.3, we know that

$$\begin{aligned}
\hat{p}_1^{(1)}\pi i &= \log\alpha_0-\log\beta_0,\\
\hat{p}_{k+1}^{(1)}\pi i &= -\log\alpha_k+\log\beta_k,\\
\hat{p}_k^{(2)}\pi i &= \log\alpha_k-\log\beta_k-\log\alpha_{k-1}+\log\beta_{k-1},\\
\hat{p}_0^{(3)}\pi i &= -\log\alpha_0+\log\beta_0,\\
\hat{p}_k^{(3)}\pi i &= \log\alpha_k-\log\beta_k,
\end{aligned}$$

for $k \in I$. Using these equations, (2.2), and Lemma 3.1, we can finish the proof as follows:

$$
\begin{aligned}
&\sum_{\substack{k\in I\cup\{0,n+1\}\\ j\in J}} \sigma_k^{(j)} \hat{p}_k^{(j)} \pi i \log(1-z_k^{(j)}) \\
&= -\hat{p}_1^{(1)}\pi i \log(1-\frac{1}{z_0}) + \sum_{k\in I} \hat{p}_{k+1}^{(1)} \pi i \log(1-z_k) \\
&\qquad - \sum_{k\in I} \hat{p}_k^{(2)} \pi i \log(1-\frac{z_{k-1}}{z_k}) + \hat{p}_0^{(3)}\pi i \log(1-z_0) + \sum_{k\in I} \hat{p}_k^{(3)} \pi i \log(1-\frac{1}{z_k}) \\
&= (-\log\alpha_0 + \log\beta_0)\left\{\log(1-\frac{1}{z_0}) - \log(1-\frac{z_0}{z_1}) + \log(1-z_0)\right\} \\
&\qquad + \sum_{k=1}^{n-1} (\log\alpha_k - \log\beta_k)\left\{-\log(1-z_k) - \log(1-\frac{z_{k-1}}{z_k})\right. \\
&\qquad\qquad\qquad \left. + \log(1-\frac{z_k}{z_{k+1}}) + \log(1-\frac{1}{z_k})\right\} \\
&\qquad + (\log\alpha_n - \log\beta_n)\left\{-\log(1-z_n) - \log(1-\frac{z_{n-1}}{z_n}) + \log(1-\frac{1}{z_n})\right\} \\
&\equiv -\sum_{k\in I\cup\{0\}} r_k \pi i \log z_k = -(q_1^{(2)} - q_0^{(3)} - q_1^{(1)})\pi i \log z_0 \\
&\qquad - \sum_{k=1}^{n-1} (q_k^{(3)} - q_k^{(2)} + q_{k+1}^{(2)} - q_{k+1}^{(1)})\pi i \log z_k - (q_n^{(3)} - q_{n+1}^{(1)} - q_n^{(2)})\pi i \log z_n \\
&\equiv \sum_{\substack{k\in I\cup\{0,n+1\}\\ j\in J}} \sigma_k^{(j)} q_k^{(j)} \pi i \log z_k^{(j)} \pmod{2\pi^2}.
\end{aligned}
$$

Applying Lemma 3.2 to (3.1), the proof of the theorem is finished. □

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