PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 10, October 2009, Pages 3227–3233
S 0002-9939(09)09913-4
Article electronically published on May 27, 2009

AFFINE ALGEBRAIC MONOIDS AS ENDOMORPHISMS' MONOIDS OF FINITE-DIMENSIONAL ALGEBRAS

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(Communicated by Birge Huisgen-Zimmermann)

ABSTRACT. We prove that any affine algebraic monoid can be obtained as the endomorphisms' monoid of a finite-dimensional (nonassociative) algebra.

1. INTRODUCTION

Let K be an algebraically closed field of arbitrary characteristic. Recall that an *affine algebraic semigroup* is an affine variety M over K with an associative product $\mu\colon M \times M \to M$, which is a morphism of algebraic varieties. Denote an element $\mu(a, b)$ by ab. A semigroup is called a *monoid* if it contains an identity element $e \in M$ such that $em = me = m$ for any $m \in M$. An element $0 \in M$ is called *zero* if $0m = m0 = 0$ for any $m \in M$. Obviously, a monoid cannot contain more than one zero. It is well known that every affine algebraic monoid is isomorphic to a Zariski closed submonoid of the monoid $\mathrm{L}(V)$ of all linear operators on some finite-dimensional vector space V; e.g. see [4, Theorem 3.8] or [1, Lemma 1.11]. A systematic account of the theory of affine algebraic monoids is given in [3] and [4]. The classification of irreducible affine monoids, whose unit group is reductive, is obtained in [5] and [6].

Let A be a finite-dimensional algebra over the field K, i.e. a finite-dimensional vector space A with a bilinear map $\alpha\colon A \times A \to A$. Note that the associativity or commutativity of the map α is not assumed. It is convenient to denote by $\mathrm{vect}(A)$ the underlying vector space of an algebra A. By an ideal of an algebra A we mean a two-sided ideal. An algebra A is called *simple* if it does not contain proper ideals. The set of all endomorphisms of A,

$$\mathrm{End}(A) := \{\phi \in \mathrm{L}(\mathrm{vect}(A)) \mid \alpha(\phi(a), \phi(b)) = \phi(\alpha(a, b)) \text{ for } a, b \in A\},$$

is a monoid with respect to composition. It is easy to check that this monoid is Zariski closed in $\mathrm{L}(\mathrm{vect}(A))$; therefore it is an affine algebraic monoid.

It is shown in [2] that any affine algebraic group can be realized as the group of automorphisms of some finite-dimensional simple algebra. This paper aims to obtain a similar realization of an arbitrary affine algebraic monoid M as the endomorphisms' monoid of a finite-dimensional algebra A. In this case two differences occur. First, we cannot assume that A is simple, since the kernel of any endomorphism is an ideal of A. Second, the monoid $\mathrm{End}(A)$ contains the zero $\mathfrak{z} \in \mathrm{End}(A)$,

Received by the editors September 13, 2008.
2000 *Mathematics Subject Classification.* Primary 17A36, 20M20; Secondary 16W22, 20G20.

$\mathfrak{z}(a) = 0$ for any $a \in A$, while M does not necessarily contain a zero. Under these circumstances we obtain the following result.

Theorem 1.1. *For any affine algebraic monoid M there exists a finite-dimensional algebra A such that* $\mathrm{End}(A) \cong M \sqcup \{\mathfrak{z}\}$*, where $\{\mathfrak{z}\}$ is an (isolated) component of the monoid* $\mathrm{End}(A)$*.*

Particularly, if M is an affine algebraic group, then there exists an algebra A such that $\mathrm{Aut}(A) \cong M$ (see [2]).

Example 1.2. Let us consider the monoid $M = \mathrm{L}(V)$ for a finite-dimensional space V. Then we may take the algebra A constructed in the following way. First, let e be a left identity of A and

$$\mathrm{vect}(A) := \langle e \rangle \oplus V,$$

where $\langle X \rangle$ stands for the linear span of a set X. Next, for any $v, w \in V$ put $\alpha(v, w) = 0$, $\alpha(v, e) = \lambda v$, where $\lambda \in \mathrm{K} \setminus \{0, 1\}$. Taking into account equations $\alpha(e, v) = v$ and $\alpha(e, e) = e$, we obtain the multiplication table for A.

Note that any endomorphism sends e to e or 0, since these two are the only idempotents of A. This way, the reader will easily prove that $\mathrm{End}(A) \cong \mathrm{L}(V) \sqcup \{\mathfrak{z}\}$.

Example 1.3. Assume $\mathrm{char}\,\mathrm{K} \neq 2$. Consider a two-dimensional space V over K with a basis $\{v_1, v_2\}$ and the exterior algebra $\Lambda(V)$ with a basis $\{1, v_1, v_2, v_1 \wedge v_2\}$. Let us take a monoid $M \subset \mathrm{L}(\mathrm{vect}(\Lambda(V)))$,

$$M := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ 0 & c_1 & c_2 & d \end{pmatrix} \middle| \, d = \det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, b_{ij}, c_i \in \mathrm{K} \right\}.$$

One may prove that M acts on $\Lambda(V)$ by endomorphisms. Moreover, $\mathrm{End}(\Lambda(V)) = M \sqcup \{\mathfrak{z}\}$. Generally, a similar equation holds for the exterior algebra of an arbitrary space.

The proof of Theorem 1.1 consists of two steps. First, for every finite-dimensional space U and its subspace S we construct a finite-dimensional algebra A such that $\mathrm{End}(A)$ is isomorphic to $\mathrm{L}(U)_S \sqcup \{\mathfrak{z}\}$, where $\mathrm{L}(U)_S$ is the normalizer of some vector subspace S of a special $\mathrm{L}(U)$-module. Second, an arbitrary affine algebraic monoid M is represented as $\mathrm{L}(U)_S$ for appropriate U and S. Overall, we follow the scheme of the proof in [2], but the ideas of the first step are significantly changed.

2. Some special algebras

In this section we define and study some finite-dimensional algebras to be used hereafter.

2.1. **Algebra $A(V, S)$.** Let V be a nonzero finite-dimensional vector space. Denote by $\mathrm{T}(V)$ the tensor algebra of V and by $\mathrm{T}(V)_+$ its maximal homogeneous ideal,

$$\mathrm{T}(V)_+ := \textstyle\bigoplus_{i \geqslant 1} V^{\otimes i}, \tag{2.1}$$

endowed with the natural $\mathrm{L}(V)$-structure

$$g \cdot t_i := g^{\otimes i}(t_i), \quad g \in \mathrm{L}(V),\ t_i \in V^{\otimes i}. \tag{2.2}$$

Thus, $\mathrm{L}(V)$ acts on $\mathrm{T}(V)_+$ faithfully by endomorphisms. Therefore we may identify $\mathrm{L}(V)$ with the corresponding submonoid of $\mathrm{End}(\mathrm{T}(V)_+)$.

Fix an integer $r > 1$. For an arbitrary subspace $S \subseteq V^{\otimes r}$ we define

$$I(S) := S \oplus (\bigoplus_{i>r} V^{\otimes i}). \tag{2.3}$$

It is an ideal of $\mathrm{T}(V)_+$. Define $A(V,S)$ as the factor algebra modulo this ideal,

$$A(V,S) := \mathrm{T}(V)_+/I(S). \tag{2.4}$$

Then

$$\mathrm{vect}(A(V,S)) = (\bigoplus_{i=1}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S). \tag{2.5}$$

We may consider $\mathrm{L}(V)_S := \{\phi \in \mathrm{L}(V) \mid \phi(S) \subseteq S\} \subset \mathrm{L}(V)$.

Proposition 2.1. $\{\sigma \in \mathrm{End}(A(V,S)) \mid \sigma(V) \subseteq V\} = \mathrm{L}(V)_S$.

Proof. By definition, elements of $A(V,S)$ are equivalence classes $x + I(S)$, $x \in \mathrm{T}(V)_+$. Let us prove the inclusion $\subseteq$. Consider $\sigma \in \mathrm{End}(A(V,S))$ such that $\sigma(V) \subseteq V$. Then the σ-action coincides with the action of $\widetilde{\sigma} := \sigma|_V \in \mathrm{L}(V)$ on $A(V,S)$ in accordance with (2.2), since the algebra $A(V,S)$ is generated by V. The σ-action preserves the zero of $A(V,S)$; hence $\widetilde{\sigma}(I(S)) \subseteq I(S)$ and $\sigma \in \mathrm{L}(V)_S$.

Now we prove the inverse inclusion. For arbitrary subsets $X, Y \subset \mathrm{T}(V)$ define $X \otimes Y := \{x \otimes y \mid x \in X, y \in Y\} \subset \mathrm{T}(V)$. Let $\sigma \in \mathrm{L}(V)_S$. Then $\sigma((x + I(S)) \otimes (y + I(S))) \subseteq \sigma(x \otimes y) + I(S) = \sigma(x) \otimes \sigma(y) + I(S)$ by definition of the $\mathrm{L}(V)$-action on $\mathrm{T}(V)_+$. Hence $\sigma \in \mathrm{End}(A(V,S))$. □

2.2. **Algebra** $D(P,U,S,\gamma)$**.**

Lemma 2.2. *Let A be an algebra with a left identity $e \in A$ such that* $\mathrm{vect}(A) = \langle e \rangle \oplus A_1 \oplus \cdots \oplus A_r$*, where A_i is the eigenspace with an eigenvalue $\alpha_i \neq 0, 1$ for the operator of right multiplication of A by e. Assume that* 0 *and e are the only idempotents in A. Then*

(i) *e is the unique left identity in A;*
(ii) *if $\sigma \in \mathrm{End}(A)$, then either $\sigma(e) = e$ and $\sigma(A_i) \subseteq A_i$ for any i, or $\sigma = \mathfrak{z}$.*

Proof. (i) The left identity is a nonzero idempotent. Hence it is unique.

(ii) Since the image of an idempotent is an idempotent, $\sigma(e) = 0$ or $\sigma(e) = e$. If $\sigma(e) = 0$, then $\sigma(a) = \sigma(ea) = \sigma(e)\sigma(a) = 0$, i.e. $\sigma = \mathfrak{z}$. Now assume that $\sigma(e) = e$. Then $\sigma(A_i)$ is the eigenspace with an eigenvalue $\alpha_i \neq 0, 1$ for the operator of right multiplication by e. Hence $\sigma(A_i) \subseteq A_i$. □

Let P be a two-dimensional vector space with a basis $\{p_1, p_2\}$, U be a nonzero finite-dimensional space, and

$$V := P \oplus U. \tag{2.6}$$

Fix an integer $r > 1$ as well as

(i) a subspace $S \subset V^{\otimes r}$;
(ii) a sequence $\gamma = (\gamma_1, \ldots, \gamma_6) \in (\mathrm{K} \setminus \{0,1\})^6$, $\gamma_i \neq \gamma_j$ for $i \neq j$.

Define an algebra $D(P,U,S,\gamma)$ in the following way. First, $A(V,S)$ is the subalgebra of $D(P,U,S,\gamma)$ and elements $b, c, d, e \in D(P,U,S,\gamma)$ are such that

$$\mathrm{vect}(D(P,U,S,\gamma)) = \langle e \rangle \oplus \langle b \rangle \oplus \langle c \rangle \oplus \langle d \rangle \oplus \mathrm{vect}(A(V,S)). \tag{2.7}$$

Second, the following conditions hold:

(D1) e is the left identity of $D(P,U,S,\gamma)$.

(D2) $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ as well as $P, U \subset V = V^{\otimes 1} \subset A(V,S)$ and $(\bigoplus_{i=2}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S) \subset A(V,S)$ are the eigenspaces with the eigenvalues $\gamma_1, \dots, \gamma_6$, respectively, of the operator of right multiplication by e.

(D3) The multiplication table for b, c, d is

$$\begin{array}{lll} b\cdot b := 0, & b\cdot c := c + \gamma_{bc} b, & b\cdot d := 0, \\ c\cdot b := -c, & c\cdot c := b, & c\cdot d := e, \\ d\cdot b := p_1, & d\cdot c := d, & d\cdot d := p_2, \end{array} \tag{2.8}$$

where $\gamma_{bc} = \frac{\gamma_2-\gamma_1}{\gamma_2-\gamma_3}$.

(D4) $\langle b,c,d\rangle \cdot A(V,S) = A(V,S)\cdot \langle b,c,d\rangle = 0$.

Define the action of $g \in \mathrm{L}(V)_S$ on $\mathrm{vect}(D(P,U,S,\gamma))$ as follows: $g|_{\langle b\rangle} = g|_{\langle c\rangle} = g|_{\langle d\rangle} = g|_{\langle e\rangle} = \mathrm{id}$, $g|_V$ is the natural $\mathrm{L}(V)$-action on V, and on other summands of $A(V,S)$ it is defined by (2.2). By Proposition 2.1 we may identify $\mathrm{L}(V)_S$ with the corresponding submonoid of $\mathrm{L}(\mathrm{vect}(D(P,U,S,\gamma)))$. Further, we may consider an embedding $\mathrm{L}(U) \hookrightarrow \mathrm{L}(V)$, $h \mapsto \mathrm{id}|_P \oplus h$. Thus, $\mathrm{L}(U)_S \subseteq \mathrm{L}(V)_S$, and we obtain the $\mathrm{L}(U)_S$-action on $\mathrm{vect}(D(P,U,S,\gamma))$.

Proposition 2.3. *We have*

$$\mathrm{End}(D(P,U,S,\gamma)) = \mathrm{L}(U)_S \sqcup \{\mathfrak{z}\},$$

where $\{\mathfrak{z}\}$ *is an (isolated) component of the monoid* $\mathrm{End}(D(P,U,S,\gamma))$.

Proof. First of all, we show that 0 and e are the only idempotents of $D(P,U,S,\gamma)$. Indeed, let $\varepsilon = \lambda_e e + \lambda_b b + \lambda_c c + \lambda_d d + a$, where $a \in A(V,S)$. Then

$$\begin{aligned} \varepsilon^2 &= (\lambda_e^2 + \lambda_c\lambda_d)e + (\lambda_b\lambda_e(1+\gamma_1) + \lambda_c^2 + \lambda_b\lambda_c\gamma_{bc})b \\ &\quad + \lambda_c\lambda_e(1+\gamma_2)c + ((1+\gamma_3)\lambda_d\lambda_e + \lambda_d\lambda_c)d + a' \\ &= \lambda_1 e + \lambda_2 d + \lambda_3 c + \lambda_4 b + a, \text{ where } a, a' \in A(V,S). \end{aligned} \tag{2.9}$$

Hence

$$\lambda_e = \lambda_e^2 + \lambda_c\lambda_d, \tag{2.10}$$

$$\lambda_b = \lambda_b\lambda_e(1+\gamma_1) + \lambda_c^2 + \lambda_b\lambda_c\gamma_{bc}, \tag{2.11}$$

$$\lambda_c = \lambda_c\lambda_e(1+\gamma_2), \tag{2.12}$$

$$\lambda_d = \lambda_d\lambda_e(1+\gamma_3) + \lambda_c\lambda_d. \tag{2.13}$$

Assume $\lambda_c \neq 0$. By (2.12), $1+\gamma_2 \neq 0, \lambda_e = \frac{1}{1+\gamma_2}$ and $\lambda_c\lambda_d = \lambda_e - \lambda_e^2 \neq 0$, so $\lambda_d \neq 0$. Hence equation (2.13) implies $\lambda_c = 1 - \lambda_e(1+\gamma_3) = \frac{\gamma_2-\gamma_3}{1+\gamma_2}$. Finally, by (2.11) we have $\lambda_c^2 = \lambda_b(1 - \lambda_e(1+\gamma_1) - \lambda_c\gamma_{bc}) = \lambda_b(\frac{\gamma_2-\gamma_1}{1+\gamma_2} - \frac{\gamma_2-\gamma_3}{1+\gamma_2}\cdot\frac{\gamma_2-\gamma_1}{\gamma_2-\gamma_3}) = 0$. From this contradiction we deduce $\lambda_c = 0$.

Moreover, $\lambda_e = 0$ or $\lambda_e = 1$ by (2.10). If $\lambda_e = 0$, then $\lambda_b = \lambda_d = 0$, $\varepsilon = a \in A(V,S)$ and $\varepsilon = 0$, since zero is the only idempotent of $A(V,S)$. Now assume $\lambda_e = 1$. From equations (2.11) and (2.13) accordingly follow $\lambda_b = 0$ and $\lambda_d = 0$. Thus, $\varepsilon = e + a$, $a \in A(V,S)$.

Let $a = a_P + a_U + a_\Sigma$, where $a_P \in P, a_U \in U, a_\Sigma \in (\bigoplus_{i=2}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S)$. Then

$$\varepsilon^2 = e + (1+\gamma_4)a_P + (1+\gamma_5)a_U + a'_\Sigma = e + a_P + a_U + a_\Sigma, \tag{2.14}$$

where $a'_\Sigma \in (\bigoplus_{i=2}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S)$. Hence $a_U = a_P = 0$. Assume $a_\Sigma \neq 0$. Then we may write $a_\Sigma = a_k + \ldots + a_r$, $a_k \neq 0$, where $a_i \in V^{\otimes i}$ for $i < r$ and $a_r \in V^{\otimes r}/S$. This way,

$$(e + a_k + \ldots + a_r)^2 = e + (1 + \gamma_6)a_k + a'' = e + a_k + \ldots + a_r, \tag{2.15}$$

where $a'' \in (\bigoplus_{i=k+1}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S)$ for $k < r$ and $a'' = 0$ for $k = r$. This implies $a_k = 0$, a contradiction. Hence $a_\Sigma = 0$ and $\varepsilon = e$.

Thus, $D(P, U, S, \gamma)$ contains no idempotents different from 0 and e. Let $\sigma \in \mathrm{End}(D(P, U, S, \gamma)) \setminus \{\mathfrak{z}\}$. By Lemma 2.2, $\sigma(e) = e$ and $\langle b \rangle$, $\langle c \rangle$, $\langle d \rangle$, P, U, $A(V, S)$ are σ-invariant. Let $\sigma(b) = \delta_b b$, $\sigma(c) = \delta_c c$, $\sigma(d) = \delta_d d$. The equations $cd = e$, $dc = d$, $cb = -c$ imply $\delta_c\delta_d = 1$, $\delta_c\delta_d = \delta_d$, $\delta_b\delta_c = \delta_c$. One may check that $\delta_b = \delta_c = \delta_d = 1$. Finally, the equations $db = p_1$, $dd = p_2$ imply $\sigma|_P = \mathrm{id}_P$.

Since V and $A(V, S)$ are σ-invariant, $\sigma|_{A(V,S)} \in \mathrm{L}(V)_S$ by Proposition 2.1. Taking into account $\sigma|_P = \mathrm{id}_P$ and $\sigma(U) \subseteq U$, we obtain $\sigma \in \mathrm{L}(U)_S$. □

3. Affine monoids as the normalizers of linear subspaces

Proposition 3.1. *Let M be an affine algebraic monoid. There is a finite-dimensional vector space U and an integer $r > 1$ such that the following holds. Let P be a two-dimensional vector space with a trivial $\mathrm{L}(U)$-action. Then the $\mathrm{L}(U)$-module $(P \oplus U)^{\otimes r}$ contains a linear subspace S such that $\mathrm{L}(U)_S \cong M$.*

Proof. Since there exists a closed embedding $M \hookrightarrow \mathrm{L}(U)$ for some finite-dimensional space U, we may suppose $M \subseteq \mathrm{L}(U)$. Consider the action of $\mathrm{L}(U)$ on itself by left multiplication. Additionally, consider the $\mathrm{L}(U)$-action on the algebra $\mathrm{K}[\mathrm{L}(U)]$ of regular functions on $\mathrm{L}(U)$,

$$(g \cdot f)(u) := f(ug), \quad g, u \in \mathrm{L}(U), f \in \mathrm{K}[\mathrm{L}(U)]. \tag{3.1}$$

Denote $d := \dim U$. Note that the $\mathrm{L}(U)$-modules $\mathrm{K}[\mathrm{L}(U)]$ and $\mathrm{Sym}(U^{\oplus d})$ are isomorphic. To prove this, it suffices to associate a linear function on $\mathrm{L}(U)$ to every vector $(u_1, \ldots, u_d) \in U^{\oplus d}$, since $\mathrm{K}[\mathrm{L}(U)] = \mathrm{Sym}(\mathrm{L}(U)^*)$. Identify U with K^d, $\mathrm{L}(U)$ with $\mathrm{Mat}_{d\times d}(\mathrm{K})$; let A be in $\mathrm{L}(U)$, B be a matrix with columns $u_1, \ldots, u_d$. Set $l_{u_1,\ldots,u_d}(A) := \mathrm{tr}\, AB$. Then $(g \cdot l_{u_1,\ldots,u_d})(A) = \mathrm{tr}\, AgB = l_{gu_1,\ldots,gu_d}(A)$; i.e. we have an $\mathrm{L}(U)$-equivariant isomorphism.

By the definition of a symmetric algebra fix a natural epimorphism

$$\xi \colon \mathrm{T}(U^{\oplus d}) \to \mathrm{Sym}(U^{\oplus d}) \cong \mathrm{K}[\mathrm{L}(U)]. \tag{3.2}$$

There is a finite-dimensional subspace $W \subset \mathrm{K}[\mathrm{L}(U)]$ such that

$$\mathrm{L}(U)_W = M. \tag{3.3}$$

In order to prove this, one may show that a linear span of an $\mathrm{L}(U)$-'orbit' of an arbitrary function $f \in \mathrm{K}[\mathrm{L}(U)]$ is finite-dimensional. Indeed, since the $\mathrm{L}(U)$-action is a morphism, $(g \cdot f)(u) = f(ug) \in \mathrm{K}[\mathrm{L}(U) \times \mathrm{L}(U)] = \mathrm{K}[\mathrm{L}(U)] \otimes \mathrm{K}[\mathrm{L}(U)]$, where $u, g \in \mathrm{L}(U)$, there are functions $F_j, H_j \in \mathrm{K}[\mathrm{L}(U)]$ such that

$$(g \cdot f)(u) = \sum_{j=1}^{n} F_j(u)H_j(g). \tag{3.4}$$

Therefore, the $\mathrm{L}(U)$-'orbit' of the function f is contained in the finite-dimensional subspace $\langle F_1, \ldots, F_n \rangle$.

Let $I(M) = (f_1, \dots, f_t) \lhd \mathrm{K}[\mathrm{L}(U)]$ be the ideal of functions vanishing on M. Summing the linear spans of $\mathrm{L}(U)$-'orbits' of the functions f_i we obtain a finite-dimensional $\mathrm{L}(U)$-invariant subspace $V \subset \mathrm{K}[\mathrm{L}(U)]$. Define $W = I(M) \cap V$. First, it contains $f_1, \dots, f_t$. Second, it is M-invariant, since the ideal $I(M)$ is M-invariant. Obviously, $g \in M$ implies $g \cdot W \subseteq W$. On the other hand, suppose that $g \cdot W \subseteq W$, where $g \in \mathrm{L}(U)$. Then $f_i(g) = (g \cdot f_i)(E) = 0$ for $i = 1, \dots, t$, where E is the identity of $\mathrm{L}(U)$ and is automatically contained in M. Therefore, $g \in M$. This proves (3.3).

Further, since the space W is finite-dimensional, there is an integer $h \in \mathbb{Z}_+$ such that

$$W \subseteq \xi(\textstyle\bigoplus_{i \leqslant h} (U^{\oplus d})^{\otimes i}). \tag{3.5}$$

Define $W' := \xi^{-1}(W) \cap (\bigoplus_{i \leqslant h} (U^{\oplus d})^{\otimes i})$. The $\mathrm{L}(U)$-equivariance of ξ implies

$$\mathrm{L}(U)_{W'} = \mathrm{L}(U)_W. \tag{3.6}$$

Fix a basis $\{p_1, p_2\}$ of the space P. There exists an embedding of $\mathrm{L}(U)$-modules

$$\iota \colon \mathrm{T}(U^{\oplus d}) \hookrightarrow \mathrm{T}(\langle p_1 \rangle \oplus U). \tag{3.7}$$

Indeed, let U_i be the ith summand of $U^{\oplus d}$. Consider an arbitrary basis $\{f_{ij} \mid j = 1, \dots, d\}$ of U_i and define an embedding as follows:

$$\iota(f_{i_1 j_1} \otimes \ldots \otimes f_{i_t j_t}) := p_1^{\otimes i_1} \otimes f'_{i_1 j_1} \otimes \ldots \otimes p_1^{\otimes i_t} \otimes f'_{i_t j_t}, \tag{3.8}$$

where f'_{ij} is the image of f_{ij} under the identity isomorphism $U_i \to U$. It is easy to check that the embedding $\iota \colon \mathrm{T}(U^{\oplus d}) \to \mathrm{T}(\langle p_1 \rangle \oplus U)$ defined on the basis of $\mathrm{T}(U^{\oplus d})_+$ by formula (3.8) and sending 1 to 1 is the one required.

Now we may consider a space $W'' := \iota(W')$,

$$\mathrm{L}(U)_{W''} = \mathrm{L}(U)_{W'}. \tag{3.9}$$

Since W'' is finite-dimensional, there exists an integer $b \in \mathbb{N}$ such that

$$W'' \subseteq \textstyle\bigoplus_{i \leqslant b} (\langle p_1 \rangle \oplus U)^{\otimes i}. \tag{3.10}$$

Take $r \geqslant b$ such that $r > 1$ and consider a linear mapping

$$\iota_r \colon \textstyle\bigoplus_{i \leqslant b} (\langle p_1 \rangle \oplus U)^{\otimes i} \to (P \oplus U)^{\otimes r}, \quad f_i \mapsto p_2^{\otimes (r-i)} \otimes f_i, f_i \in (\langle p_1 \rangle \oplus U)^{\otimes i}. \tag{3.11}$$

Obviously, ι_r is an embedding of $\mathrm{L}(U)$-modules. Define $S = \iota_r(W'')$. Then

$$\mathrm{L}(U)_S = \mathrm{L}(U)_{W''}. \tag{3.12}$$

Now the claim follows from equations (3.3), (3.6), (3.9) and (3.12). □

Proof of Theorem 1.1. Let M be an arbitrary affine algebraic monoid, U, b, r, P, S be as in Proposition 3.1. Fix some set $\gamma \in (\mathrm{K} \setminus \{0,1\})^6$ such that $\gamma_i \neq \gamma_j$ for $i \neq j$, and consider the algebra $D(P, U, S, \gamma)$. It follows from Proposition 3.1 and Proposition 2.3 that $\mathrm{End}(D(P, U, S, \gamma)) \cong M \sqcup \{\mathfrak{z}\}$.

Acknowledgement

The author expresses his sincere thanks to I.V. Arzhantsev for posing the problem and for useful discussions.

References

[1] I.V. Arzhantsev, *Affine embeddings of homogeneous spaces*, in *"Surveys in Geometry and Number Theory"*, N. Young (Editor), London Math. Soc. Lecture Notes Series **338**, Cambridge Univ. Press, Cambridge, 2007, 1–51. MR2306139 (2008d:14074)

[2] N.L. Gordeev and V.L. Popov, *Automorphism groups of finite dimensional simple algebras*, Annals of Mathematics **158** (2003), 1041–1065. MR2031860 (2005b:20086)

[3] M.S. Putcha, *Linear algebraic monoids*, London Math. Soc. Lecture Notes Series **133**, Cambridge Univ. Press, Cambridge, 1988. MR964690 (90a:20003)

[4] L. Renner, *Linear algebraic monoids*, Encyclopaedia of Mathematical Sciences **134**, Springer-Verlag, Berlin-Heidelberg, 2005. MR2134980 (2006a:20002)

[5] A. Rittatore, *Algebraic monoids and affine embeddings*, Transform. Groups **3** (1998), no. 4, 375–396. MR1657536 (2000a:14056)

[6] E.B. Vinberg, *On reductive algebraic semigroups*, Amer. Math. Soc. Transl. (2) **169** (1995), 145–182. MR1364458 (97d:20057)

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