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EXISTENCE RESULTS FOR ABSTRACT NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we discuss the existence of solutions for a class of abstract partial neutral functional differential equations.

1. Introduction

We study the existence of solutions for a class of abstract neutral functional differential equations of the form

$$\frac{d}{dt}\left[x(t)+g(t,x(t-r_1))\right] = Ax(t)+f(t,x_t), \qquad t\in[0,a], \tag{1.1}$$

$$x_0 = \varphi\in\mathcal{C}=C([-r,0];X), \tag{1.2}$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X,\|\cdot\|)$, $0<r_1\leq r$, and $g:[0,a]\times X\to X$, $f:[0,a]\times\mathcal{C}\to X$ are appropriate functions.

In Datko [10] and Adimy and Ezzinbi [1] some linear neutral systems similar to (1.1)-(1.2) are studied under the strong assumption that the range of g is contained in $D(A)$. If A is the generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ (the case studied by Datko), this assumption arises from the associated integral equation

$$\begin{aligned} u(t) &= T(t)[\varphi(0)+g(0,u(t-r_1))]-g(t,u(t-r_1)) \\ &\quad -\int_0^t AT(t-s)g(s,u(s-r_1))ds+\int_0^t T(t-s)f(s,u_s)ds, \end{aligned}$$

since except in trivial cases the operator function $s\to AT(s)$ is not integrable in the operator topology on $[0,b]$, for $b>0$. The same reason explains the use of this assumption in [1], where the case in which A is a Hille-Yosida type operator is studied. We mention the papers [2, 3, 4, 5, 6] (among several works) where an alternative assumption has been used to treat some abstract neutral systems. In these works, A is the generator of a C_0-semigroup, and it is assumed that the set $\{AT(t):t\in(0,b]\}$, $(b>0)$, is bounded in the operator topology. However, as was pointed out in [18], this condition is valid if and only if A is a bounded, which restricts the applications to ordinary neutral differential equations.

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In this paper we discuss the existence of solutions for (1.1)-(1.2) without using the related restrictions. Our results are proved assuming some temporal and spatial "regularity" type conditions for the function $t \to g(t, \varphi(t - r_1))$, which permits us to study some neutral systems which have not been considered in the literature.

Neutral differential equations arise in many areas. The literature on related ordinary neutral functional differential equations is very extensive, and we refer the reader to the Hale and Lunel book [15] for details. Partial neutral differential equations arise, for instance, in transmission line theory. Wu and Xia have shown in [25] that a ring array of identical resistively coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling which exhibits various types of discrete waves. By taking a natural limit, they obtain from this system of neutral equations a scalar partial neutral differential equation defined on the unit circle. Such a partial neutral differential equation is also investigated by Hale in [16] under the more general form

$$\begin{aligned} \frac{d}{dt}\mathcal{D}u_t(x) &= \frac{\partial^2}{\partial x^2}\mathcal{D}u_t(x) + f(u_t)(x), \qquad t \geq 0, \\ u_0 &= \varphi \in C([-r, 0]; C(S^1; \mathbb{R})), \end{aligned}$$

where k is a constant, $\mathcal{D}(\psi)(s) := \psi(0)(s) - \int_{-r}^{0}[d\eta(\theta)]\psi(\theta)(s)$ for $s \in S^1$, $\psi \in C([-r, 0]; C(S^1; \mathbb{R}))$ and η is a function of bounded variation.

We also find abstract neutral systems in the theory of heat conduction. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depends linearly on the temperature u and on its gradient ∇u. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [14, 23], the internal energy and the heat flux are described as functionals of u and u_x. The next system (see [7, 8, 9, 22]) has been frequently used to describe this phenomena:

$$\begin{aligned} \frac{d}{dt}\left[u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)ds\right] &= c\triangle u(t,x) + \int_{-\infty}^{t} k_2(t-s)\triangle u(s,x)ds, \\ u(t,x) &= 0, \qquad x \in \partial\Omega. \end{aligned}$$

In this system, $\Omega \subset \mathbb{R}^n$ is open, bounded and has smooth boundary, $(t, x) \in [0, \infty) \times \Omega$, $u(t, x)$ represents the temperature in x at the time t, c is a physical constant and $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are the internal energy and the heat flux relaxation respectively. By assuming the solution u is known on $(-\infty, 0]$ and $k_2 \equiv 0$, we can transform this system into the abstract form

$$\frac{d}{dt}\left[x(t) + g(t, x_t)\right] = Ax(t) + f(t, x_t).$$

There exists an extensive literature on ordinary neutral differential equations in the theory of population dynamics; see [12, 20, 11, 21, 11] and their references. Looking at these papers, it is natural to think that the abstract system (2.1)-(2.2) can be used to describe spatial diffusion phenomena, which arise because of the natural tendency of biological populations to migrate from high population density regions to regions with minor density.

2. Existence of solutions

In this section we discuss the existence of solutions for systems in the form

$$\frac{d}{dt}\left[x(t)+g(t,x(t-r_1))\right] = Ax(t)+f(t,x_t), \qquad t\in[\sigma,\sigma+b], \tag{2.1}$$

$$x_\sigma = \varphi\in C([-r+\sigma,\sigma];X), \tag{2.2}$$

where $A : D(A)\subseteq X\to X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space $(X,\|\cdot\|)$.

In what follows, $[D(A)]$ is the space $D(A)$ endowed with the graph norm, $M>0$ is such that $\| T(t) \|\leq M$ for all $t\in[0,a]$ and $\|\cdot\|_{\mathcal{C}}$ denotes the sup-norm in $\mathcal{C}=C([-r,0];X)$. For Banach spaces Z,W, we use the notation $(\mathcal{L}(Z,W),\|\cdot\|_{Z,W})$ for the space of bounded linear operators from Z into W endowed with the uniform operator norm $\|\cdot\|_{Z,W}$, and we write $(\mathcal{L}(Z),\|\cdot\|_Z)$ when $Z=W$.

Definition 2.1. A function $u\in C([-r+\sigma,\sigma+b];X)$, $b>0$, $\sigma\in\mathbb{R}$, is called a mild solution of the neutral differential system (2.1)-(2.2) if $u_\sigma=\varphi$, the function $s\to AT(t-s)g(s,u(s-r_1))$ is integrable on $[\sigma,t)$ for every $\sigma<t\leq\sigma+b$ and

$$\begin{aligned} u(t) &= T(t-\sigma)[\varphi(0)+g(\sigma,\varphi(-r_1))]-g(t,u(t-r_1)) \\ &\quad -\int_\sigma^t AT(t-s)g(s,u(s-r_1))ds+\int_\sigma^t T(t-s)f(s,u_s)ds, \ \ t\in[\sigma,\sigma+b]. \end{aligned}$$

Definition 2.2. A function $u\in C([-r+\sigma,\sigma+b];X)$, $b>0$, $\sigma\in\mathbb{R}$, is called an S-mild solution of the neutral system (2.1)-(2.2) if $u_\sigma=\varphi$, the function $s\to g(s,u(s-r_1))$ is differentiable *a.e.* on $[\sigma,\sigma+b]$, $s\to\frac{d}{ds}g(s,u(s-r_1))\in L^1([\sigma,\sigma+b],X)$ and

$$u(t)=T(t-\sigma)\varphi(0)+\int_\sigma^t T(t-s)[-\frac{d}{ds}g(s,u(s-r_1))+f(s,u_s)]ds, \ \ t\in[\sigma,\sigma+b].$$

Definition 2.3. A function $u\in C([-r+\sigma,\sigma+b];X)$, $b>0$, $\sigma\in\mathbb{R}$, is said to be a classical solution of the neutral system (2.1)-(2.2) if the function $t\to g(t,u(t-r_1))$ belongs to $C^1([\sigma,\sigma+b];X)$, $u\in C^1([-r+\sigma,\sigma+b];X)\cap C([-r+\sigma,\sigma+b];[D(A)])$ and (2.1)-(2.2) are satisfied on $[-r+\sigma,\sigma+b]$.

We can now establish our first existence result.

Theorem 2.1. *Assume that the following conditions are satisfied:*

- (**a**) *There exist a natural number $n\geq 2$ and Banach spaces $(Y_i,\|\cdot\|_{Y_i})$, $i=1,\dots,2n$, such that $Y_{i+1}\hookrightarrow Y_i\hookrightarrow X$ for all $i=1,\dots,2n-1$, $AT(\cdot)\in L^1([0,a],\mathcal{L}(Y_{i+1},Y_i))$ for all $i=1,\dots,2n-1$, and $T(\cdot)\in L^1([0,a],\mathcal{L}(X,Y_i))$ for each $i=1,\dots,2n$.*
- (**b**) *The functions g,f are continuous, $g\in C([0,a]\times Y_{i+1};Y_i)$ for every $i=1,\dots,2n-1$, and there exists $L_f>0$ such that*

$$\|f(t,\psi_1)-f(t,\psi_2)\| \leq L_f\|\psi_1-\psi_2\|_{\mathcal{C}}, \qquad t\in[0,a],\ \psi_i\in\mathcal{C}.$$

If $\varphi\in C([-r,0];Y_{2n})$, then there exists a unique mild solution of the neutral system (1.1)-(1.2) on $[-r,nr_1\wedge a]$.

Proof. Assume $nr_1\leq a$. Let $\Gamma : C([-r,r_1];X)\to C([-r,r_1];X)$ be the map defined by $(\Gamma u)_0=\varphi$ and

$$\begin{aligned} \Gamma u(t) &= T(t)[\varphi(0)+g(0,\varphi(-r_1))]-g(t,\varphi(t-r_1)) \\ &\quad -\int_0^t AT(t-s)g(s,\varphi(s-r_1))ds+\int_0^t T(t-s)f(s,u_s)ds, \ \ t\in[0,r_1]. \end{aligned}$$

From Bochner's Theorem for integrable functions and the estimate

$$\| AT(t-s)g(s,\varphi(s-r_1)) \| \leq \| AT(t-s) \|_{\mathcal{L}(Y_{2n-1},X)} \| g(s,\varphi(s-r_1)) \|_{Y_{2n-1}},$$

we infer that the function $s \to AT(t-s)g(s,\varphi(s-r_1))$ is integrable on $[0,r_1]$, which implies that $\Gamma u \in C([-r,r_1];X)$.

On the other hand, from the inequality

$$\sup_{\theta\in[0,t]} \| \Gamma^k u(\theta) - \Gamma^k v(\theta) \| \quad \leq \quad \frac{(ML_f)^k}{k!} \int_0^t \sup_{\theta\in[0,s]} \| u(\theta) - v(\theta) \| \, ds, \quad t \in [0,r_1],$$

it follows that Γ^k is a contraction for k large enough and there exists a unique fixed point u^1 of Γ. Obviously, u^1 is a mild solution of (1.1)-(1.2) on the interval $[-r,r_1]$, and from the inequality

$$\begin{aligned} \| u^1(t) \|_{Y_{2n-2}} \quad \leq \quad & \| T(t) \|_{Y_{2n-1},Y_{2n-2}} \| \varphi(0) + g(0,\varphi(-r_1)) \|_{Y_{2n-1}} \\ & + \| g(t,\varphi(t-r_1)) \|_{Y_{2n-2}} \\ & + \int_0^a \| AT(s) \|_{Y_{2n-1},Y_{2n-2}} \| g(s,\varphi(s-r_1)) \|_{Y_{2n-1}} \, ds \\ & + \int_0^t \| T(s) \|_{X,Y_{2n-2}} \| f(s,u^1_s) \| \, ds \end{aligned}$$

we infer that $u^1 \in C([-r,r_1];Y_{2n-2})$ and $s \to g(s,u^1(s-r_1)) \in C([r_1,2r_1];Y_{2n-3})$.

We can now repeat the above procedure with $(u^1)_{r_1}$ and Y_{2n-2} in place of φ and Y_{2n}, and we deduce the existence of a unique mild solution u^2 for

$$\frac{d}{dt}\left[x(t) + g(t,x(t-r_1))\right] \quad = \quad Ax(t) + f(t,x_t), \quad t \in [r_1,2r_1], \tag{2.3}$$

$$x_{r_1} \quad = \quad (u^1)_{r_1}, \tag{2.4}$$

such that $u^2 \in C([r_1-r,2r_1];Y_{2n-4})$ and $t \to g(t,u^2(t-r_1)) \in C([2r_1,3r_1];Y_{2n-5})$.

From the above development, for each $i = 1,\dots,n$ there exists a unique mild solution $u^i \in C([(i-1)r_1-r,ir_1];X)$ of the neutral system

$$\begin{aligned} \frac{d}{dt}\left[x(t) + g(t,x(t-r_1))\right] \quad &= \quad Ax(t) + f(t,x_t), \qquad t \in [(i-1)r_1,ir_1], \\ x_{(i-1)r_1} \quad &= \quad (u^{i-1})_{(i-1)r_1}. \end{aligned}$$

It is easy to see that the function $u : [-r,nr_1] \to X$ defined by $u(t) = \varphi(t)$ for $t \leq 0$ and $u(t) = u^i(t)$ for $t \in [(i-1)r_1,ir_1]$, $i = 1,2,\dots,n$, is the unique mild solution of (1.1)-(1.2) on $[-r,nr_1]$. The proof is complete. □

Corollary 2.1. *Assume there exists a sequence of spaces $(Y_i)_{i\in\mathbb{N}}$ such that conditions* (**a**) *and* (**b**) *in Theorem* 2.1 *hold for all $n \in \mathbb{N}$. If $\varphi \in \bigcap_{n\in\mathbb{N}} C([-r,0];Y_n)$, then there exists a mild solution of* (1.1)-(1.2) *on $[-r,\infty)$.*

Theorem 2.2. *Assume condition* (**a**) *of Theorem* 2.1 *holds, the semigroup $(T(t))_{t\geq 0}$ is compact and $g \in C([0,a] \times Y_{i+1};Y_i)$ for $i = 1,2,3$. Suppose, in addition:*

- **(a)** *The function f is continuous and there exist a non-decreasing function $W : [0,\infty) \to (0,\infty)$ and a continuous function $m : [0,a] \to [0,\infty)$ such that $\| f(t,\psi) \| \leq m(t)W(\| \psi \|)$ for every $(t,\psi) \in [0,a] \times \mathcal{C}$.*

(b) $\varphi \in C([-r,0];Y_4)$ *and* $M\int_0^{r_1} m(s)ds < \int_{C(\varphi)}^{\infty} \frac{ds}{W(s)}$, *where*

$$\begin{aligned} C(\varphi) &= M(\| \varphi(0) + g(0,\varphi(-r_1)) \| + \sup_{\theta\in[0,r_1]} \| g(\theta,\varphi(\theta-r_1)) + \| \varphi \|_{\mathcal{C}} \\ &\quad + \sup_{\theta\in[0,r_1]} \| g(\theta,\varphi(\theta-r_1)) \|_{Y_3} \| AT(\cdot) \|_{L^1([0,r_1],\mathcal{L}(Y_3,X))} . \end{aligned}$$

Then there exists a mild solution of (1.1)-(1.2) *on* $[-r_1,b]$ *for some* $r_1 < b \le a$.

Proof. Let Γ be the map defined in the proof of Theorem 2.1. A standard argument using the Lebesque dominated convergence theorem and the Azcoli-Arzela criterion allows us to prove that Γ is completely continuous. In order to apply the Leray-Schauder Alternative Theorem ([13, Theorem 6.5.4]), we construct *a priori* estimates for the solutions of $z = \lambda\Gamma z$, $\lambda \in (0,1)$. Let $\lambda \in (0,1)$, z^λ be a solution of $z = \lambda\Gamma z$ and $\alpha^\lambda(t) = \| \varphi \|_{\mathcal{C}} + \sup_{\theta\in[0,t]} \| z^\lambda(\theta) \|$, $t \in [0,r_1]$. Then, for $t \in [0,r_1]$ we get

$$\begin{aligned} \| z^\lambda(t) \| &\le M \| \varphi(0) + g(0,\varphi(-r_1)) \| + \| g(t,\varphi(t-r_1)) \| \\ &\quad + \sup_{\theta\in[0,r_1]} \| g(\theta,\varphi(\theta-r_1)) \|_{Y_3} \| AT(\cdot) \|_{L^1([0,r_1],\mathcal{L}(Y_3,X))} \\ &\quad + M\int_0^t m(s)W(\| z_s^\lambda \|_{\mathcal{C}})ds, \end{aligned}$$

and hence,

$$\alpha^\lambda(t) \le C(\varphi) + M\int_0^t m(s)W(\alpha^\lambda(s))ds. \tag{2.5}$$

If we denote by $\beta_\lambda(t)$ the left hand side of (2.5), then $\beta_\lambda'(t) \le Mm(t)W(\beta_\lambda(t))$ and

$$\int_{C(\varphi)}^{\beta_\lambda(t)} \frac{ds}{W(s)} \le M\int_0^{r_1} m(s)ds < \int_{C(\varphi)}^{\infty} \frac{ds}{W(s)}ds,$$

which shows that the set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded in $C([0,r_1])$ and, as a consequence, that the set $\{z^\lambda : \lambda \in (0,1)\}$ is bounded in $C([-r,r_1];X)$.

From [13, Theorem 6.5.4], there exists a fixed point $u^1 \in C([-r,r_1];X)$ of Γ. Moreover, arguing as in the proof of Theorem 2.1 we obtain that $u^1 \in C([0,r_1];Y_2)$ and $s \to g(s,u^1(s-r_1)) \in C([r_1,2r_1];Y_1)$.

Let $r_1 < b \le 2r_1$ be such that $M\int_{r_1}^b m(s)ds < \int_{C(u^1)}^{\infty} \frac{ds}{W(s)}$, where

$$\begin{aligned} C(u^1) &= M \| u^1(r_1) + g(r_1,u^1(0)) \| + \sup_{\theta\in[r_1,2r_1]} \| g(\theta,u^1(\theta-r_1)) \| \\ &\quad + \| (u^1)_{r_1} \|_{\mathcal{C}} + \sup_{\theta\in[r_1,2r_1]} \| g(\theta,u^1(\theta-r_1)) \|_{Y_1} \| AT(\cdot) \|_{L^1([r_1,2r_1],\mathcal{L}(Y_1,X))} . \end{aligned}$$

Arguing as in the first part of this proof, we infer the existence of a mild solution $u^2 \in C([r_1-r,b];X)$ of

$$\begin{aligned} \frac{d}{dt}[x(t) + g(t,x(t-r_1))] &= Ax(t) + f(t,x_t), \quad t \in [r_1,b], \\ x_{r_1} &= (u^1)_{r_1}. \end{aligned}$$

It is clear that the function obtained by pasting the functions u^1 and u^2 is a mild solution of (1.1)-(1.2) on the interval $[-r,b]$. This completes the proof. □

In the rest of this section, we assume that $(T(t))_{t\geq 0}$ is analytic and $0 \in \rho(A)$. In this case, it is possible to define the fractional power $(-A)^\alpha$, $\alpha > 0$, as a closed linear operator with domain $D((-A)^\alpha)$. If X_α is the space $D((-A)^\alpha)$ endowed with the norm $\|x\|_\alpha = \|(-A)^\alpha x\|$, then X_α is a Banach space, $X_\alpha \hookrightarrow X_\beta$ for $\beta < \alpha$ and $C_\alpha^k = \sup_{t>0} \| t^{-(k+\alpha)}(-A)^{k+\alpha}T(t)\| < \infty$ for all $k \in \mathbb{N}$; see [24] for details.

In what follows, $y : [-r, a] \to X$ is the function defined by $y_0 = \varphi$ and $y(t) = T(t)\varphi(0)$ for $t \in [0, a]$, and $\|\cdot\|_{C_\gamma}$ is the sup-norm in the space $C_\gamma = C([-r, 0]; X_\gamma)$.

Theorem 2.3. *Assume that the following conditions are satisfied:*

- **(a)** *There exists $\mu \in (0, 1)$ such that $t \to y_t \in C^\mu([0, a]; \mathcal{C})$ and $\varphi(0) \in D(A)$.*
- **(b)** *There exists $p > 1$ such that $s \to \frac{d^2}{ds^2} g(s, \varphi(s - r_1)) \in L^p([0, r_1], X)$. For $\psi \in C([r_1, 2r_1]; X)$ with $\psi' \in L^p([r_1, 2r_1], X)$, there exists $\nu(p) \geq 1$ such that the function $s \to g(s, \psi(s - r_1))$ is differentiable a.e. on $[r_1, 2r_1]$ and $\frac{d}{ds} g(s, \psi(s - r_1)) \in L^{\nu(p)}([r_1, 2r_1], X)$.*
- **(c)** *There exist $\gamma \in (0, 1)$ and $L_f \in C([0, \infty); (0, \infty))$ such that*
$$\| f(t, \psi_1) - f(s, \psi_2) \| \leq L_f(r)(| t - s |^\gamma + \| \psi_1 - \psi_2 \|_{\mathcal{C}})$$
for every $t, s \in [0, a]$, $r > 0$ and all $\psi_i \in B_r(\varphi, \mathcal{C})$.

If $\Theta(r, r_1) = M \int_0^{r_1} \| \frac{d}{ds} g(s, \varphi(s-r_1)) + f(s, y_s) \| ds + ML_f(r + M \| \varphi \|) r r_1 < r$ for some $r > 0$, or L_f is constant and $ML_f(0)r_1 < 1$, then there exists an S-mild solution of (1.1)-(1.2) on $[-r_1, b]$ for some $r_1 < b \leq a$.

Proof. Assume $\Theta(r, r_1) < r$ for some $r > 0$. It follows from this condition that the map $\Gamma : B_r(y|_{[-r,r_1]}, C([-r, r_1]; X)) \to C([-r, r_1]; X)$ defined by $(\Gamma u)_0 = \varphi$ and

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t - s)[-\frac{d}{ds} g(s, \varphi(s - r_1)) + f(s, u_s)]ds, \ t \in [0, r_1]$$

is a contraction from $B_r(y|_{[-r,r_1]}, C([-r, r_1]; X))$ into itself. Let u^1 be the unique fixed point of Γ. In the next steps, $q > 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Step 1. Let $\alpha \in (0, 1)$ be such that $1 > q\alpha$. Then $u^1 \in C^{\min\{\mu,\alpha\}}([0, r_1]; X)$.

Let H be the function $H(s) = \frac{d}{ds} g(s, \varphi(s - r_1)) + f(s, u_s^1)$, $\theta = \frac{(1-\alpha q)}{q}$ and $\widetilde{\theta} = (1 - \alpha q)^{\frac{1}{q}}$. Then, for $t \in [0, r_1)$ and $h > 0$ we get

$$\begin{aligned}
\| u(t + h) - u(t) \| &\leq c_1 h^\mu + \| \int_0^t T(t - s)(T(h) - I)H(s)ds \| \\
&\quad + M \| g(s, \varphi(s - r_1)) + f(s, u_s^1) \|_{L^p([0,r_1])} h^{\frac{1}{q}} \\
&\leq c_2 h^{\min\{\mu, \frac{1}{q}\}} + \| \int_0^t \int_0^h (-A)T(t + \xi - s)H(s)d\xi ds \| \\
&\leq c_2 h^{\min\{\mu, \frac{1}{q}\}} + C_0^1 \int_0^t \frac{\| H(s) \|}{(t - s)^\alpha} \int_0^h \frac{d\xi ds}{(t + \xi - s)^{1-\alpha}} \\
&\leq c_2 h^{\min\{\mu, \frac{1}{q}\}} + c_3 h^\alpha \| H(s) \|_{L^p([0,r_1])} \frac{(r_1)^\theta}{\widetilde{\theta}},
\end{aligned}$$

where c_i, $i = 1, \ldots, 3$, are constants independent of t and h. This proves the assertion.

Step 2. Let $\alpha \in (0, 1)$ with $1 > q\alpha$. Then the function $s \to f(s, u_s^1)$ belongs to $C^{\min\{\mu,\alpha,\gamma\}}([0, r_1]; X)$.

Arguing as in the proof of step 1, we obtain $v = u^1 - y \in C^\alpha([0, r_1]; X)$. Now using the relation $u_t = y_t + v_t$ and the fact that $v_0 = 0$, for $t \in [0, r_1)$, and $h > 0$ such that $t + h \in [0, r_1]$, we have

$$\begin{aligned} \| u^1_{t+h} - u^1_t \|_{\mathcal{C}} &\leq \| y_h - \varphi \|_{\mathcal{C}} + \| v_h \|_{\mathcal{C}} + \sup_{\theta \in [h, r_1]} \| u^1(\theta + h) - u^1(\theta) \| \\ &\leq c_4 h^\mu + \sup_{\theta \in [0,h]} \| v(\theta) \| + c_5 h^{\min\{\mu, \alpha\}}, \end{aligned}$$

which shows our claim from condition (**b**).

Step 3. The function u^1 is differentiable *a.e.* on $[0, r_1]$, $\frac{d}{dt} u^1 \in L^p([0, r_1], X)$ and $t \to \frac{d}{dt} g(t, u^1(t - r_1)) \in L^{\nu(p)}([r_1, 2r_1], X)$.

Let $z_1, z_2 \in C([0, r_1]; X)$ be the mild solution of the systems

$$x'(t) = Ax(t) + f(t, (u^1)_t), \qquad t \in [0, r_1], \tag{2.6}$$

$$x(0) = \varphi(0), \tag{2.7}$$

and

$$x'(t) = Ax(t) + \frac{d}{dt} g(t, \varphi(t - r_1)), \qquad t \in [0, r_1], \tag{2.8}$$

$$x(0) = 0, \tag{2.9}$$

respectively. From [24, Theorem 4.3.5] and [24, Corollary 4.2.10] it follows that z_1 is a classical solution of (2.6)-(2.7) and z_2 is a strong solution of (2.8)-(2.9). Consequently, $z_1 \in C^1([0, r_1]; X) \cap C([0, r_1]; [D(A)])$, z_2 is differentiable *a.e.* on $[0, r_1]$ and $z_2' \in L^1([0, r_1], X)$. Moreover, from the development in [24, Section 4.3] we can also infer that $z_2' \in L^p([0, r_1], X)$. Since $u^1(t) = z_1(t) + z_2(t)$, it follows that $\frac{d}{dt} u^1 \in L^p([0, r_1], X)$ and $t \to \frac{d}{dt} g(t, u^1(t - r_1)) \in L^{\nu(p)}([r_1, 2r_1], X)$.

From the above and the contraction mapping principle, we deduce the existence of $r_1 < b \leq 2r_1$ such that there exists a mild solution $u^2 \in C([r_1 - r, b]; X)$ of

$$x'(t) = Ax(t) + \frac{d}{dt} g(t, u^1(t - r_1)) + f(t, x_t), \qquad t \in [r_1, b], \tag{2.10}$$

$$x_{r_1} = (u^1)_{r_1} \in C([r_1 - r, r_1]; X). \tag{2.11}$$

Obviously, the function u obtained by pasting the functions u^i, $i = 1, 2$, is a mild solution of (1.1)-(1.2) on $[-r_1, b]$.

When L_f is constant function, the function Γ is a contraction on $C([-r, r_1]; X)$ and the proof can be completed as above. The proof is complete. □

Next, we discuss the existence of a classical solution. In what follows, A_W is the infinitesimal generator of the semigroup $(W(t))_{t \geq 0}$ on $\mathcal{C}$ defined by $W(t)\psi(\theta) = T(t + \theta)\psi(0)$ for $-t \leq \theta \leq 0$ and $W(t)\psi(\theta) = \psi(t + \theta)$ for $-\infty < \theta < -t$.

Proposition 2.1. *Assume $g \in C^1([0, a] \times X; X)$, $f \in C^1([0, a] \times \mathcal{C}; X)$ and f is Lipschitz on $[0, r_1] \times \mathcal{C}$. If $\varphi \in D(A_W)$, $s \to g(s, \varphi(s - r_1)) \in C^2([0, r_1]; X)$ and $\frac{d}{ds} g(s, \varphi(s - r_1))_{|s=0} + f(0, \varphi) = 0$, then there exists a classical solution of* (1.1)-(1.2) *on $[-r_1, b]$ for some $r_1 < b \leq a$.*

Proof. It is easy to see that there exists a unique S-mild solution u^1 of (1.1)-(1.2) on the interval $[-r, r_1]$. Let $v \in C([-r, r_1]; X)$ be the unique solution of

$$\begin{aligned} z(t) &= AT(t)\varphi(0) - \int_0^t T(t-s)\frac{d^2}{ds^2}g(s, \varphi(s-r_1))ds \\ &\quad + \int_0^t T(t-s)\left(D_1 f(s, u_s^1) + D_2 f(s, u_s^1) z_s\right) ds, \qquad t \in [0, r_1], \\ z_0 &= A_W \varphi. \end{aligned}$$

Arguing as in the proof of [19, Theorem 4], we can prove that $u^1 \in C^1([0, r_1]; X)$ (in fact, $\frac{d}{dt}u^1 = v$), which implies that $s \to g(s, u^1(s-r_1)) \in C^1([r_1, 2r_1]; X)$.

Since f is locally Lipschitz, there is a unique mild solution $u^2 \in C([r_1 - r, b]; X)$ of (2.10)-(2.11) for some $r_1 < b \le 2r_1$. Let u be the function obtained by pasting the functions u^1 and u^2. From [19, Theorem 4] we infer that u satisfies (1.1) on $[0, b]$ and $u \in C^1([0, b]; X) \bigcap C([0, b]; [D(A)])$. This completes the proof. □

2.1. **An application.** To finish this section, we briefly study the system

$$\frac{\partial}{\partial t}\left(u(t,\xi) + \alpha(t)u(t-r_1)\right) = \frac{\partial^2}{\partial \xi^2}u(t,\xi) + \sum_{i=1}^m \beta_i(t)u(t-r_i^1), \tag{2.12}$$

$$u(t,0) = u(t,\pi) = 0, \tag{2.13}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \qquad \theta \in [-r, 0],\ \xi \in [0,\pi], \tag{2.14}$$

where $\alpha, \beta_i : [0, a] \to \mathbb{R}$ are continuous and r_1, r_i^1 are real numbers with $0 < r_1 < r$.

To treat this system, we choose the space $X = L^2([0,\pi])$ and the operator $A : D(A) \subset X \to X$ defined by $Ax = x''$ on $D(A) := \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the generator of an analytic semigroup $(T(t))_{t\ge 0}$ on X.

The next results are consequences of Corollary 2.1 and Theorem 2.3 respectively.

Proposition 2.2. *Let $\gamma > 0$ and $\varphi \in C([-r, 0]; X_\gamma)$. Then there exists a unique mild solution u of* (2.12)-(2.14) *on $[-r, \infty)$.*

Proposition 2.3. *Assume $t \to y_t \in C^\mu([0, a]; X)$, $\varphi \in W^{2,p}([-r, 0], X)$ and $\varphi(0) \in D(A)$ for some $\mu \in (0, 1)$ and $p \ge 1$. If $\sum_{i=1}^m \mid \beta_i \mid_\infty r_1 < 1$, then there exists a unique S-mild solution u of* (2.12)-(2.14) *on $[-r, b]$ for some $r_1 < b$. If, in addition, $\varphi \in D(A_W)$, the function $s \to \alpha(s)\varphi(s - r_1)$ is of class C^2 on $[0, r_1]$ and $\frac{d}{ds}[\alpha(s)\varphi(s-r_1)]_{|s=0} + \sum_{i=1}^m \beta_i(0)\varphi(-r_i) = 0$, then u is a classical solution.*

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