TOEPLITZ AND HANKEL OPERATORS
AND DIXMIER TRACES ON THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. We compute the Dixmier trace of pseudo-Toeplitz operators on the Fock space. As an application we find a formula for the Dixmier trace of the product of commutators of Toeplitz operators on the Hardy and weighted Bergman spaces on the unit ball of $\mathbb{C}^d$. This generalizes an earlier work of Helton-Howe for the usual trace of the anti-symmetrization of Toeplitz operators.

1. Introduction

In the present paper we will study the Dixmier trace of a class of Toeplitz and Hankel operators on the Hardy and weighted Bergman spaces on the unit ball of $\mathbb{C}^d$. We give a brief account of our problem and explain some motivations. Consider the Bergman space $L^2_a(D)$ of holomorphic functions on the unit disk $D$ in the complex plane. For a bounded function $f$ let $T_f$ be the Toeplitz operator on $L^2_a(D)$. It is well-known that for a holomorphic function $f$ in a neighborhood of $D$ the commutator $[T^{*}_{f}, T_f]$ is of trace class and the trace is given by the square of the Dirichlet norm of $f$,
\[ \text{tr}[T^{*}_{f}, T_f] = \int_{D} |f'(z)|^2 \, dm(z), \]
which is one of the best known Möbius invariant integrals. This formula actually holds for Toeplitz operators on any Bergman space on a bounded domain with the area measure replaced by any reasonable measure [2]. There is a significant difference between Toeplitz operators on the unit disk and on the unit ball $B = B^d$ in $\mathbb{C}^d$, $d > 1$. Let $L^p$ be the Schatten - von Neumann class of $p$-summable operators. The commutator $[T^{*}_{f}, T_f]$ on the weighted Bergman space, say for holomorphic functions $f$ in a neighborhood of the closed unit disk, is in the Schatten - von Neumann class $L^p$, for $p > \frac{1}{2}$, and is zero if it is in $L^p$, for $p \leq \frac{1}{2}$, $\frac{1}{2}$ being called the cut-off; on the Hardy space $[T^{*}_{f}, T_f]$ can be in any Schatten - von Neumann class...
in $L^p$, for $p > 0$. See [14] and [16] for the case of a Hardy space and [1] for the case of a weighted Bergman space. However for $d > 1$, it is in $L^p$ for $p > d$, with $p = d$ being the cut-off, both on the weighted Bergman spaces and on the Hardy space. Thus no trace formula was expected for the commutators. Nevertheless Helton and Howe [9] were able to find an analogue of the previous formula. They showed, for smooth functions $f_1, \cdots, f_{2d}$ on the closed unit ball, that the anti-symmetrization $[T_{f_1}, T_{f_2}, \cdots, T_{f_{2d}}]$ of the $2d$ operators $T_{f_1}, T_{f_2}, \cdots, T_{f_{2d}}$ is of trace class and found that

$$\text{tr}[T_{f_1}, T_{f_2}, \cdots, T_{f_{2d}}] = \int_B df_1 \wedge df_2 \cdots \wedge df_{2d}.$$ 

On the other hand, we observe that $[T_f, T_g]$ is, for smooth functions $f$ and $g$, in the Macaev class $L^{d, \infty}$ (which is an analogue of the Lorentz space $L^{d, \infty}$); thus the product of $d$ such commutators $[T_{f_1}, T_{g_1}][T_{f_2}, T_{g_2}] \cdots [T_{f_d}, T_{g_d}]$ is in $L^{1, \infty}$ and hence has a Dixmier trace. One of the goals of the present paper is to prove the following formula for the Dixmier trace of this product of commutators:

$$\text{tr}_\omega[T_{f_1}, T_{g_1}] \cdots [T_{f_d}, T_{g_d}] = \frac{1}{d!} \int_S \{f_1, g_1\} \cdots \{f_d, g_d\}.$$ 

Here $\{f, g\}$ is the Poisson bracket of $f$ and $g$; its restriction to the boundary $S$ of $B$ depends only on the boundary values of $f$ and $g$ and can be expressed in terms of the boundary $C\mathcal{R}$ operators. This can be viewed as a generalization of the Helton-Howe theorem. We apply our result also to Hankel operators and obtain a formula for the Dixmier trace of the $d$-th power of the square modulus of the Hankel operators $H^*_f H_f$ for holomorphic functions $f$. Namely we have

$$\text{tr}_\omega |H_f|^d = \text{tr}_\omega([T_f, T_f]^d) = \frac{1}{d!} \int_S (|\nabla f|^2 - |Rf|^2)^d.$$ 

This provides a boundary $L^{d, \infty}$ result for the Schatten - von Neumann $L^p$ ($p > d$) properties of the square modulus of the Hankel operators (see [3], [5] and [19]). We mention also, besides the above results on exact norms, that there are exact formulas proved by Janson, Upmeier and Wallstén [12] on the Schatten - von Neumann $L^p$-norm of the Hankel operators on the unit circle for $p = 2, 4, 6$, and by Peetre [13] on $L^d$-norms of Hankel forms on Fock spaces [11].

There has been an intensive study of Dixmier trace and residue trace of pseudo-differential operators, mostly on compact manifolds where the analysis is relatively easier; see e.g. [4] and [15] and the references therein. Thus the Toeplitz operators on Hardy spaces on the boundary of a bounded strictly pseudo-convex domain can be treated using the techniques developed there. The Hankel and Toeplitz operators on Bergman spaces, generally speaking, behave rather differently from those on a Hardy space, and the result of Howe [10] roughly speaking proves that Toeplitz operators of certain classes can be treated similarly to those in the Hardy space case (also called the de Monvel - Howe compactification [6]). Our result can thus be viewed as a generalization of the compactification to weighted Bergman spaces and an application of the ideas in [7] of computing Dixmier traces. In particular our Theorem 4.1 is closely related to the results in [4], where the residue trace of pseudo-differential operators of a certain class is computed; here we use the Weyl transforms and they differ from pseudo-differential operators of lower order, so that Theorem 4.1 can also be obtained from [4] provided one proves that the lower order terms are of trace class.
In another paper we will study the Dixmier trace for Toeplitz operators on a general strongly pseudo-convex domain. One of the authors, G. Zhang, would also like to thank Professor Richard Rochberg and Professor Harald Upmeier for introducing him to the work of Connes [8, Chapter IV.2] on Dixmier traces of pseudo-differential operators.

2. Toeplitz operators on Bergman spaces and their realization as pseudo-Toeplitz operators on Fock spaces

Let \( dm(z) \) be the Lebesgue measure on \( \mathbb{C}^d \) and consider the weighted measure
\[
d\mu_\nu = C_\nu (1 - |z|^2)^{\nu - d - 1} dm(z),
\]
where \( C_\nu \) is the normalizing constant to make \( d\mu_\nu \) a probability measure and \( \nu > d \).

We let \( H_\nu \) be the corresponding Bergman space of holomorphic functions on \( B \). We will also consider the Hardy space of square integrable functions on \( S \) which are holomorphic on \( B \). This can be viewed as the analytic continuation of \( H_\nu \) at \( \nu = d \).

Thus we assume throughout this paper that \( \nu \geq d \).

Let \( f \) be a bounded smooth function on \( \overline{B} \), the closure of \( B \). The Toeplitz operator \( T_f \) on \( H_\nu \) with symbol \( f \) is defined by
\[
T_f g = P(fg),
\]
where \( P \) is the Bergman or the Hardy projection for \( \nu > d \) and \( \nu = d \), respectively.

As was shown by Howe [10] there is a more flexible and effective way of studying the spectral properties of Toeplitz operators with smooth symbol, by using the theories of representations of the Heisenberg group and of pseudo-differential operators. We will adopt that approach. We will be very brief and refer to [10] and [18, Chapter XII] for details. So let \( H_n = \mathbb{C}^d \times T \) be the Heisenberg group as in loc. cit. The Heisenberg group has an irreducible representation, \( \rho \), on the Fock space \( F \) consisting of entire functions \( f \) on \( \mathbb{C}^d \) such that
\[
\int_{\mathbb{C}^d} |f(z)|^2 e^{-\pi |z|^2} dm(z) < \infty.
\]
The action of the Heisenberg group is explicitly given as follows. For \( w \in \mathbb{C}^d \) viewed as an element in \( H_d \),
\[
\rho(w)f(w') = e^{-\pi/2|w|^2 + \pi w' \cdot \overline{w}} f(w' - w),
\]
where \( w' \cdot \overline{w} \) is the Hermitian inner product on \( \mathbb{C}^d \). The action of \( T \) is by rotation.

Identifying the Lie algebra \( \mathfrak{h} \) of the Heisenberg group with \( \mathbb{R}^{2n} \oplus \mathbb{R} \) and thus \( \mathbb{R}^{2n} \) with a subspace of the Lie algebra, we get an action of \( \mathbb{R}^{2n} \) as holomorphic differential operators on \( F \), which extends from \( \mathfrak{h} \) to the whole enveloping algebra \( \mathfrak{U}(\mathfrak{h}) \) and which will also be denoted by \( \rho \). In particular, taking the basis elements \( \partial_j = \partial/\partial w_j \) and \( \overline{\partial}_j = \partial/\partial \overline{w}_j \) of \( \mathbb{R}^{2n} \) we have
\[
\rho(\partial_j)f(w) = -\partial_j f(w), \quad \rho(\overline{\partial}_j)f(w) = \pi w_j f(w).
\]

Following the notation in [10], let \( \Delta \in \mathfrak{U}(\mathfrak{h}) \) be the element
\[
\Delta = \frac{1}{2}(\partial_j \cdot \overline{\partial}_j + \overline{\partial}_j \cdot \partial_j).
\]
Then $\rho(\Delta)$ acts on $F$ as a diagonal self-adjoint operator \[10\], under the orthogonal basis $\{w^\alpha, \alpha = (\alpha_1, \cdots, \alpha_d}\}$, viz.

$$\rho(\Delta)w^\alpha = -\pi(|\alpha| + \frac{d}{2})w^\alpha.$$  

Let $F(z)$ be a function on $\mathbb{C}^d$ (viewed as a function on the Heisenberg group). The Weyl transform $\rho(F)$ of $F$ is defined by

$$\rho(F) = \int_{\mathbb{C}^d} F(w)\rho(w)dm(w).$$

To understand the operator theoretic properties of $\rho(F)$ we will need the Fourier transform of $F$. Let $\hat{F}$ be the (symplectic-) Fourier transform of $F$,  

$$\hat{F}(w') = 2^{-d} \int_{\mathbb{C}^d} F(w)e^{\pi i \text{Im}(w \cdot w')}dm(w),$$

and let $F * G$ be the symplectic convolution

$$F * G(w) = \int_{\mathbb{C}^d} F(z)G(w - z)e^{\pi i \text{Im}(w \cdot z)}dm(z).$$

We recall that

$$\overline{\hat{F} * \hat{G}} = \hat{F} * \hat{G}$$

and

$$\rho(F)\rho(G) = \rho(F * G)$$

for an appropriate class of functions. A well-known theorem of Calderón-Vaillancourt states that if $\hat{F}$ and all its derivatives are bounded, then $\rho(F)$ can be defined as a bounded operator on $F$.

We will need a finer class of symbols introduced by Howe, corresponding to the so-called pseudo-Toeplitz operators. Let

$$\mathcal{P}T(m, \mu) = \{F \in S^*(\mathbb{C}^d) : |\partial^\alpha \partial^\beta \hat{F}| \leq C_{\alpha \beta}(1 + |w|)^{m-\mu(|\alpha|+|\beta|)}\}$$

and

$$\mathcal{P}T_{\text{rad}}(m, \mu) = \{F \in \mathcal{P}T(m, \mu) : \hat{F} = (1 - g(|w|))\psi\left(\frac{w}{|w|}\right)|w|^m + D_1, D_1 \in \mathcal{P}T(m - \mu, \mu)\}.$$

Here $g$ is a smooth function on $\mathbb{R}$ such that $0 \leq g(t) \leq 1$ on $\mathbb{R}$, $g(t) = 0$ for $|t| \geq 2$ and $g(t) = 1$ for $0 \leq t \leq 1$.

For $F \in \mathcal{P}T_{\text{rad}}(m, \mu)$ we will call

$$\sigma_m(F) := \psi\left(\frac{w}{|w|}\right)|w|^m$$

its principal symbol. It can be obtained, up to the factor $|w|^m$, by

$$\psi(w) = \lim_{t \to \infty} t^{-m}\hat{F}(tw), \quad w \in S.$$

Following Howe \[10\], we will call $\rho(F), F \in \mathcal{P}T(m, \mu)$, a pseudo-Toeplitz operator of order $m$ and smoothness $\mu$. One has \[10\] Lemma 4.2.2

$$F \in \mathcal{P}T(m_1, \mu), G \in \mathcal{P}T(m_2, \mu) \implies F * G \in \mathcal{P}T(m_1 + m_2, \mu).$$
We will realize the Toeplitz operators $T_f$ on $H_\nu$ for $f$ on $B$ (or on $S$ for the Hardy space) as Weyl transforms $\rho(F)$ of certain symbols $F$ on $\mathbb{C}^d$. First we notice that

$$e_\beta := \left(\frac{(\nu_{|\beta|})}{\beta!}\right)^{\frac{1}{2}} z^\beta$$

form an orthonormal basis of $H_\nu$, and so do

$$E_\beta := \left(\frac{1}{\pi|\beta|3!}\right)^{\frac{1}{2}} w^\beta$$

for $F$. (Here $(\nu)_j := \nu(\nu + 1)\ldots(\nu + j - 1)$ is the usual Pochhammer symbol.)

Thus the map

$$(2.5) \quad U : e_\beta \rightarrow E_\beta$$

is a unitary operator. First we will find the action of the elementary Toeplitz operators $T_{\alpha}$ under the intertwining map $U$.

**Lemma 2.1.** The operator $UT_{\alpha}U^*$ on $F$ is given by

$$(2.6) \quad UT_{\alpha}U^* = \rho(z)^{\alpha} \rho \left(\pi|\alpha| \left(\nu - \frac{d}{2} - \frac{1}{\pi}\Delta|\alpha|\right)\right)^{-1/2}.$$ 

This can be proved by direct computation. Indeed we have

$$T_{\alpha}e_\beta = \left(\frac{(\beta)_n}{(\nu + |\beta|)|\alpha|}\right)^{\frac{1}{2}} e_{\beta + \alpha},$$

and the right hand side $2.6$ can be easily computed by $2.1$ and $2.2$.

By using the previous lemma we have then the following result, which was proved by Howe [10] Proposition 4.2.3] in the case when $\nu = d + 1$; the general case of $\nu \geq d$ is essentially the same.

** Proposition 2.2.** Let $f \in C^\infty(S)$ and let $\tilde{f}$ be a $C^\infty$ extension to $B$ and $T_f$ the Toeplitz operator on $H_\nu$. Then under the unitary equivalence of $H_\nu$ and the Fock space $F$ on $\mathbb{C}^d$, the Toeplitz operators are pseudo-Toeplitz operators with radial asymptotic limits $PT_{\alpha}$. More precisely, there exists $F \in PT_{\alpha}(0,1)$ such that $UT_fU^* = \rho(F)$, and $f(\zeta) = \lim_{n \rightarrow \infty} \tilde{F}(t\zeta)$ for each $\zeta \in \mathbb{C}$.

3. **Schatten - von Neumann properties of pseudo-Toeplitz operators**

Recall that the Schatten - von Neumann class $L^p$, $p \geq 1$, consists of compact operators $T$ such that the eigenvalues $\{\mu_n\}$ of $|T| = (T^*T)^{\frac{1}{2}}$ are $p$-summable, $\sum \mu_n^p < \infty$. In particular $L^2$ is the Hilbert-Schmidt class, $L^1$ the trace class and $L^\infty$ are the compact operators. For $1 < p < \infty$, $1 \leq q \leq \infty$, the Macaev class $L^{p,q}$ is obtained by the real interpolation between $L^1$ and $L^\infty$. However, we will need the Macaev class $L^{1,\infty}$, which consists of all compact operators such that, if $\mu_1 \geq \mu_2 \geq \ldots$,

$$\sum_{n=1}^N \mu_n = O(\log N).$$

There exists a linear functional on the space $L^{1,\infty}$ that resembles the usual trace, called the Dixmier trace. Its definition is rather involved and we refer to [8].

1 Sometimes this ideal is denoted by $\mathcal{L}_1$, the notation $L^{1,\infty}$ being reserved for the (smaller) class of operators for which $\mu_n = O(1/n)$. Our notation follows Connes’ book [6].
Chapter IV [for details. Let $C_0(\mathbb{R}_+)$ be the space of bounded continuous functions on $\mathbb{R}_+$ and $C_0(\mathbb{R}_+^+)$ the subspace of functions vanishing at $\infty$. Let $\omega$ be a positive linear functional on the quotient space $C_0(\mathbb{R}_+^+)/C_0(\mathbb{R}_+)$ such that $\omega(1) = 1$. For a positive compact operator $T \in \mathcal{L}^{1,\infty}$ with eigenvalues $\{\mu_n\}$, extend $\mu_n$ to a step function on $\mathbb{R}_+$ and let $M_T(\lambda)$ be its Cesàro mean, which is a bounded continuous function on $\mathbb{R}_+^+$. The Dixmier trace of $T$ is then defined by

$$\text{tr}_\omega T = \omega(M_T).$$

It is then extended to all of $\mathcal{L}^{1,\infty}$ by linearity. In particular it is bounded and vanishes on trace class operators. The fact that we will need is that

$$\text{tr}_\omega T = \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T)$$

if $T$ is a positive operator and if the right hand side exists.

**Lemma 3.1.** For any $c \geq 0$ the operator $(c - \rho(\Delta))^{-d} = \rho(c\delta_0 - \Delta)^{-d}$ is in the Macaev class $\mathcal{L}^{1,\infty}$.

**Proof.** It follows from (2.2) that the eigenvalues of $(c - \rho(\Delta))^d$ are $\{c + \pi(m + \frac{d}{2})\}^d$, $m = 0, 1, \cdots$, each of multiplicity $d_m := \dim\{w^\alpha, |\alpha| = m\} = (d+m-1) \approx m^{d-1}$. The partial sums thus satisfy

$$\sum_{m \leq N} (c + \pi(m + \frac{d}{2}))^{-d} \approx \frac{1}{(d-1)!} \sum_{m \leq N} (c + \pi(m + \frac{d}{2}))^{-d} m^{d-1} \approx \frac{1}{(d-1)!} \log N,$$

completing the proof. \qed

**Proposition 3.2.** Let $F \in \mathcal{PT}(-2d,1)$. Then the Weyl transform $\rho(F)$ is in the Macaev class $\mathcal{L}^{1,\infty}$.

**Proof.** By (3.5.6) in [10],

$$(3.1) \quad \hat{\Delta} = -\frac{\pi^2}{4} |w|^2,$$

so $-\Delta \in \mathcal{PT}(2,1)$, whence by (2.4) $(-\Delta)^{d} \in \mathcal{PT}(2d,1)$ and $(-\Delta)^{d*}F \in \mathcal{PT}(0,1)$. By the Calderón-Vaillancourt theorem [10 Theorem 3.1.3], the corresponding Weyl transform, $\rho(-\Delta)^d\rho(F)$, is bounded. Hence by the previous lemma, $\rho(F) \in \mathcal{L}^{1,\infty}$, since the Macaev class $\mathcal{L}^{1,\infty}$ is an ideal. \qed

4. **Dixmier trace formula for Toeplitz operators**

**Theorem 4.1.** Let $F \in \mathcal{PT}_{rad}(-2d,1)$ with the principal symbol $\sigma_{-2d}(\hat{F})$ as defined in (2.3). Then the Dixmier trace $\text{tr}_\omega \rho(F)$ is independent of $\omega$ and is given by

$$\text{tr}_\omega \rho(F) = \frac{\pi^d}{4^d d!} \int_S \sigma_{-2d}(F)(w),$$

where $\int_S$ is the normalized integral over the unit sphere.
Proof. The proof is quite similar to that of Connes [7] for pseudo-differential operators on compact manifolds. Namely, by [10, Theorem 4.2.5] and the definition of \( \mathcal{PT}_{\text{rad}} \), the Dixmier trace of \( \rho(F) \) depends only on the leading symbol of \( \sigma_{-2d}(\hat{F}) \) and defines a positive measure on the unit sphere \( S \) in \( \mathbb{C}^d \). By the unitary invariance of \( \rho(F) \), the measure has to be a constant multiple of the area measure. To find the constant we note that the symbol of \( c\delta_0 - \Delta, c > 0 \), is absolutely elliptic in the sense of (4.2.20) in [10], and thus by pp. 246–247 in [10], we can construct \( F_0 \in \mathcal{PT}_{\text{rad}}(-2d,1) \) such that \( \rho(F_0) = (c - \rho(\Delta))^{-d} \). The eigenvalue of \( \rho(F_0) \) on the space of all \( m \)-homogeneous polynomials is, by the proof of Lemma 3.1, 

\[
\frac{1}{(c + \pi(m + \frac{d}{2}))^d}.
\]

Its Dixmier trace exists and is (noticing that the dimension of the space of homogeneous polynomials of degree \( m \leq N \) is \( \approx N^d \))

\[
\text{tr}_\omega \rho(F_0) = \frac{1}{\pi^d d!}.
\]

On the other hand, the principal symbol \( \sigma_{-2d}(F_0) \) is the constant function \( (4/\pi^2)^d |w|^{-2d} \) by the definition (cf. (3.1)), whose integration over the sphere is \( (4/\pi^2)^d \). This completes the proof. \( \square \)

To apply our result to Toeplitz operators we need to introduce some more notation. We let \( \partial_b^j = \partial_j - \bar{z}_j R, \bar{\partial}_b^j = \bar{\partial}_j - z_j \bar{R} \) be the boundary Cauchy-Riemann operators [17], where \( R = \sum_{j=1}^d z_j \partial_j \) is the holomorphic radial derivative. As vector fields they are linearly dependent, to wit,

\[
\sum_{j=1}^d z_j \partial_b^j = 0, \quad \sum_{j=1}^d \bar{z}_j \bar{\partial}_b^j = 0.
\]

**Definition 4.2.** We define a bracket \( \{f,g\}_b \) for smooth functions \( f \) and \( g \) on \( S \) by

\[
\{f,g\}_b := \sum_{j=1}^d (\partial_b^j f \bar{\partial}_b^j g - \bar{\partial}_b^j f \partial_b^j g)
\]

and call it the boundary Poisson bracket.

**Lemma 4.3.** Let \( F \) and \( G \) be two functions in \( \mathcal{PT}_{\text{rad}}(0,\mu) \) with principal symbols

\[
\sigma_0(F)(z) = f \left( \frac{z}{|z|^2} \right), \quad \sigma_0(G)(z) = g \left( \frac{z}{|z|^2} \right)
\]

for \( f \) and \( g \) in \( C^\infty(S) \). Then the principal symbol of \( F \ast G - G \ast F \) is given by

\[
\sigma_{-2}(F \ast G - G \ast F)(z) = \frac{4}{\pi} \{f,g\}_b \left( \frac{z}{|z|^2} \right) |z|^{-2}.
\]
Proof. By the general result for the symbol calculus for pseudo-Toeplitz operators (cf. (2.2.5) in [10]), we have \( F * G - G * F \in \mathcal{PT}_{rad}(-2\mu, \mu) \) with the principal symbol
\[
\sigma_{-2}(F * G - G * F)(z) = \frac{4}{\pi} \{ \sigma_0(F), \sigma_0(G) \}(z),
\]
where \( \{ \cdot, \cdot \} \) is the ordinary Poisson bracket in complex coordinates
\[
\{ \Psi, \Phi \} := \sum_{j=1}^{d} (\partial_j \Psi \overline{\partial_j \Phi} - \partial_j \Phi \overline{\partial_j \Psi}).
\]
The function \( \sigma_{-2}(F * G - G * F)(z) \) is positive homogeneous of degree \(-2\). We need only to compute it for \( z \in S \). Defining the Reeb vector field \( E \) and the outward normal vector field \( N \) in terms of the radial derivative \( R \),
\[
E := \frac{1}{2}(R - R), \quad N := R + R,
\]
we can write
\[
R = -E + \frac{N}{2},
\]
Note that \( E \) is well-defined on \( S \). The vector field \( \partial_j^b + z_j E \) is thus a well-defined vector field on \( S \), and for any function \( \Phi(z) = \phi(\frac{z}{|z|}) \) we have
\[
\partial_j \Phi(z) = (\partial_j^b + z_j R) \Phi(z) = (\partial_j^b - z_j E + \frac{z_j}{2} N) \Phi(z) = (\partial_j^b - z_j E) \phi(z),
\]
since \( N \Phi(z) = 0 \) by homogeneity. Similarly \( \overline{\partial_j} \Phi = (\overline{\partial_j}^b + z_j E) \phi \) on \( S \). From this it follows that for \( z \in S \),
\[
\{ \sigma_0(F), \sigma_0(G) \}(z) = \sum_{j=1}^{d} \left( (\partial_j^b f(z) - z_j Ef(z)) (\overline{\partial_j^b} g(z) + z_j Eg(z)) - (\overline{\partial_j^b} f(z) - z_j Ef(z)) (\partial_j^b g(z) + z_j Eg(z)) \right) = \{ f, g \}_b.
\]
by using (4.1). \( \square \)

Theorem 4.4. Let \( f_1, g_1, \ldots, f_d, g_d \) be smooth functions on \( S \), \( \tilde{f}_1, \tilde{g}_1, \ldots, \tilde{f}_d, \tilde{g}_d \) their smooth extensions to \( B \) and \( T_{\tilde{f}_1}, T_{\tilde{g}_1}, \ldots, T_{\tilde{f}_d}, T_{\tilde{g}_d} \) the associated Toeplitz operators on \( \mathcal{H}_\nu \), for \( \nu \geq d \). Then the product \( \prod_{j=1}^{d}[T_{\tilde{f}_j}, T_{\tilde{g}_j}] \) is in the Macaev class and its Dixmier trace is given by
\[
\text{tr}_\omega \prod_{j=1}^{d}[T_{\tilde{f}_j}, T_{\tilde{g}_j}] = \frac{1}{d!} \int_S \prod_{j=1}^{d}\{ f_j, g_j \}_b.
\]

Proof. The proof is straightforward from the preceding lemma, formula (2.2.5) in [10] and Theorem 4.1. \( \square \)

We apply our result to Hankel operators with anti-holomorphic symbols. Let \( f \) be a holomorphic function in a neighborhood of \( B \) and \( H_f g = (I - P)\tilde{f}_g \), \( g \in \mathcal{H}_\nu \) the Hankel operator. Then
\[
[T_{\tilde{f}}, T_{\tilde{f}}] = [T_{\tilde{f}}^*, T_{\tilde{f}}] = |H_f|^2 = H_f^* H_f.
Corollary 4.5. Let \( f \) be as above. Then the Hankel operator is in \( \mathcal{L}^{2d, \infty} \); equivalently the commutator \([T_f, T_f]\) is in \( \mathcal{L}^{d, \infty} \) and we have

\[
\text{tr}_\omega |H_f|^2 = \text{tr}_\omega ([T_f, T_f]^d) = \frac{1}{d!} \int_S (|\nabla f|^2 - |Rf|^2)^d.
\]

Notice that \( H_f \) is in the Schatten class \( \mathcal{L}^p \) for \( p > 2d \) and that its Schatten norm is

\[
||H_f||_p \approx \int_B (1 - |z|^2)^p (|\nabla f|^2 - |Rf|^2) \overline{r} \, dm(z);
\]

see [3] and [19] for the Bergman space case \( (\nu = d + 1) \) and the Hardy space case \( (\nu = d) \). Our resulting formula thus provides a limiting result of the above estimates, and it is interesting to note that the estimate has an equality as its limit for \( p \to 2d \).

References


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