PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 10, October 2009, Pages 3207–3217
S 0002-9939(09)09992-4
Article electronically published on June 5, 2009

# ON THE CLIQUE NUMBER OF THE GENERATING GRAPH OF A FINITE GROUP

ANDREA LUCCHINI AND ATTILA MARÓTI

(Communicated by Jonathan I. Hall)

Abstract. The generating graph $\Gamma(G)$ of a finite group $G$ is the graph defined on the elements of $G$ with an edge connecting two distinct vertices if and only if they generate $G$. The maximum size of a complete subgraph in $\Gamma(G)$ is denoted by $\omega(G)$. We prove that if $G$ is a non-cyclic finite group of Fitting height at most 2 that can be generated by 2 elements, then $\omega(G) = q + 1$, where $q$ is the size of a smallest chief factor of $G$ which has more than one complement. We also show that if $S$ is a non-abelian finite simple group and $G$ is the largest direct power of $S$ that can be generated by 2 elements, then $\omega(G) \leq (1 + o(1))m(S)$, where $m(S)$ denotes the minimal index of a proper subgroup in $S$.

## 1. Introduction

The generating graph $\Gamma(G)$ of a finite group $G$ is the graph defined on the elements of $G$ with an edge connecting two distinct vertices if and only if they generate $G$. By the solution of Dixon's conjecture, it is known that $\Gamma(S)$ has "many" edges for $S$ a non-abelian finite simple group. In particular, Liebeck and Shalev [9] proved that there exists a universal positive constant $c$ such that the maximal size of a complete subgraph in $\Gamma(S)$ is at least $c \cdot m(S)$ where $m(S)$ is the minimal index of a proper subgroup in $S$. This result, in general, is best possible. Indeed, by a result of Dye [6], the group $S = Sp_{2n}(2)$ is the union of all conjugates of the maximal subgroups $O^+_{2n}(2)$ and $O^-_{2n}(2)$, and so $\omega(S) \leq 2^{2n} = (2+o(1))m(S)$.

This result of Liebeck and Shalev together with the above-mentioned remark on the symplectic group justifies the following definitions. For a finite group $G$ let the maximum size of a complete subgraph in $\Gamma(G)$ be denoted by $\omega(G)$. For a non-cyclic finite group $G$ let $\sigma(G)$ denote the least number of proper subgroups of $G$ whose union is $G$. Clearly, $\omega(G) \leq \sigma(G)$. Moreover, if $\chi(G)$ denotes the chromatic number of $\Gamma(G)$ (that is, the least number of colors needed to color the vertices of $\Gamma(G)$ in such a way that the endpoints of each edge receive different colors), then we also have $\omega(G) \leq \chi(G) \leq \sigma(G)$, where the second inequality follows from the fact that $\Gamma(G)$ is $\sigma(G)$-colorable since its vertex set is the union of $\sigma(G)$ empty subgraphs.

Received by the editors July 22, 2008.
2000 *Mathematics Subject Classification.* Primary 05C25, 20D10, 20P05.
The research of the second author was supported by OTKA NK72523, OTKA T049841, NSF Grant DMS 0140578, and by a fellowship of the Mathematical Sciences Research Institute.

©2009 American Mathematical Society

The function $\sigma$ has been much investigated. For example, for a finite solvable group $G$, Tomkinson [14] showed that $\sigma(G) = q + 1$, where $q$ is the minimal size of a chief factor of $G$ having more than one complement. Our first result is

**Theorem 1.1.** *Let $G$ be a finite group with Fitting height at most 2. Then $\omega(G) = \chi(G)$. Moreover, if the minimal number of generators of $G$ is 2, then $\omega(G) = \sigma(G)$.*

It is not known whether the conclusions of Theorem 1.1 are true for an arbitrary finite solvable group $G$. Blackburn [3] showed that $\omega(\mathrm{Sym}(n)) = \sigma(\mathrm{Sym}(n)) = 2^{n-1}$ for $n$ a sufficiently large odd positive integer, and also that $\omega(\mathrm{Alt}(n)) = \sigma(\mathrm{Alt}(n)) = 2^{n-2}$ for $n$ a sufficiently large even integer not divisible by 4. However, by [11], there are infinitely many non-abelian finite simple groups $S$ with $\omega(S) < \chi(S) < \sigma(S)$.

Still $\omega(S)$ and $\sigma(S)$ do not seem to be "far apart" for a non-abelian finite simple group $S$. In fact, Blackburn [3] asked whether $\omega(S)/\sigma(S)$ tends to 1 as the size of the non-abelian finite simple group $S$ tends to infinity. Our second result shows that there is an infinite sequence of 2-generated finite groups $G$ such that $\omega(G)/\sigma(G)$ tends to 0 as the size of $G$ tends to infinity.

**Theorem 1.2.** *Let $S$ be a non-abelian finite simple group, let $m(S)$ be the minimal index of a proper subgroup in $S$ and let $G$ be the largest direct power of $S$ that can be generated by 2 elements. Then $\omega(G) \leq m(S) + O(m(S)^{14/15})$ if $S$ is a group of Lie type and $\omega(G) \leq m(S) + O(1)$ otherwise. In particular, if $S = \mathrm{Alt}(n)$, then $\omega(G)/\sigma(G) \leq (n + O(1))/2^{n-2}$.*

## 2. Groups of fitting height at most 2

In this section we prove Theorem 1.1.

Let $V$ be a finite dimensional vector space over a finite field of prime order. Let $H$ be a linear solvable group acting irreducibly and faithfully on $V$. Suppose that $H$ can be generated by 2 elements. For a positive integer $t$ we consider the semidirect product $G = V^t \rtimes H$, where $H$ acts in the same way on each of the $t$ direct factors. We would like to derive some information about $\omega(G)$. Put $F = \mathrm{End}_H(V)$.

**Proposition 2.1.** *Assume $H = \langle x, y\rangle$ and let $(u_1, \dots, u_t)$, $(w_1, \dots, w_t) \in V^t$. The following are equivalent:*

(1) *$G \neq \langle x(u_1, \dots, u_t), y(w_1, \dots, w_t)\rangle$;*
(2) *there exist $\lambda_1, \dots, \lambda_t \in F$ and $w \in V$ with $(\lambda_1, \dots, \lambda_t, w) \neq (0, \dots, 0, 0)$ such that $\sum_i \lambda_i u_i = w - wx$ and $\sum_i \lambda_i w_i = w - wy$.*

*Proof.* Let $a = x(u_1, \dots, u_t)$, $b = y(w_1, \dots, w_t)$, $K = \langle a, b\rangle$. First we prove, by induction on $t$, that if $K \neq G$, then (2) holds. Let $\bar{a} = x(u_1, \dots, u_{t-1}, 0)$, $\bar{b} = y(w_1, \dots, w_{t-1}, 0)$, $\bar{K} = \langle \bar{a}, \bar{b}\rangle$. If $\bar{K} \ncong V^{t-1}H$, then, by induction, there exist $\lambda_1, \dots, \lambda_{t-1} \in F$ and $w \in V$ with $(\lambda_1, \dots, \lambda_{t-1}, w) \neq (0, \dots, 0, 0)$ such that $\sum_i \lambda_i u_i = w - wx$ and $\sum_i \lambda_i w_i = w - wy$. In this case $\lambda_1, \dots, \lambda_{t-1}, 0$ and $w$ are the requested elements. So we may assume $\bar{K} \cong V^{t-1}H$. Set $V_t = \{(0, \dots, 0, v) \mid v \in V\}$. We have $\bar{K}V_t = KV_t = G$ and $K \neq G$; this implies that $K$ is a complement of $V_t$ in $G$ and therefore there exists $\delta \in \mathrm{Der}(\bar{K}, V_t)$ such that $\delta(\bar{a}) = u_t$ and $\delta(\bar{b}) = w_t$. However, by Propositions 2.7 and 2.10 of [2], we have $H^1(\bar{K}, V_t) \cong F^{t-1}$. More precisely if $\delta \in \mathrm{Der}(\bar{K}, V_t)$, then there exist an inner derivation $\delta_w \in \mathrm{Der}(H, V)$ and $\lambda_1, \dots, \lambda_{t-1} \in F$ such that for each

$g(v_1,\dots,v_{t-1},0)\in\bar{K}$ we have $\delta(g(v_1,\dots,v_{t-1},0))=\delta_w(g)+\lambda_1v_1+\cdots+\lambda_{t-1}v_{t-1}=wg-w+\lambda_1v_1+\cdots+\lambda_{t-1}v_{t-1}$. In particular $u_t=wx-w+\lambda_1u_1+\cdots+\lambda_{t-1}u_{t-1}$ and $w_t=wx-w+\lambda_1w_1+\cdots+\lambda_{t-1}w_{t-1}$.

Conversely, if (2) holds, then $\langle h(v_1,\dots,v_t)\mid w-wh=\lambda_1v_1+\cdots+\lambda_tv_t\rangle$ is a proper subgroup of $G$ containing $K$. □

Let $n$ be the dimension of $V$ over $F$. We may identify $H=\langle x,y\rangle$ with a subgroup of $GL(n,F)$. In this identification $x$ and $y$ become two $n\times n$ matrices $X$ and $Y$ with coefficients in $F$. Let $(u_1,\dots,u_t)$, $(w_1,\dots,w_t)\in V^t$. Then every $u_i$ and $w_j$ can be viewed as a $1\times n$ matrix. Denote the $t\times n$ matrix with rows $u_1,\dots,u_t$ (resp. $w_1,\dots,w_t$) by $A$ (resp. $B$). By Proposition 2.1, the elements $x(u_1,\dots,u_t)$, $y(w_1,\dots,w_t)$ generate a proper subgroup of $G$ if and only if there exists a non-zero vector $(\lambda_1,\dots,\lambda_t;\mu_1,\dots\mu_n)$ in $F^{t+n}$ such that

$$\begin{cases}(\lambda_1,\dots,\lambda_t)A=(\mu_1,\dots,\mu_n)(1-X)\\(\lambda_1,\dots,\lambda_t)B=(\mu_1,\dots,\mu_n)(1-Y)\end{cases}.$$

This is equivalent to saying that there exist elements $\bar{X}$ and $\bar{Y}$ in $G$ such that $\langle\bar{X},\bar{Y}\rangle=G$ with the property that $\bar{X}$ maps to $x$ and $\bar{Y}$ maps to $y$ under the projection from $G$ to $H$ if and only if there exist $t\times n$ matrices $A$ and $B$ with

$$\operatorname{rank}\begin{pmatrix}1-X & 1-Y\\ A & B\end{pmatrix}=n+t. \tag{1}$$

From this it immediately follows that $G$ cannot be generated by 2 elements if $t>n$ (hence $\omega(G)=1$ in this case). Notice also that if $X$ and $Y$ are two $n\times n$ matrices generating the matrix group $H$, then the linear map $\alpha:F^n\to F^n\times F^n$, $w\mapsto(w(1-X),w(1-Y))$ is injective (if $w\in\ker\alpha$, then $wX=wY=w$ against the fact that $X$ and $Y$ generate a non-trivial irreducible group); the matrix $\begin{pmatrix}1-X & 1-Y\end{pmatrix}$ has rank $n$, and so it is possible to find $A$ and $B$ satisfying (1) whenever $t\le n$. Hence $3\le\omega(V^n\rtimes H)\le\omega(G)$ whenever $t\le n$. The case $t=n$ is of special importance. In this case our observations yield

**Proposition 2.2.** *Let $t=n$. Assume that $X_1,\dots,X_\omega$ pairwise generate $H$. Then there exist elements $\bar{X}_1,\dots,\bar{X}_\omega$ pairwise generating $G$ (so that for all $i$ with $1\le i\le\omega$ the element $X_i$ is the projection of $\bar{X}_i$ under the projection from $G$ to $H$) if and only if there exist $n\times n$ matrices $A_1,\dots,A_\omega$ such that for all $i$ and $j$ with $1\le i<j\le\omega$ we have*

$$\det\begin{pmatrix}1-X_i & 1-X_j\\ A_i & A_j\end{pmatrix}\neq 0.$$

From now on let $H$ be a nilpotent finite group that can be generated by 2 elements with an irreducible (but not necessarily faithful) action $\rho:H\to GL(V)$. Let $F=\mathrm{End}_H(V)$ and let $n=\dim_F(V)$. The Sylow subgroups of $H$ are either cyclic or non-cyclic and 2-generated. Let $\pi_1$ be the set consisting of those prime divisors of $|H|$ whose corresponding Sylow subgroups are not cyclic, and let $\pi_2$ be the set of all other prime divisors of $|H|$. Let $p$ be the smallest prime in $\pi_1$. (If $\pi_1=\emptyset$, then set $p=\infty$.) We can find two generators $x$ and $y$ of $H$ such that $|x|$ is divisible only by primes in $\pi_1$ (if $\pi_1=\emptyset$ we take $x=1$). Let $X=x^\rho,Y=y^\rho$, and $u=\min(p,|V|)$. Clearly $\sigma(V^t\rtimes H^\rho)\le u+1$.

**Proposition 2.3.** *With the notation and assumptions above we have $\omega(V^t\rtimes H^\rho)=u+1$ if $t\le n$ and $\omega(V^t\rtimes H^\rho)=1$ otherwise.*

*Proof.* By our observations above, to prove Proposition 2.3, it is sufficient to show that $u+1 \leq \omega(V^n \rtimes H^\rho)$. To see this it is sufficient to verify that there exist $A$, $B_0, \dots, B_{u-1} \in V^n$ such that the elements $X$, $YB_0$, $XYB_1$, $X^2YB_2, \dots,$ $X^{u-1}YB_{u-1}$ pairwise generate $V^n \rtimes H^\rho$.

We need to consider two different cases.

*Case* 1: $X \neq 1$.

Notice that $Z(H^\rho) \leq (\mathrm{End}_H(V))^*$; hence $Z(H^\rho)$ is a subgroup of $F^*$. This implies

- $|X|$ divides $|F|-1$ (in particular $p \leq |F|-1$);
- for any $h \in H$, $V$ is a completely reducible $\langle h \rangle$-module (indeed any prime divisor of $|H^\rho|$ divides $|Z(H^\rho)|$, and hence it is coprime with $|F|$).

The second remark implies that we may write $x$ in the form

$$X = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix},$$

where 1 denotes the identity $\ell \times \ell$ matrix for some non-negative integer $\ell$ with $\ell < n$ and $C$ is an invertible $(n-\ell) \times (n-\ell)$ matrix which does not admit 1 as an eigenvalue. Decompose $Y$ and $1-Y$ as block matrices in the following way:

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

and

$$1 - Y = \begin{pmatrix} T_1 - Y_1 \\ T_2 - Y_2 \end{pmatrix},$$

where $Y_1$ and $T_1 - Y_1$ denote the matrices consisting of the first $\ell$ rows of $Y$ and $1-Y$ respectively and $Y_2$ and $T_2 - Y_2$ denote the matrices consisting of the remaining rows of $Y$ and $1-Y$ respectively. Since

$$\mathrm{rank}(1-X \quad 1-Y) = \mathrm{rank}\begin{pmatrix} 0 & T_1 - Y_1 \\ 1-C & T_2 - Y_2 \end{pmatrix} = n$$

we deduce that $\mathrm{rank}(T_1 - Y_1) = \ell$. Let $D$ be an $(n-\ell) \times n$ matrix such that

$$\det \begin{pmatrix} T_1 - Y_1 \\ D \end{pmatrix} \neq 0.$$

By Theorem 2.2, we look for $A, B_0, \dots, B_{p-1}$ such that

$$\det \begin{pmatrix} 1-X & 1-X^rY \\ A & B_r \end{pmatrix} \neq 0 \quad \text{and} \quad \det \begin{pmatrix} 1-X^rY & 1-X^sY \\ B_r & B_s \end{pmatrix} \neq 0$$

for all $r$ and $s$ such that $0 \leq r \leq s \leq p-1$. Since $p$ divides $|F|-1$, there exist $p$ pairwise distinct elements $b_0, \dots, b_{p-1} \in F^*$. Consider the following $p \times p$ matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} b_i Y_1 \\ D \end{pmatrix}$$

for all $i$ with $0 \leq i \leq p-1$, where 1 in the definition of $A$ denotes the $\ell \times \ell$ identity matrix. We prove that $A, B_0, \dots, B_{p-1}$ are the matrices we are looking for. Notice that

$$X^rY = \begin{pmatrix} Y_1 \\ C^rY_2 \end{pmatrix};$$

hence

$$\det\begin{pmatrix}1-X & 1-X^rY\\ A & B_r\end{pmatrix} = \det\begin{pmatrix}0 & 0 & T_1-Y_1\\ 0 & 1-C & *\\ 1 & 0 & *\\ 0 & 0 & D\end{pmatrix}$$
$$= \pm\det(1-C)\det\begin{pmatrix}T_1-Y_1\\ D\end{pmatrix}\neq 0.$$

On the other hand, if $r\neq s$, then

$$\det\begin{pmatrix}1-X^rY & 1-X^sY\\ B_r & B_s\end{pmatrix} = \det\begin{pmatrix}1-X^rY & X^rY-X^sY\\ B_r & B_s-B_r\end{pmatrix}$$
$$= \det\begin{pmatrix}T_1-Y_1 & 0\\ * & (C^r-C^s)Y_2\\ b_rY_1 & (b_s-b_r)Y_1\\ D & 0\end{pmatrix} = \det\begin{pmatrix}T_1-Y_1\\ D\end{pmatrix}\det\begin{pmatrix}(C^r-C^s)Y_2\\ (b_s-b_r)Y_1\end{pmatrix}$$
$$= (b_s-b_r)\det(C^r-C^s)\det\begin{pmatrix}T_1-Y_1\\ D\end{pmatrix}\det\begin{pmatrix}Y_2\\ Y_1\end{pmatrix},$$

which is non-zero if and only if $\det(C^r-C^s)=\det(C^r(1-C^{s-r}))\neq 0$. To show that the matrix $1-C^{s-r}$ is non-singular it is sufficient to see that 1 is not an eigenvalue of $C^{s-r}$. Since $V$ is a completely reducible $F\langle X\rangle$-module, $C$ can be diagonalised over a suitable field extension of $F$. Let $\beta$ be an arbitrary eigenvalue of $C^{s-r}$. Then $\beta=\gamma^{s-r}$ for some eigenvalue $\gamma$ of $C$. Now $\gamma$ is different from 1 by our choice of $C$. Finally since $0<s-r<p$ and since no prime smaller than $p$ divides $|X|$ we see that $\gamma^{s-r}$ cannot be 1. This settles Case 1.

*Case* 2: $X=1$.

In this case $H^\rho=\langle Y\rangle$ is a cyclic group and $V$ is an absolutely irreducible $FH$-module. Hence $V=F$ and $n=1$. We have $u\leq|F|$ and if $0\neq A\in V$ and $B_0,B_1,\ldots,B_{u-1}$ are distinct elements of $V$, then, by Proposition 2.1, $A$, $YB_0$, $YB_1,\ldots,YB_{u-1}$ pairwise generate $V\rtimes H^\rho$. This proves Proposition 2.3. □

Let $G$ be a finite solvable group, and let $\mathcal{A}$ be a set of representatives for the irreducible $G$-groups that are $G$-isomorphic to a complemented chief factor of $G$. For $A\in\mathcal{A}$, let $R_G(A)$ be the smallest normal subgroup contained in $C_G(A)$ with the property that $C_G(A)/R_G(A)$ is $G$-isomorphic to a direct product of copies of $A$ and it has a complement in $G/R_G(A)$. The factor group $C_G(A)/R_G(A)$ is called the $A$-crown of $G$. The non-negative integer $\delta_G(A)$ defined by $C_G(A)/R_G(A)\cong_G A^{\delta_G(A)}$ is called the $A$-rank of $G$ and it coincides with the number of complemented factors in any chief series of $G$ that are $G$-isomorphic to $A$. If $\delta_G(A)\neq 0$, then the $A$-crown is the socle of $G/R_G(A)$. The notion of crown was introduced by Gaschütz in [7].

**Proposition 2.4.** *Let $G$ and $\mathcal{A}$ be as above. Let $x_1,\ldots,x_u$ be elements of $G$ such that $\langle x_1,\ldots,x_u,R_G(A)\rangle=G$ for any $A\in\mathcal{A}$. Then $\langle x_1,\ldots,x_u\rangle=G$.*

*Proof.* Let $H=\langle x_1,\ldots,x_u\rangle$ and suppose that $HR_G(A)=G$ for any $A\in\mathcal{A}$. There exists a normal subgroup $N$ of $G$ of minimum order with respect to the property $HN=G$. Assume by contradiction that $N\neq 1$ and choose $M$ such that $A=N/M$ is a chief factor of $G$. Since $HM\neq G$, we have that $A$ is a complemented chief factor and $(HM/M)(R_G(A)M/M)=G/M=(HM/M)(N/M)$. By Proposition 11

of [5] and the fact that $R_{G/M}(A) = R_G(A)M/M$, we deduce $HM = G$, against the choice of $N$. □

Let $d(X)$ denote the minimal number of generators of the finite group $X$.

**Proposition 2.5.** *Let $G$ be a finite group of Fitting height equal to 2. If $d(G) = 2$, then $\omega(G) = \sigma(G)$.*

*Proof.* Clearly we may assume that the Frattini subgroup $\mathrm{Frat}(G)$ of $G$ is trivial. Then the Fitting subgroup $\mathrm{Fit}(G)$ coincides with the direct product of the minimal normal subgroups of $G$ (see [13, 5.2.15]). Let $V$ be the subgroup of $\mathrm{Fit}(G)$ generated by the non-central minimal normal subgroups of $G$. Now $V$ is an abelian normal subgroup of a finite group with trivial Frattini subgroup, so $V$ is complemented in $G$ (see [13, 5.2.13]). So we have that $G = V \rtimes H$ for some nilpotent group $H$ with $d(H) \leq 2$. Let $\mathcal{Z}$ be the set of $G$-irreducible modules $G$-isomorphic to some factor of $V$. We have that $V = \prod_{M\in\mathcal{Z}} V_M$, where $V_M$ is the product of the minimal normal subgroups $G$-isomorphic to $M$. If $M \in \mathcal{Z}$ and $\rho_M : H \to GL(M)$ is the action of $H$ on $M$, then $R_G(M) = C_H(M) \times \prod_{L\in\mathcal{Z},L\neq M} V_L$ and $G/R_G(M) \cong V_M \rtimes H^{\rho_M} = M^{t_M} \rtimes H^{\rho_M}$ for some positive integer $t_M$. Notice that $d(M^{t_M} \rtimes H^{\rho_M}) \leq 2$ for all $M \in \mathcal{Z}$.

For the finite nilpotent group $H$ choose $p$, $x$, and $y$ as in the preceding paragraph of the statement of Proposition 2.3. Put $\tau = \min_{M\in\mathcal{Z}}\{|M|\}$. (Note that $\mathcal{Z} \neq \emptyset$, for otherwise $V = 1$ and $G = H$ is nilpotent.) Then $\sigma(G) = 1 + \min\{\tau, p\}$. Put $\sigma = \sigma(G)$. By Proposition 2.3, for any $M \in \mathcal{Z}$ there exist $A_M$, $B_{0,M}, \ldots, B_{\sigma-2,M}$ such that the $\sigma$ elements

$$x^{\rho_M} A_M, y^{\rho_M} B_{0,M}, \ldots, (x^{\sigma-2}y)^{\rho_M} B_{\sigma-2,M}$$

pairwise generate $M^{t_M} \rtimes H^{\rho_M}$. Put

$$a = \prod_{M\in\mathcal{Z}} A_M, \quad \text{and} \quad b_i = \prod_{M\in\mathcal{Z}} B_{i,M}$$

for all $i$ such that $0 \leq i \leq \sigma - 2$. Finally consider the set

$$\Omega = \{xa, yb_0, xyb_1, \ldots, x^{\sigma-2}yb_{\sigma-2}\}.$$

We claim that two distinct elements $\omega_1, \omega_2$ of $\Omega$ generate $G$. Indeed, take $M \in \mathcal{A}$. If $G$ centralizes $M$, then $V \leq R_G(M)$; otherwise $M \in \mathcal{Z}$. In both cases $\langle \omega_1, \omega_2, R_G(M)\rangle = G$; hence, by Proposition 2.4, we have $\langle \omega_1, \omega_2\rangle = G$. This proves Proposition 2.5. □

We are now in the position to prove Theorem 1.1.

Let $G$ be as in the statement of Theorem 1.1. If $d(G) > 2$, then $\Gamma(G)$ is the empty graph and so $\omega(G) = \chi(G) = 1$. So assume that $d(G) \leq 2$. If the Frattini subgroup of $G$ is denoted by $\mathrm{Frat}(G)$, then $\omega(G) = \omega(G/\mathrm{Frat}(G))$ and $\chi(G) = \chi(G/\mathrm{Frat}(G))$. Moreover, if $G$ is non-cyclic, then $\sigma(G) = \sigma(G/\mathrm{Frat}(G))$. Hence we may assume that $\mathrm{Frat}(G) = 1$.

Let $G$ be cyclic. Since $\mathrm{Frat}(G) = 1$, the cyclic group $G$ is the direct product of say $t$ cyclic groups of distinct prime orders. Let $S$ be the set of generators of $G$. In the graph $\Gamma(G)$ every vertex in $S$ is connected to every other vertex in $\Gamma(G)$. Thus, if $\Gamma(G) \setminus S$ denotes the graph obtained from $\Gamma(G)$ by removing all vertices from $S$ together with all edges having an endpoint in $S$, then $\omega(G)$ equals the maximum size of a complete subgraph in the graph $\Gamma(G) \setminus S$ plus $|S|$, and $\chi(G)$ equals the

chromatic number of the graph $\Gamma(G) \setminus S$ plus $|S|$. Now $G$ has $t$ maximal subgroups each of which is cyclic. We may choose a generator from each of these maximal subgroups. Since any distinct pair of these elements generate $G$, we have a complete subgraph of size $t$ in the graph $\Gamma(G) \setminus S$. On the other hand, the graph $\Gamma(G) \setminus S$ can be expressed as the union of $t$ empty subgraphs (coming from the $t$ maximal subgroups of $G$); hence it is $t$-colorable and so the chromatic number of $\Gamma(G) \setminus S$ is at most $t$. These observations yield $t + |S| \leq \omega(G) \leq \chi(G) \leq t + |S|$; hence $\omega(G) = \chi(G)$.

We may now also assume that $d(G) = 2$. Also, by Proposition 2.5, we assume that $G$ is nilpotent. Then, since $\mathrm{Frat}(G) = 1$, we have $G = C \times N_{p_1} \times \ldots \times N_{p_t}$ for some positive integer $t$, where $p_1 < \ldots < p_t$ are distinct primes, $N_{p_j} = C_{p_j} \times C_{p_j}$ for all $j$ with $1 \leq j \leq t$, and $C$ is a cyclic group that is a direct product of cyclic groups of prime orders different from $p_j$ for $j$ with $1 \leq j \leq t$. Let $N$ be the normal subgroup of $G$ for which $G/N \cong N_{p_1} = C_{p_1} \times C_{p_1}$. Then $\sigma(G) \leq \sigma(G/N) \leq p_1 + 1$. For each $j$ with $1 \leq j \leq t$ let $a_{1,j}, a_{2,j}, \ldots, a_{p_1+1,j}$ be non-identity elements from $N_{p_j}$ generating distinct cyclic subgroups in $N_{p_j}$. Let $c$ be a generator from $C$. For any $i$ with $1 \leq i \leq p_1 + 1$ let $a_i$ be the element $(c, a_{i,1}, \ldots, a_{i,t})$ from $G$. Clearly, $\{a_1, \ldots, a_{p_1+1}\}$ spans a complete subgraph in $\Gamma(G)$. Hence $p_1 + 1 \leq \omega(G) \leq \sigma(G) \leq p_1 + 1$, that is, $\omega(G) = \sigma(G)$.

## 3. Direct products of non-abelian simple groups

In this section we prove Theorem 1.2.

Our first result (Proposition 3.1) was also proved (independently) by Abdollahi and Jafarian Amiri in [1].

**Proposition 3.1.** *Let $S$ be a non-abelian finite simple group. Then for any positive integer $n$ we have $\sigma(S^n) = \sigma(S)$, where $S^n$ denotes the direct product of $n$ copies of $S$.*

*Proof.* The inequality $\sigma(S^n) \leq \sigma(S)$ follows at once from the observation that if $\{M_i\}$ is a set of proper subgroups of $S$ with $S = \bigcup_i M_i$, then $\{M_i \times S^{n-1}\}$ is a set of proper subgroups of $S^n$ with $S^n = \bigcup_i (M_i \times S^{n-1})$.

Let $\{Y_1, \ldots, Y_\tau\}$ be a set of proper subgroups of $S^n$ such that $S^n = \bigcup_{i=1}^{\tau} Y_i$. Suppose also that $\tau$ is as small as possible (that is, $\tau = \sigma(S^n)$). Put $\sigma = \sigma(S)$. We need to show that $\sigma \leq \tau$.

We may assume that the $Y_i$'s are maximal subgroups of $S^n$. What are the maximal subgroups of $S^n$? They are of the following two kinds:

- product type: $P_{M,i} = \{(x_1, \ldots, x_n) \in S^n \mid x_i \in M\}$, where $M$ is a maximal subgroup of $S$;
- diagonal type: $D_{i,j,\phi} = \{(x_1, \ldots, x_n) \mid x_j = x_i^{\phi}\}$, where $\phi \in \mathrm{Aut}(S)$.

Without loss of generality assume that $Y_i$ is of product type if $i \leq a$ and $Y_i$ is of diagonal type if $a < i \leq \tau$ for some non-negative integer $a$ at most $\tau$. We may assume that $a < \sigma$ for otherwise $\sigma \leq a \leq \tau$ in which case we are done.

Let $I$ be the set of those indices $i$ with $1 \leq i \leq n$ for which there exists a maximal subgroup $M$ of $S$ and an index $j$ with $1 \leq j \leq a$ such that $Y_j = P_{M,i}$. For every $i \in I$ let $\mathcal{M}_i$ be the set of those maximal subgroups $M$ of $S$ for which there exists an index $j$ with $1 \leq j \leq a$ such that $Y_j = P_{M,i}$. Define $\Omega_i = S \setminus (\bigcup_{M \in \mathcal{M}_i} M)$. Note that $\Omega_i$ has cardinality at least $\sigma - a$. Now for each index $j$ with $1 \leq j \leq n$, let $\Delta_j$ be a subset of $S$ of cardinality $\sigma - a$ with the property that $\Delta_i \subseteq \Omega_i$ whenever

$i \in I$. Consider the subset $\Gamma = \prod_{j=1}^{n} \Delta_j$ of $S^n$. Clearly $|\Gamma| = (\sigma - a)^n$. Since $\Gamma \cap (\bigcup_{j \leq a} Y_j) = \emptyset$, we must have $\Gamma \subseteq \bigcup_{j>a} Y_j$. Notice that $|M \cap \Gamma| \leq (\sigma - a)^{n-1}$ for any maximal subgroup of $S^n$ of diagonal type. This means that $\Gamma$ is a subset of no less than $\sigma - a$ maximal subgroups of diagonal type. Hence $\sigma - a \leq \tau - a$, which is exactly what we wanted. □

Let $S$ be a non-abelian finite simple group. Define $\delta = \delta(S)$ to be the largest positive integer $r$ such that $S^r$, the direct product of $r$ copies of $S$, can be generated by 2 elements. (The positive integer $\delta$ is well-defined. To see this, first note that it is known that every non-abelian finite simple group can be generated by 2 elements. Also, for any positive integer $d$, the group $S^r$ cannot be generated by $d$ elements whenever $r$ is larger than the number of $\mathrm{Aut}(S)$-orbits on the set of $d$-tuples generating $S$. This latter claim follows from the combination of the definition of a maximal subgroup of product type and the Pigeonhole Principle.) Let us denote $S^\delta$ by $G$. (Actually, $\delta$ is equal to the number of $\mathrm{Aut}(S)$-orbits on ordered pairs of generators for $S$, and for arbitrary elements $x = (x_1, \ldots, x_\delta)$ and $y = (y_1, \ldots, y_\delta)$ of $G$ we have that $G = \langle x, y \rangle$ if and only if the pairs $(x_i, y_i)$ are distinct representatives for these orbits for $i$ with $1 \leq i \leq \delta$.)

Consider $A = \mathrm{Aut}(G) \cong \mathrm{Aut}(S) \wr \mathrm{Sym}(\delta)$ and let $(x, y)$ be a fixed pair of generators for $G$ with $x = (x_1, \ldots, x_\delta)$ and $y = (y_1, \ldots, y_\delta)$, where the $x_i$'s and $y_i$'s are elements of $S$. Since $\langle x, y \rangle = G$, the elements $(x_1, y_1), \ldots, (x_\delta, y_\delta)$ form a set of representatives for the $\mathrm{Aut}(S)$-orbits of the set of generating pairs for $S$. From this it is easy to see that $G$ has the following relevant property: *($\mathcal{P}$) if $G = \langle \bar{x}, \bar{y} \rangle$, then there exists $a \in A$ with $(\bar{x}, \bar{y}) = (x^a, y^a)$.*

Now we can define a graph $\Gamma$ in which the set of vertices $V$ is the set of all $A$-conjugates of $x$ and two vertices $\bar{x}_1$, $\bar{x}_2$ are connected by an edge if and only if $G = \langle \bar{x}_1, \bar{x}_2 \rangle$. Note that $\Gamma$ is obtained from $\Gamma(G)$ just by removing all isolated vertices. By property ($\mathcal{P}$), the graph $\Gamma$ is vertex-transitive and edge-transitive. Let $\alpha = |A|$, $C = C_A(x)$, and $\gamma = |C|$. The number of vertices in $V$ is $\alpha/\gamma$, and the number of edges in $\Gamma$ is $\alpha/2$ (since the action of $A$ on the pairs of generators is regular and 2 in the denominator comes from the fact that the edges of $\Gamma$ are unoriented).

In the remainder of this section we wish to give an upper bound for $\omega(G)$ which is precisely the clique number of $\Gamma$.

We will use Corollary 4 of [4] which states that if $X$ is a clique and $Y$ a coclique (an empty subgraph) in a vertex-transitive graph on $m$ vertices, then $|X||Y| \leq m$.

We also need a definition. Let $s$ be an element of $S$ and let $\omega(s)$ be the number of indices $i$ with $1 \leq i \leq \delta$ such that $x_i$ and $s$ are $\mathrm{Aut}(S)$-conjugate. We have $\omega(s) = \rho(s)/|C_{\mathrm{Aut}(S)}(s)|$, where $\rho(s)$ is the number of elements $t$ in $S$ such that $\langle s, t \rangle = S$ (this is because for any $t$ with $\langle s, t \rangle = S$ there exists a unique index $i$ with $1 \leq i \leq \delta$ and a unique automorphism $a \in \mathrm{Aut}(S)$ such that $(s^a, t^a) = (x_i, y_i)$).

Now take $M$ to be a maximal subgroup of $S$ and put

$$Y_M = \{v = (z_1, \ldots, z_\delta) \in V \mid \pi_1(v) = z_1 \in M\},$$

where $\pi_1$ is the natural projection from $G$ to the first direct factor. Since $Y_M$ is a coclique in $\Gamma$ we have $\omega(G) \leq |V|/|Y_M|$.

For any $z \in M$ with $z \neq 1$ there exists a vertex $v_z$ in $V$ such that $\pi_1(v_z) = z$. (This follows from the corollary on page 745 of [8], which states that any non-trivial element of a finite almost simple group $G$ belongs to a pair of elements

generating at least the socle of $G$.) Other vertices $v$ with the property that $\pi_1(v) = z$ can be obtained by conjugating $v_z$ by automorphisms from the subgroup $\bar{A} \cong \mathrm{Aut}(S) \wr \mathrm{Sym}(\delta - 1)$ of $A$. So if we define $C_z$ to be $C_{\bar{A}}(v_z)$, then we obtain

$$\sum_{z \in M, z \neq 1} \frac{|\bar{A}|}{|C_z|} \leq |Y_M|.$$

This implies

$$\omega(G) \leq \frac{|V|}{\sum_{z \in M, z \neq 1} \frac{|\bar{A}|}{|C_z|}} = \left( \sum_{z \in M, z \neq 1} \frac{|\bar{A}||C|}{|A||C_z|} \right)^{-1}.$$

Clearly $|A|/|\bar{A}| = \delta |\mathrm{Aut}(S)|$. Now assume that $\{u_1, \ldots, u_l\}$ is a set of representatives for the orbits of the action of $\mathrm{Aut}(S)$ on $S \setminus \{1\}$. For $C = C_A(x)$ we have

$$C \cong \prod_{i=1}^{l} C_{\mathrm{Aut}(S)}(u_i) \wr \mathrm{Sym}(\omega(u_i)).$$

On the other hand, if $z \in u_j^{\mathrm{Aut}(S)}$, we have

$$C_z \cong \left( \prod_{i \neq j} C_{\mathrm{Aut}(S)}(u_i) \wr \mathrm{Sym}(\omega(u_i)) \right) \times \left( C_{\mathrm{Aut}(S)}(u_i) \wr \mathrm{Sym}(\omega(u_i) - 1) \right).$$

It follows that $|C|/|C_z| = |C_{\mathrm{Aut}(S)}(z)| \cdot \omega(z) = \rho(z)$ and

$$\omega(G) \leq \left( \sum_{z \in M, z \neq 1} \frac{\rho(z)}{|\mathrm{Aut}(S)|\delta} \right)^{-1}.$$

Note that $|\mathrm{Aut}(S)|\delta$ is the number of ordered pairs $(s,t)$ generating $S$, while $\sum_{z \in M, z \neq 1} \rho(z)$ is the number of ordered pairs $(s,t)$ generating $S$ such that $s \in M$. So if we define $P_M$ to be the conditional probability that $(s,t) \in M \times S$ given that $\langle s,t \rangle = S$, then $\omega(G) \leq 1/P_M$. We may also write $P_M$ in the form

$$P_M = \frac{P(\langle x,y \rangle = S \mid x \in M) \cdot P(x \in M)}{P(\langle x,y \rangle = S)} \geq P(\langle x,y \rangle = S \mid x \in M) \cdot \frac{|M|}{|S|},$$

where $Q_M = P(\langle x,y \rangle = S \mid x \in M)$ is the conditional probability that the ordered pair $(x,y)$ generates $S$ given that $x \in M$, where $P(x \in M) = |M|/|S|$ is the probability that $x \in M$ and where $P(\langle x,y \rangle = S)$ is the probability that the ordered pair $(x,y)$ generates $S$. Clearly, $\omega(G) \leq 1/P_M \leq |S : M|/Q_M$. We need a lower bound for $Q_M$. In what follows $m(S)$ denotes the minimal index of a proper subgroup in $S$.

**Proposition 3.2.** *Let $M \leq S$ with $|S : M| = m(S)$. Then $1 - O(m(S)^{-1/15}) \leq Q_M$. Moreover if $S = \mathrm{Alt}(n)$, then $1 - O(n^{-1}) \leq Q_M$.*

*Proof.* If $(m,s) \in M \times S$, then $\langle m,s \rangle \neq S$ if and only if $(m,s) \in (K \cap M) \times K$ for some maximal subgroup $K$ of $S$. This allows us to deduce

$$1 - \sum_K \frac{1}{|S : K||M : K \cap M|} \leq Q_M,$$

where $K$ runs through the set of maximal subgroups of $S$.

Now use the notation of Section 6 of [9]. There exist positive real numbers $\delta$ and $b$ with $\delta > 1$ such that the set $\mathcal{A}$ of maximal subgroups whose index is smaller than $b \cdot m(S)^\delta$ is known (and $|\mathcal{A}|$ is "small"). The values of $\delta$ and $b$ together with the description of $\mathcal{A}$ are given in [9] when $S$ is a simple group of Lie type. If $S = \mathrm{Alt}(n)$ and $n$ is large enough, then any subgroup of $\mathrm{Alt}(n)$ different from a point-stabilizer has index at least $n(n-1)/2$, so for any $\delta$ with $1 < \delta \leq 2$ there exists $b > 0$ with the property that any maximal subgroup of $\mathrm{Alt}(n)$ with index smaller than $b \cdot n^\delta$ is a point-stabilizer. By [9], we may take $\delta = 16/15$ if $S$ is a group of Lie type, and, by the remarks above, we may take $\delta = 2$ if $S$ is an alternating group. Let $\mathcal{B}$ be the set of those maximal subgroups of $S$ which do not belong to $\mathcal{A}$. Note that

$$\frac{1}{|S:K||M:K\cap M|} \leq \frac{m(S)}{|S:K|^2}.$$

We will make use of the identity

$$\sum_{K\in\mathcal{B}} |S:K|^{-2} = O(m(S)^{-\delta}),$$

which, for exceptional groups $S$ of Lie type, is found in line $-2$ of the proof of Lemma 6.7 in [9], and which, for classical groups $S$, follows from Theorem 3.1 of [10] by noting that we may replace 2 by $\delta$ since $\delta \leq 2$. This implies

$$\sum_{K\in\mathcal{B}} |S:K|^{-1}|M:K\cap M|^{-1} = O(m(S)^{-\delta+1}).$$

Hence

$$1 - \sum_{K\in\mathcal{A}} \frac{1}{|S:K||M:K\cap M|} - O(m(S)^{-\delta+1}) \leq Q_M.$$

Now let $\{K_1, \ldots, K_t\}$ be a set of representatives for the $S$-conjugacy classes of all members of $\mathcal{A}$. For every $i$ with $1 \leq i \leq t$ let $s_i$ be the number of $M$-orbits on the coset space $(S : K_i)$. Note (see the proof of Lemma 6.10 in [9]) that for every $i$ with $1 \leq i \leq t$ we have

$$\sum_{K\in K_i^G} \frac{1}{|S:K||M:K\cap M|} = \frac{s_i}{|S:K_i|} \leq \frac{s_i}{m(S)}.$$

We conclude that

$$1 - \sum_{i=1}^{t} \frac{s_i}{m(S)} - O(m(S)^{-\delta+1}) \leq Q_M. \tag{2}$$

We now have to show that $\sum_{i=1}^t s_i$ is "small". If $S$ is a group of Lie type, then, by [9], $t \leq 3$ and either $s_i \leq 3$ for all $i$ with $1 \leq i \leq t$ or $S = P\Omega_{2m}^{\pm}(q)$ in which case there exists a constant $c_1$ such that $s_i \leq c_1 q$ for all $i$ with $1 \leq i \leq t$ (see the last part of the proof of Lemma 6.7 in [9]). Finally, if $S = \mathrm{Alt}(n)$ and $n \neq 6$, then $t = 1$ and $s_1$ is the number of orbits of the point-stabilizer $M$ on the coset space $(S : K_1)$ where $K_1$ is another point-stabilizer. In this case $s_1 = 2$ since $\mathrm{Alt}(n)$ is 2-transitive. By these remarks and by inequality (2), we get

$$1 - O(m(S)^{-\delta+1}) \leq Q_M,$$

which is exactly what we wanted. □

By the inequality $\omega(G) \leq |S : M|/Q_M$ and by Proposition 3.2, we conclude that $\omega(G) \leq m(S) + O(m(S)^{14/15})$ if $S$ is a finite simple group of Lie type and $\omega(G) \leq m(S) + O(1)$ otherwise. Now let $S = \mathrm{Alt}(n)$. Then, by [12], we have $2^{n-2} \leq \sigma(S)$ unless $n = 7$ or 9. Hence, by Proposition 3.1, $2^{n-2} \leq \sigma(G)$ unless $n = 7$ or 9. From this it follows that $\omega(G)/\sigma(G) \leq (n + O(1))/2^{n-2}$. The proof of Theorem 1.2 is now complete.

## References


[1] Abdollahi, A.; Jafarian Amiri, S. M. Minimal coverings of completely reducible groups. *Publ. Math. Debrecen* **72**/1-2 (2008), 167-172. MR2376867 (2008k:20044)

[2] Aschbacher, M.; Guralnick, R. M. Some applications of the first cohomology group. *J. Algebra* **90** (1984), 446–460. MR760022 (86m:20060)

[3] Blackburn, S. Sets of permutations that generate the symmetric group pairwise. *J. Combin. Theory Ser. A* **113** (2006), no. 7, 1572-1581. MR2259081 (2007e:20005)

[4] Cameron, P. J.; Ku, C. Y. Intersecting families of permutations. *European J. Combin.* **24** (2003), no. 7, 881–890. MR2009400 (2004g:20003)

[5] Detomi, E.; Lucchini, A. Crowns and factorization of the probabilistic zeta function of a finite group. *J. Algebra* **265** (2003), no. 2, 651–668. MR1987022 (2004e:20119)

[6] Dye, R. H. Interrelations of symplectic and orthogonal groups in characteristic two. *J. Algebra* **59** (1979), no. 1, 202–221. MR541675 (81c:20028)

[7] Gaschütz, W. Praefrattinigruppen. *Arch. Math. (Basel)* **13** (1962), 418–426. MR0146262 (26:3784)

[8] Guralnick, R. M.; Kantor, W. M. Probabilistic generation of finite simple groups. Special issue in honor of Helmut Wielandt. *J. Algebra* **234** (2000), no. 2, 743–792. MR1800754 (2002f:20038)

[9] Liebeck, M. W.; Shalev, A. Simple groups, probabilistic methods, and a conjecture of Kantor and Lubotzky. *J. Algebra* **184** (1996), no. 1, 31–57. MR1402569 (97e:20106b)

[10] Liebeck, M. W.; Shalev, A. Classical groups, probabilistic methods, and the $(2, 3)$-generation problem. *Ann. of Math. (2)* **144** (1996), no. 1, 77–125. MR1405944 (97e:20106a)

[11] Lucchini, A.; Maróti, A. On finite simple groups and Kneser graphs, *J. Algebraic Combin.*, to appear.

[12] Maróti, A. Covering the symmetric groups with proper subgroups. *J. Combin. Theory Ser. A* **110** (2005), no. 1, 97–111. MR2128968 (2005m:20009)

[13] Robinson, D. J. S. A course in the theory of groups, Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1993. MR1261639 (94m:20001)

[14] Tomkinson, M. J. Groups as the union of proper subgroups. *Math. Scand.* **81** (1997), 191-198. MR1613772 (99g:20042)



Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Trieste 63, 35121 Padova, Italy

*E-mail address*: lucchini@math.unipd.it

Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda utca 13-15, H-1053, Budapest, Hungary

*E-mail address*: maroti@renyi.hu