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A NOTE ON TRANSITIVE LOCALIZING ALGEBRAS

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Abstract. A simple proof is provided for a theorem of Troitsky that every nonzero quasinilpotent operator on a Banach space whose commutant is a localizing algebra has a nontrivial hyperinvariant subspace.

The concept of a localizing algebra has been introduced recently by Lomonosov, Radjavi and Troitsky [10] as a condition to obtain invariant subspaces for operators on Banach spaces. A slightly stronger version of this notion had already been considered by Androulakis [1] as an assumption in the method of minimal vectors, a procedure to find invariant subspaces that was designed not long ago by Enflo [7] and that has been studied ever since then by many authors [1, 2, 3, 4, 5, 6, 8, 10, 12].

Definition 1. Let X be a Banach space and let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on X. A subalgebra $\mathcal{R}$ of $\mathcal{B}(X)$ is said to be *localizing* if there is a closed ball $B \subseteq X$ such that $0 \notin B$ and for every sequence (x_n) in B there is a subsequence (x_{n_j}) and a sequence (R_j) in $\mathcal{R}$ such that $\|R_j\| \leq 1$ and $(R_j x_{n_j})$ converges in norm to a nonzero vector.

Recall that the commutant of a subset $\mathcal{S} \subseteq \mathcal{B}(X)$ is the algebra $\mathcal{S}'$ of all operators that commute with every element of $\mathcal{S}$. A subspace $Y \subseteq X$ is said to be invariant under an operator $T \in \mathcal{B}(X)$ if $TY \subseteq Y$. A subspace $Y \subseteq X$ is said to be invariant under a subalgebra $\mathcal{R} \subseteq \mathcal{B}(X)$ if Y is invariant under every $R \in \mathcal{R}$. A subspace $Y \subseteq X$ is said to be hyperinvariant under an operator $T \in \mathcal{B}(X)$ if Y is invariant under the subalgebra $\{T\}'$. A subalgebra $\mathcal{R} \subseteq \mathcal{B}(X)$ is said to be transitive if the only closed subspaces invariant under $\mathcal{R}$ are the trivial ones, $Y = \{0\}$ and $Y = X$. It turns out that a subalgebra $\mathcal{R} \subseteq \mathcal{B}(X)$ is transitive if and only if for each nonzero vector $x \in X$, the orbit $\mathcal{R}x = \{Rx : R \in \mathcal{R}\}$ is a dense subspace of X. Using the method of minimal vectors, Troitsky [12] obtained the following result.

Theorem 2. *If T is a nonzero quasinilpotent operator on a Banach space and $\{T\}'$ is a localizing algebra, then T has a nontrivial hyperinvariant subspace.*

As pointed out by Lomonosov, Radjavi and Troitsky [10], the above result easily extends to algebras of operators as follows.

Theorem 3. *If $\mathcal{R}$ is a transitive localizing subalgebra of $\mathcal{B}(X)$, then $\mathcal{R}'$ does not contain any nonzero quasinilpotent operator.*

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The purpose of this paper is to provide a proof for this result that depends only on the spectral radius formula. This proof represents a simplification of the original one because it does not rely on the method of minimal vectors.

The first part of our proof is a claim isolated from the beginning of the proof for Theorem 2.3 in the paper of Lomonosov, Radjavi and Troitsky [10]. Its proof is included here for the sake of clarity and completeness.

The second part of our proof resembles an argument at the end of the proof in Hilden's simplification for the striking theorem of Lomonosov [9] that any nonzero compact operator on a complex Banach space has a nontrivial invariant subspace. We refer to the book of Rudin [11] for this argument.

Proof of Theorem 3. Let $T \in \mathcal{R}'$ be a nonzero operator. We must show that T is not quasinilpotent. Since $\mathcal{R}$ is transitive and $\ker T$ is a closed proper subspace of X invariant under $\mathcal{R}$, it follows that $\ker T = \{0\}$, so that T is injective.

Let $B \subseteq X$ be a ball as in Definition 1. We claim that there exists $c > 0$ such that for every $x \in B$ there is an operator $R \in \mathcal{R}$ with $\|R\| \leq c$ and $RTx \in B$. If this is not so, then for every $n \geq 1$, there is a vector $x_n \in B$ such that $\|R\| \geq n$, whenever $R \in \mathcal{R}$ and $RTx_n \in B$. Since $\mathcal{R}$ is localizing, there is a subsequence (x_{n_j}) and a sequence (R_j) in $\mathcal{R}$ such that $\|R_j\| \leq 1$ and $(R_j x_{n_j})$ converges in norm to some nonzero vector $x \in X$. We have $TR_j = R_j T$ for all $j \geq 1$, so that $(R_j T x_{n_j})$ converges to Tx in norm. Now $Tx \neq 0$ because T is injective and $x \neq 0$. Since $\mathcal{R}$ is transitive, there is an operator $R \in \mathcal{R}$ such that $RTx \in \operatorname{int} B$. It follows that there is a $j_0 \geq 1$ such that $RR_j T x_{n_j} \in \operatorname{int} B$ for every $j \geq j_0$. Since $RR_j \in \mathcal{R}$, the choice of the sequence (x_n) implies that $\|RR_j\| \geq n_j$ for every $j \geq j_0$, and this is a contradiction because $\|RR_j\| \leq \|R\|$ for every $j \geq 1$.

Take a vector $x_0 \in B$ and choose an operator $R_1 \in \mathcal{R}$ with $\|R_1\| \leq c$ and such that $R_1 T x_0 \in B$. Now choose another operator $R_2 \in \mathcal{R}$ with $\|R_2\| \leq c$ and such that $R_2 T R_1 T x_0 \in B$. Continue this ping-pong game to obtain a sequence of vectors (x_n) in B and a sequence of operators (R_n) in $\mathcal{R}$ such that $\|R_n\| \leq c$ and

$$x_n = R_n T \cdots R_1 T x_0 = R_n \cdots R_1 T^n x_0.$$

Finally, let $d = \min\{\|x\| : x \in B\}$. It is plain that $d > 0$ because $0 \notin B$. Hence,

$$d \leq \|x_n\| \leq c^n \|T^n\| \cdot \|x_0\|,$$

and this gives information on the spectral radius of T, namely,

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \geq \frac{1}{c} > 0.$$

This shows that T fails to be quasinilpotent, as we wanted. □

It was shown by Troitsky [12] that if a subalgebra $\mathcal{R}$ of $\mathcal{B}(X)$ contains a nonzero compact operator, then $\mathcal{R}$ is localizing. The following example goes to show that a localizing algebra may not contain nonzero compact operators. Let $\varphi \in C[0,1]$ and consider the multiplication operator M_φ defined on $C[0,1]$ by the expression $(M_\varphi f)(t) = \varphi(t) f(t)$ for each $f \in C[0,1]$, so that $\|M_\varphi\| = \|\varphi\|_\infty$. Then consider the algebra of all multiplications $\mathcal{R} = \{M_\varphi : \varphi \in C[0,1]\}$. We claim that $\mathcal{R}$ is a localizing algebra. Consider the ball $B = \{f \in C[0,1] : \|f - \chi_{[0,1]}\|_\infty \leq 1/2\}$, so that $0 \notin B$. Now take any sequence (f_n) in B and notice that $|f_n(t)| \geq 1/2$ for each $t \in [0,1]$. Hence, $\varphi_n = 1/(2f_n)$ is a continuous function with $\|\varphi_n\|_\infty \leq 1$, so that $M_{\varphi_n} \in \mathcal{R}$ and $\|M_{\varphi_n}\| \leq 1$. Moreover, the sequence $(M_{\varphi_n} f_n)$ converges in norm to a nonzero function because $M_{\varphi_n} f_n \equiv 1/2$ for every $n \geq 1$. We finally show

that $\mathcal{R}$ does not contain nonzero compact operators. Let $M_\varphi \in \mathcal{R}$ be a nonzero multiplication, that is, $\varphi \neq 0$. Consider the sequence of functions $f_n(t) = \cos 2\pi nt$ and notice that $\|f_n\|_\infty \leq 1$ but $(M_\varphi f_n)$ has no pointwise convergent subsequence. This shows that in fact the operator M_φ fails to be weakly compact. We have been told by Luis Rodríguez-Piazza in a private communication that M_φ fails to be strictly singular, that is, that there is an infinite-dimensional subspace such that the restriction of M_φ to it is invertible.

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