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DEFINING ADDITIVE SUBGROUPS OF THE REALS FROM CONVEX SUBSETS

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ABSTRACT. Let G be a subgroup of the additive group of real numbers and let $C \subseteq G$ be infinite and convex in G. We show that G is definable in $(\mathbb{R}, +, \cdot, C)$ and that $\mathbb{Z}$ is definable if G has finite rank. This has a number of consequences for expansions of certain o-minimal structures on the real field by multiplicative groups of complex numbers.

We answer some questions about expansions of o-minimal structures on the real field $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot\,)$ by various groups. For more detailed treatment of some of the issues raised below, see van den Dries and Günaydın [4] and Miller [6, 7]. In this paper, definable means first-order definable with parameters from the appropriate underlying set. Throughout, let K $(= (K, +, \cdot, <\,))$ be an ordered subfield of $\overline{\mathbb{R}}$, and let G be a nontrivial additive subgroup of K.

Proposition 1. *Let $C \subseteq G$ be infinite and convex (in G). Then G is definable in (K, C).*

Proof. If C is unbounded, then $G = C \cup (-C) \cup (a + C)$ for some $a \in K$. Suppose now that C is bounded. Then G is not discrete and is thus dense in K. By translation and division by some nonzero element of G, we reduce to the case that $1 \in G$ and $C = G \cap [-1, 1]$. Put $S = \{\, s \in K : \exists \varepsilon > 0,\ s\big(C \cap [0, \varepsilon)\big) \subseteq C \,\}$. Evidently, (K, C) defines S, and S is a subgroup of $(K, +)$ containing 1. Hence, it suffices to show that $S \cap (0, 1) \subseteq C$, for then $G = S + C$. Let $s \in S \cap (0, 1)$ and $\varepsilon > 0$ be such that $s\big(C \cap [0, \varepsilon)\big) \subseteq C$. By density of G, there exists $c \in C \cap (0, \varepsilon)$. Note that $sc \in C \subseteq G$. Let k be the smallest positive integer such that $kc > 1 - \varepsilon$. Then $skc \in G$ and $1 - kc \in C \cap (-\varepsilon, \varepsilon)$, so that $s(1 - kc) \in G$ as well. Since $s = s(1 - kc) + skc$, we have $s \in G$ as required. ☐

Proposition 2. *If G has finite rank (equivalently, if the $\mathbb{Q}$-linear span of G is finite dimensional) and $C \subseteq G$ is infinite and convex, then (K, C) defines $\mathbb{Z}$.*

Proof. By Proposition 1 and division by some nonzero element of G, we reduce to the case that $C = G$ and $1 \in G$. Put $R = \{\, r \in K : rG \subseteq G \,\}$. Observe that R is a subring of K contained in G and is definable in (K, G). Since $R \subseteq G$, it has finite rank as an additive group. Hence, the fraction field J of R is a finite-degree algebraic extension of $\mathbb{Q}$. By J. Robinson [10], $\mathbb{Z}$ is definable in $(J, +, \cdot)$. Since J is definable in (K, R), so is $\mathbb{Z}$. ☐

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Remarks. (i) If G is finitely generated and $1 \in G$, then (K, G) defines $\mathbb{Z}$ for all *subrings* K of $\mathbb{R}$ containing G [6, 6.1]. (ii) The set R contains S as in the proof of Proposition 1; equality holds if and only if G is dense. (iii) Of course, undecidability of $\mathrm{Th}(K, C)$ follows, but much more is true when $K = \mathbb{R}$: every real Borel set is definable in $(\overline{\mathbb{R}}, \mathbb{Z})$. (iv) Any conditions on G that force the field J to be either a finite-degree algebraic extension of $\mathbb{Q}$ or a purely transcendental extension of some $J' \subseteq J$ yield definability of $\mathbb{Z}$, again by [10] for the former and by R. Robinson [11] for the latter. A more recent result of Poonen [9] shows that it is sufficient for J to be a finitely generated field extension of $\mathbb{Q}$. (v) If K is real closed and G is a real-closed subfield of K, then (K, G) does not define $\mathbb{Z}$ by van den Dries [3]. We do not know of any G other than real-closed subfields that do not define $\mathbb{Z}$ over $\overline{\mathbb{R}}$.

We now concentrate on the case that $K = \mathbb{R}$ and consider expansions of $\overline{\mathbb{R}}$ by multiplicative subgroups of $\mathbb{C}\setminus\{0\}$. We make the usual identifications $\mathbb{R} \hookrightarrow \mathbb{C} \cong \mathbb{R}^2$. Given $Z \subseteq \mathbb{C}$, let e^Z denote the image of Z under complex exponentiation. We focus here on expansions of structures of the form $(\overline{\mathbb{R}}, (e^{wG})_{w\in W})$ where $W \subseteq \mathbb{C}$. At present, there are essentially only two positive results known:

- [6, 7]. If G is cyclic and $\mathfrak{R}$ is an o-minimal expansion of $\overline{\mathbb{R}}$ that defines no irrational power functions, then $(\mathfrak{R}, e^G)$ does not define $\mathbb{Z}$. Indeed, every definable set (of any arity) is a boolean combination of open sets (but much more is true). Write $G = \alpha\mathbb{Z}$, put $\omega = 2\pi/\alpha$, and suppose moreover that $\mathfrak{R}$ defines the restriction of the function $t \mapsto \sin(\omega \log t)\colon \mathbb{R}^{>0} \to \mathbb{R}$ to some nontrivial subinterval of $\mathbb{R}^{>0}$. Then $(\mathfrak{R}, e^G)$ defines the group $e^{(1+i\omega)\mathbb{R}}$. By [2], the expansion of $\overline{\mathbb{R}}$ by the restriction $\sin(\omega \log t)\restriction[1, 2]$ is o-minimal and defines no irrational power functions. This means that $(\overline{\mathbb{R}}, e^{i\mathbb{R}})$ does not define $\mathbb{Z}$.
- [4]. If G is noncyclic and has finite rank, then neither $(\overline{\mathbb{R}}, e^G)$ nor $(\overline{\mathbb{R}}, e^{iG})$ defines $\mathbb{Z}$. Indeed, every definable set (of any arity) is a boolean combination of F_σ sets.

Let $\mathbb{R}_e$ and $\mathbb{R}_s$ denote the expansions of $\overline{\mathbb{R}}$ by the restrictions $\exp\restriction[0, 1]$ and $\sin\restriction[0, 2\pi]$ of the real exponential and sine functions. The intervals of definition are picked for convenience: by addition formulas, for any bounded $[a, b] \subseteq \mathbb{R}$ we have $\exp\restriction[a, b]$ definable in $\mathbb{R}_e$ and $\sin\restriction[a, b]$ definable in $\mathbb{R}_s$. By [2], both $\mathbb{R}_e$ and $\mathbb{R}_s$ are o-minimal and define no irrational power functions (indeed, the same is true of their amalgam). It is worth noting that, by Bianconi [1], $\mathbb{R}_s$ does not define $\exp\restriction[0, 1]$ and $\mathbb{R}_e$ does not define $\sin\restriction[0, 2\pi]$. In his thesis [5], Günaydın shows that for any nonzero $\alpha \in \mathbb{R}$, both $(\overline{\mathbb{R}}_e, e^{\alpha\mathbb{Q}})$ and $(\overline{\mathbb{R}}_s, e^{i\alpha\mathbb{Q}})$ define $\mathbb{Q}$ and hence also $\mathbb{Z}$. We now extend his result.

Corollary 3. *If G is noncyclic and C is an infinite convex subset of e^G, then G is definable in $(\mathbb{R}_e, C)$. If moreover G has finite rank, then $\mathbb{Z}$ is definable.*

Proof. There exist $0 < a < b < \infty$ such that $[a, b] \cap C$ is infinite. As $\log\restriction[a, b]$ is definable in $\mathbb{R}_e$, we have $[\log a, \log b] \cap G$ definable in $(\mathbb{R}_e, C)$. Apply Propositions 1 and 2. $\square$

Corollary 4. *If e^{iG} is infinite and A is a nondegenerate arc of the unit circle, then $G + 2\pi\mathbb{Z}$ is definable in $(\mathbb{R}_s, e^{iG} \cap A)$. If moreover G has finite rank, then $\mathbb{Z}$ is definable.*

Proof. Intersect $e^{iG} \cap A$ with either the right half-plane or the left half-plane, project onto the second coordinate, and then take the image under arcsin. This set is definable and contains an infinite convex subset of $G + 2\pi\mathbb{Z}$. Apply Proposition 1. If G has finite rank, then so does $G + 2\pi\mathbb{Z}$. Apply Proposition 2. □

We have concentrated so far on expanding by single groups, but there are many reasons to be interested in expanding by collections of groups. The most fundamental question is this: For $\mathbb{Q}$-linearly independent $\alpha, \beta \in \mathbb{R}$, what can be said about $(\overline{\mathbb{R}}, e^{\alpha\mathbb{Z}}, e^{\beta\mathbb{Z}})$? Other than the obvious fact that the direct product $e^{\alpha\mathbb{Z}+\beta\mathbb{Z}}$ is definable, very little known is known as of this writing. The situation is similar for the structures $(\overline{\mathbb{R}}, e^{i\alpha\mathbb{Z}}, e^{i\beta\mathbb{Z}})$ and $(\overline{\mathbb{R}}, e^{(1+i\alpha)\mathbb{R}}, e^{(1+i\beta)\mathbb{R}})$. We now provide some partial answers.

Corollary 5. *If $\alpha, \beta \in \mathbb{R}$ are $\mathbb{Q}$-linearly independent, then $\mathbb{Z}$ is definable in each of the structures*

$$(\mathbb{R}_e, e^{\alpha\mathbb{Z}}, e^{\beta\mathbb{Z}}), (\mathbb{R}_s, e^{i\alpha\mathbb{Z}}, e^{i\beta\mathbb{Z}}), (\mathbb{R}_e, e^{(1+i\alpha)\mathbb{R}}, e^{(1+i\beta)\mathbb{R}}), (\mathbb{R}_s, e^{(1+i\alpha)\mathbb{R}}, e^{(1+i\beta)\mathbb{R}}).$$

Proof. Since the product groups are definable, the first two cases are immediate from Corollaries 3 and 4. For the third, observe that $e^{(1+i\alpha)\mathbb{R}} \cap \mathbb{R}^{>0}$ and $e^{(1+i\beta)\mathbb{R}} \cap \mathbb{R}^{>0}$ are definable, thus reducing to the first case. For the fourth, project $e^{(1+i\alpha)\mathbb{R}} \cap e^{(1+i\beta)\mathbb{R}}$ onto the unit circle and apply Corollary 4. □

As an immediate consequence, we have the following:

Corollary 6. *Let $r \in \mathbb{R} \setminus \mathbb{Q}$. Then $\mathbb{Z}$ is definable in each of the structures*

$$(\mathbb{R}_e, x^r, e^{\alpha\mathbb{Z}}), \quad (\mathbb{R}_e, x^r, e^{(1+i\alpha)\mathbb{R}}), \quad (\mathbb{R}_s, x^r, e^{(1+i\alpha)\mathbb{R}}).$$

We close with one more application. In [8], Miller and Speissegger establish a trichotomy for expansions of the structure $\mathbb{R}_{\text{an}}$ by individual trajectories of certain kinds of analytic planar vector fields. By combining Corollaries 5 and 6 with Corollary 4 of their paper, the trichotomy extends to expansions of $\mathbb{R}_{\text{an}}$ by arbitrary collections of such trajectories.

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