PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 11, November 2009, Pages 3669–3678
S 0002-9939(09)09331-9
Article electronically published on June 16, 2009

TOEPLITZ AND HANKEL OPERATORS AND DIXMIER TRACES ON THE UNIT BALL OF $\mathbb{C}^n$

MIROSLAV ENGLIŠ, KUNYU GUO, AND GENKAI ZHANG

(Communicated by Marius Junge)

ABSTRACT. We compute the Dixmier trace of pseudo-Toeplitz operators on the Fock space. As an application we find a formula for the Dixmier trace of the product of commutators of Toeplitz operators on the Hardy and weighted Bergman spaces on the unit ball of $\mathbb{C}^d$. This generalizes an earlier work of Helton-Howe for the usual trace of the anti-symmetrization of Toeplitz operators.

1. INTRODUCTION

In the present paper we will study the Dixmier trace of a class of Toeplitz and Hankel operators on the Hardy and weighted Bergman spaces on the unit ball of $\mathbb{C}^d$. We give a brief account of our problem and explain some motivations. Consider the Bergman space $L^2_a(D)$ of holomorphic functions on the unit disk D in the complex plane. For a bounded function f let T_f be the Toeplitz operator on $L^2_a(D)$. It is well-known that for a holomorphic function f in a neighborhood of D the commutator $[T_f^*, T_f]$ is of trace class and the trace is given by the square of the Dirichlet norm of f,

$$\operatorname{tr}[T_f^*, T_f] = \int_D |f'(z)|^2 \, dm(z),$$

which is one of the best known Möbius invariant integrals. This formula actually holds for Toeplitz operators on any Bergman space on a bounded domain with the area measure replaced by any reasonable measure [2]. There is a significant difference between Toeplitz operators on the unit disk and on the unit ball $B = B^d$ in $\mathbb{C}^d$, $d > 1$. Let $\mathcal{L}^p$ be the Schatten - von Neumann class of p-summable operators. The commutator $[T_f^*, T_f]$ on the weighted Bergman space, say for holomorphic functions f in a neighborhood of the closed unit disk, is in the Schatten - von Neumann class $\mathcal{L}^p$, for $p > \frac{1}{2}$, and is zero if it is in $\mathcal{L}^p$, for $p \leq \frac{1}{2}$, $\frac{1}{2}$ being called the cut-off; on the Hardy space $[T_f^*, T_f]$ can be in any Schatten - von Neumann class

Received by the editors February 19, 2007, and, in revised form, July 12, 2007.

2000 *Mathematics Subject Classification.* Primary 32A36; Secondary 47B35, 47B06.

Key words and phrases. Schatten - von Neumann classes, Macaev classes, trace, Dixmier trace, Toeplitz operators, Hankel operators, pseudo-Toeplitz operators, pseudo-differential operators, boundary CR operators, invariant Banach spaces.

The research of the first author was supported by GA ČR grant No. 201/06/128 and AV ČR Institutional Research Plan No. AV0Z10190503, of the second author by NSFC(10525106) and NKBRPC(2006CB805905), and of the third author by the Swedish Science Council (VR) and SIDA-Swedish Research Links.

$\mathcal{L}^p$, for $p > 0$. See [14] and [16] for the case of a Hardy space and [1] for the case of a weighted Bergman space. However for $d > 1$, it is in $\mathcal{L}^p$ for $p > d$, with $p = d$ being the cut-off, both on the weighted Bergman spaces and on the Hardy space. Thus no trace formula was expected for the commutators. Nevertheless Helton and Howe [9] were able to find an analogue of the previous formula. They showed, for smooth functions $f_1, \cdots, f_{2d}$ on the closed unit ball, that the anti-symmetrization $[T_{f_1}, T_{f_2}, \cdots, T_{f_{2d}}]$ of the $2d$ operators $T_{f_1}, T_{f_2}, \cdots, T_{f_{2d}}$ is of trace class and found that

$$\operatorname{tr}[T_{f_1}, T_{f_2}, \cdots, T_{f_{2d}}] = \int_B df_1 \wedge df_2 \cdots \wedge df_{2d}.$$

On the other hand, we observe that $[T_f, T_g]$ is, for smooth functions f and g, in the Macaev class $\mathcal{L}^{d,\infty}$ (which is an analogue of the Lorentz space $L^{d,\infty}$); thus the product of d such commutators $[T_{f_1}, T_{g_1}][T_{f_2}, T_{g_2}] \cdots [T_{f_d}, T_{g_d}]$ is in $\mathcal{L}^{1,\infty}$ and hence has a Dixmier trace. One of the goals of the present paper is to prove the following formula for the Dixmier trace of this product of commutators:

$$\operatorname{tr}_\omega [T_{f_1}, T_{g_1}] \cdots [T_{f_d}, T_{g_d}] = \frac{1}{d!} \int_S \{f_1, g_1\} \cdots \{f_d, g_d\}.$$

Here $\{f, g\}$ is the Poisson bracket of f and g; its restriction to the boundary S of B depends only on the boundary values of f and g and can be expressed in terms of the boundary CR operators. This can be viewed as a generalization of the Helton-Howe theorem. We apply our result also to Hankel operators and obtain a formula for the Dixmier trace of the d-th power of the square modulus of the Hankel operators $H_{\overline{f}}^* H_{\overline{f}}$ for holomorphic functions f. Namely we have

$$\operatorname{tr}_\omega |H_{\bar{f}}|^{2d} = \operatorname{tr}_\omega([T_{\bar{f}}, T_f]^d) = \frac{1}{d!} \int_S (|\nabla f|^2 - |Rf|^2)^d.$$

This provides a boundary $\mathcal{L}^{d,\infty}$ result for the Schatten - von Neumann $\mathcal{L}^p$ $(p > d)$ properties of the square modulus of the Hankel operators (see [3], [5] and [19]). We mention also, besides the above results on exact norms, that there are exact formulas proved by Janson, Upmeier and Wallstén [12] on the Schatten - von Neumann $\mathcal{L}^p$-norm of the Hankel operators on the unit circle for $p = 2, 4, 6$, and by Peetre [13] on $\mathcal{L}^4$-norms of Hankel forms on Fock spaces [11].

There has been an intensive study of Dixmier trace and residue trace of pseudo-differential operators, mostly on compact manifolds where the analysis is relatively easier; see e.g. [4] and [15] and the references therein. Thus the Toeplitz operators on Hardy spaces on the boundary of a bounded strictly pseudo-convex domain can be treated using the techniques developed there. The Hankel and Toeplitz operators on Bergman spaces, generally speaking, behave rather differently from those on a Hardy space, and the result of Howe [10] roughly speaking proves that Toeplitz operators of certain classes can be treated similarly to those in the Hardy space case (also called the de Monvel - Howe compactification [6]). Our result can thus be viewed as a generalization of the compactification to weighted Bergman spaces and an application of the ideas in [7] of computing Dixmier traces. In particular our Theorem 4.1 is closely related to the results in [4], where the residue trace of pseudo-differential operators of a certain class is computed; here we use the Weyl transforms and they differ from pseudo-differential operators of lower order, so that Theorem 4.1 can also be obtained from [4] provided one proves that the lower order terms are of trace class.

In another paper we will study the Dixmier trace for Toeplitz operators on a general strongly pseudo-convex domain. One of the authors, G. Zhang, would also like to thank Professor Richard Rochberg and Professor Harald Upmeier for introducing him to the work of Connes [8, Chapter IV.2] on Dixmier traces of pseudo-differential operators.

2. Toeplitz operators on Bergman spaces and their realization as pseudo-Toeplitz operators on Fock spaces

Let $dm(z)$ be the Lebesgue measure on $\mathbb{C}^d$ and consider the weighted measure

$$d\mu_\nu = C_\nu(1-|z|^2)^{\nu-d-1}dm(z),$$

where C_ν is the normalizing constant to make $d\mu_\nu$ a probability measure and $\nu > d$. We let $\mathcal{H}_\nu$ be the corresponding Bergman space of holomorphic functions on B. We will also consider the Hardy space of square integrable functions on S which are holomorphic on B. This can be viewed as the analytic continuation of $\mathcal{H}_\nu$ at $\nu = d$. Thus we assume throughout this paper that $\nu \geq d$.

Let f be a bounded smooth function on $\bar{B}$, the closure of B. The Toeplitz operator T_f on $\mathcal{H}_\nu$ with symbol f is defined by

$$T_f g = P(fg),$$

where P is the Bergman or the Hardy projection for $\nu > d$ and $\nu = d$, respectively.

As was shown by Howe [10] there is a more flexible and effective way of studying the spectral properties of Toeplitz operators with smooth symbol, by using the theories of representations of the Heisenberg group and of pseudo-differential operators. We will adopt that approach. We will be very brief and refer to [10] and [18, Chapter XII] for details. So let $H_n = \mathbb{C}^d \times T$ be the Heisenberg group as in loc. cit. The Heisenberg group has an irreducible representation, ρ, on the Fock space $\mathcal{F}$ consisting of entire functions f on $\mathbb{C}^d$ such that

$$\int_{\mathbb{C}^d} |f(z)|^2 e^{-\pi|z|^2} dm(z) < \infty.$$

The action of the Heisenberg group is explicitly given as follows. For $w \in \mathbb{C}^d$ viewed as an element in H_d,

$$\rho(w)f(w') = e^{-\pi/2|w|^2+\pi w'\cdot\overline{w}} f(w'-w),$$

where $w'\cdot\overline{w}$ is the Hermitian inner product on $\mathbb{C}^d$. The action of T is by rotation.

Identifying the Lie algebra $\mathfrak{h}$ of the Heisenberg group with $\mathbb{R}^{2n}\oplus\mathbb{R}$ and thus $\mathbb{R}^{2n}$ with a subspace of the Lie algebra, we get an action of $\mathbb{R}^{2n}$ as holomorphic differential operators on $\mathcal{F}$, which extends from $\mathfrak{h}$ to the whole enveloping algebra $\mathfrak{U}(\mathfrak{h})$ and which will also be denoted by ρ. In particular, taking the basis elements $\partial_j = \partial/\partial w_j$ and $\overline{\partial}_j = \partial/\partial \overline{w}_j$ of $\mathbb{R}^{2n}$ we have

$$\rho(\partial_j)f(w) = -\partial_j f(w), \qquad \rho(\overline{\partial}_j)f(w) = \pi w_j f(w). \tag{2.1}$$

Following the notation in [10], let $\Delta \in \mathfrak{U}(\mathfrak{h})$ be the element

$$\Delta = \frac{1}{2}(\partial_j\cdot\overline{\partial}_j + \overline{\partial}_j\cdot\partial_j).$$

Then $\rho(\Delta)$ acts on $\mathcal{F}$ as a diagonal self-adjoint operator [10], under the orthogonal basis $\{w^\alpha, \alpha = (\alpha_1, \cdots, \alpha_d)\}$, viz.

$$\rho(\Delta) w^\alpha = -\pi\Big(|\alpha| + \frac{d}{2}\Big) w^\alpha. \tag{2.2}$$

Let $F(z)$ be a function on $\mathbb{C}^d$ (viewed as a function on the Heisenberg group). The Weyl transform $\rho(F)$ of F is defined by

$$\rho(F) = \int_{\mathbb{C}^d} F(w)\rho(w) dm(w).$$

To understand the operator theoretic properties of $\rho(F)$ we will need the Fourier transform of F. Let $\hat{F}$ be the (symplectic-) Fourier transform of F,

$$\hat{F}(w') = 2^{-d} \int_{\mathbb{C}^d} F(w) e^{\pi i \operatorname{Im} w' \cdot \overline{w}} dm(w),$$

and let $F * G$ be the symplectic convolution

$$F * G(w) = \int_{\mathbb{C}^d} F(z) G(w - z) e^{\pi i \operatorname{Im} w \cdot \overline{z}} dm(z).$$

We recall that

$$\widehat{F * G} = F * \hat{G}$$

and

$$\rho(F)\rho(G) = \rho(F * G)$$

for an appropriate class of functions. A well-known theorem of Calderón-Vaillancourt states that if $\hat{F}$ and all its derivatives are bounded, then $\rho(F)$ can be defined as a bounded operator on $\mathcal{F}$.

We will need a finer class of symbols introduced by Howe, corresponding to the so-called pseudo-Toeplitz operators. Let

$$\mathcal{PT}(m, \mu) = \{F \in \mathcal{S}^*(\mathbb{C}^d) : |\partial^\alpha \overline{\partial}^\beta \hat{F}| \le C_{\alpha\beta}(1 + |w|)^{m - \mu(|\alpha| + |\beta|)}\}$$

and

$$\begin{aligned}\mathcal{PT}_{rad}(m, \mu) = \{F \in \mathcal{PT}(m, \mu) :& \\ &\hat{F} = (1 - g(|w|))\psi(\frac{w}{|w|})|w|^m + D_1,\ D_1 \in \mathcal{PT}(m - \mu, \mu)\}.\end{aligned}$$

Here g is a smooth function on $\mathbb{R}$ such that $0 \le g(t) \le 1$ on $\mathbb{R}$, $g(t) = 0$ for $|t| \ge 2$ and $g(t) = 1$ for $0 \le t \le 1$.

For $F \in \mathcal{PT}_{rad}(m, \mu)$ we will call

$$\sigma_m(F) := \psi(\frac{w}{|w|})|w|^m \tag{2.3}$$

its principal symbol. It can be obtained, up to the factor $|w|^m$, by

$$\psi(w) = \lim_{t \to \infty} t^{-m} \hat{F}(tw), \quad w \in S.$$

Following Howe [10], we will call $\rho(F)$, $F \in \mathcal{PT}(m, \mu)$, a pseudo-Toeplitz operator of order m and smoothness μ. One has [10, Lemma 4.2.2]

$$F \in \mathcal{PT}(m_1, \mu),\ G \in \mathcal{PT}(m_2, \mu) \implies F * G \in \mathcal{PT}(m_1 + m_2, \mu). \tag{2.4}$$

We will realize the Toeplitz operators T_f on $\mathcal{H}_\nu$ for f on B (or on S for the Hardy space) as Weyl transforms $\rho(F)$ of certain symbols F on $\mathbb{C}^d$. First we notice that

$$e_\beta := \Big(\frac{(\nu)_{|\beta|}}{\beta!}\Big)^{\frac{1}{2}} z^\beta$$

form an orthonormal basis of $\mathcal{H}_\nu$, and so do

$$E_\beta := \Big(\frac{1}{\pi^{|\beta|}\beta!}\Big)^{\frac{1}{2}} w^\beta$$

for $\mathcal{F}$. (Here $(\nu)_j := \nu(\nu+1)\dots(\nu+j-1)$ is the usual Pochhammer symbol.) Thus the map

$$U : e_\beta \to E_\beta \tag{2.5}$$

is a unitary operator. First we will find the action of the elementary Toeplitz operators T_{z^α} under the intertwining map U.

Lemma 2.1. *The operator $UT_{z^\alpha}U^*$ on $\mathcal{F}$ is given by*

$$UT_{z^\alpha}U^* = \rho(z)^\alpha \rho\left(\pi^{|\alpha|}\Big(\nu - \frac{d}{2} - \frac{1}{\pi}\Delta\Big)_{|\alpha|}\right)^{-1/2}. \tag{2.6}$$

This can be proved by direct computation. Indeed we have

$$T_{z^\alpha} e_\beta = \Big(\frac{(\beta)_\alpha}{(\nu+|\beta|)_{|\alpha|}}\Big)^{\frac{1}{2}} e_{\beta+\alpha},$$

and the right hand side (2.6) can be easily computed by (2.1) and (2.2).

By using the previous lemma we have then the following result, which was proved by Howe [10, Proposition 4.2.3] in the case when $\nu = d+1$; the general case of $\nu \geq d$ is essentially the same.

Proposition 2.2. *Let $f \in C^\infty(S)$ and let $\tilde{f}$ be a C^∞ extension to B and $T_{\tilde{f}}$ the Toeplitz operator on $\mathcal{H}_\nu$. Then under the unitary equivalence of $\mathcal{H}_\nu$ and the Fock space $\mathcal{F}$ on $\mathbb{C}^d$, the Toeplitz operators are pseudo-Toeplitz operators with radial asymptotic limits $\mathcal{PT}_{rad}(0,1)$. More precisely, there exists $F \in \mathcal{PT}_{rad}(0,1)$ such that $UT_{\tilde{f}}U^* = \rho(F)$, and $f(\zeta) = \lim_{t\to\infty} \hat{F}(t\zeta)$ for each $\zeta \in S$.*

3. Schatten - von Neumann properties of pseudo-Toeplitz operators

Recall that the Schatten - von Neumann class $\mathcal{L}^p$, $p \geq 1$, consists of compact operators T such that the eigenvalues $\{\mu_n\}$ of $|T| = (T^*T)^{\frac{1}{2}}$ are p-summable, $\sum \mu_n^p < \infty$. In particular $\mathcal{L}^2$ is the Hilbert-Schmidt class, $\mathcal{L}^1$ the trace class and $\mathcal{L}^\infty$ are the compact operators. For $1 < p < \infty$, $1 \leq q \leq \infty$, the Macaev class $\mathcal{L}^{p,q}$ is obtained by the real interpolation between $\mathcal{L}^1$ and $\mathcal{L}^\infty$. However, we will need the Macaev class $\mathcal{L}^{1,\infty}$, which consists [1] of all compact operators such that, if $\mu_1 \geq \mu_2 \geq \dots$,

$$\sum_{n=1}^N \mu_n = O(\log N).$$

There exists a linear functional on the space $\mathcal{L}^{1,\infty}$ that resembles the usual trace, called the Dixmier trace. Its definition is rather involved and we refer to [8,

[1] Sometimes this ideal is denoted by $\mathcal{L}_\Omega$, the notation $\mathcal{L}^{1,\infty}$ being reserved for the (smaller) class of operators for which $\mu_n = O(1/n)$. Our notation follows Connes' book [8].

Chapter IV] for details. Let $C_b(\mathbb{R}_+)$ be the space of bounded continuous functions on $\mathbb{R}_+$ and $C_0(\mathbb{R}_+)$ the subspace of functions vanishing at ∞. Let ω be a positive linear functional on the quotient space $C_b(\mathbb{R}_+)/C_0(\mathbb{R}_+)$ such that $\omega(1) = 1$. For a positive compact operator $T \in \mathcal{L}^{1,\infty}$ with eigenvalues $\{\mu_n\}$, extend μ_n to a step function on $\mathbb{R}_+$ and let $M_T(\lambda)$ be its Cesáro mean, which is a bounded continuous function on $\mathbb{R}^+$. The Dixmier trace of T is then defined by

$$\operatorname{tr}_\omega T = \omega(M_T).$$

It is then extended to all of $\mathcal{L}^{1,\infty}$ by linearity. In particular it is bounded and vanishes on trace class operators. The fact that we will need is that

$$\operatorname{tr}_\omega T = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T)$$

if T is a positive operator and if the right hand side exists.

Lemma 3.1. *For any $c \geq 0$ the operator $(c - \rho(\Delta))^{-d} = \rho(c\delta_0 - \Delta)^{-d}$ is in the Macaev class $\mathcal{L}^{1,\infty}$.*

Proof. It follows from (2.2) that the eigenvalues of $(c-\rho(\Delta))^d$ are $(c+\pi(m+\frac{d}{2}))^d$, $m = 0, 1, \cdots$, each of multiplicity $d_m := \dim\{w^\alpha, |\alpha| = m\} = \binom{d+m-1}{d-1} \approx m^{d-1}$. The partial sums thus satisfy

$$\sum_{m\leq N} (c + \pi(m + \frac{d}{2}))^{-d} d_m \approx \frac{1}{(d-1)!} \sum_{m\leq N} (c + \pi(m + \frac{d}{2}))^{-d} m^{d-1} \approx \frac{1}{(d-1)!} \log N,$$

completing the proof. □

Proposition 3.2. *Let $F \in \mathcal{PT}(-2d, 1)$. Then the Weyl transform $\rho(F)$ is in the Macaev class $\mathcal{L}^{1,\infty}$.*

Proof. By (3.5.6) in [10],

$$\hat{\Delta} = -\frac{\pi^2}{4}|w|^2, \tag{3.1}$$

so $-\Delta \in \mathcal{PT}(2, 1)$, whence by (2.4) $(-\Delta)^{*d} \in \mathcal{PT}(2d, 1)$ and $(-\Delta)^{*d} {*} F \in \mathcal{PT}(0, 1)$. By the Calderón-Vaillancourt theorem [10, Theorem 3.1.3], the corresponding Weyl transform, $\rho(-\Delta)^d\rho(F)$, is bounded. Hence by the previous lemma, $\rho(F) \in \mathcal{L}^{1,\infty}$, since the Macaev class $\mathcal{L}^{1,\infty}$ is an ideal. □

4. Dixmier trace formula for Toeplitz operators

Theorem 4.1. *Let $F \in \mathcal{PT}_{rad}(-2d, 1)$ with the principal symbol $\sigma_{-2d}(\hat{F})$ as defined in (2.3). Then the Dixmier trace $\operatorname{tr}_\omega \rho(F)$ is independent of ω and is given by*

$$\operatorname{tr}_\omega \rho(F) = \frac{\pi^d}{4^d d!} \int_S \hat{\sigma}_{-2d}(F)(w),$$

where $\int_S$ is the normalized integral over the unit sphere.

Proof. The proof is quite similar to that of Connes [7] for pseudo-differential operators on compact manifolds. Namely, by [10, Theorem 4.2.5] and the definition of $\mathcal{PT}_{rad}$, the Dixmier trace $\operatorname{tr}_\omega \rho(F)$ depends only on the leading symbol of $\sigma_{-2d}(\hat{F})$ and defines a positive measure on the unit sphere S in $\mathbb{C}^d$. By the unitary invariance of $\rho(F)$ the measure has to be a constant multiple of the area measure. To find the constant we note that the symbol of $c\delta_0 - \Delta$, $c > 0$, is absolutely elliptic in the sense of (4.2.20) in [10], and thus by pp. 246–247 in [10] we can construct $F_0 \in \mathcal{PT}_{rad}(-2d, 1)$ such that $\rho(F_0) = (c - \rho(\Delta))^{-d}$. The eigenvalue of $\rho(F_0)$ on the space of all m-homogeneous polynomials is, by the proof of Lemma 3.1,

$$\frac{1}{(c + \pi(m + \frac{d}{2}))^d}.$$

Its Dixmier trace exists and is (noticing that the dimension of the space of homogeneous polynomials of degree $m \le N$ is $\approx N^d$)

$$\operatorname{tr}_\omega \rho(F_0) = \frac{1}{\pi^d d!}.$$

On the other hand, the principal symbol $\sigma_{-2d}(F_0)$ is the constant function $(4/\pi^2)^d |w|^{-2d}$ by the definition (cf. (3.1)), whose integration over the sphere is $(4/\pi^2)^d$. This completes the proof. $\square$

To apply our result to Toeplitz operators we need to introduce some more notation. We let

$$\partial_j^b = \partial_j - \bar{z}_j R, \quad \bar{\partial}_j^b = \bar{\partial}_j - z_j \bar{R}$$

be the boundary Cauchy-Riemann operators [17], where $R = \sum_{j=1}^d z_j \partial_j$ is the holomorphic radial derivative. As vector fields they are linearly dependent, to wit,

$$\sum_{j=1}^d z_j \partial_j^b = 0, \quad \sum_{j=1}^d \bar{z}_j \bar{\partial}_j^b = 0. \tag{4.1}$$

Definition 4.2. We define a bracket $\{f, g\}_b$ for smooth functions f and g on S by

$$\{f, g\}_b := \sum_{j=1}^d (\partial_j^b f \, \overline{\partial}_j^b g - \overline{\partial}_j^b f \, \partial_j^b g)$$

and call it the boundary Poisson bracket.

Lemma 4.3. *Let F and G be two functions in $\mathcal{PT}_{rad}(0, \mu)$ with principal symbols*

$$\sigma_0(F)(z) = f\Big(\frac{z}{|z|}\Big), \quad \sigma_0(G)(z) = g\Big(\frac{z}{|z|}\Big)$$

for f and g in $C^\infty(S)$. Then the principal symbol of $F * G - G * F$ is given by

$$\sigma_{-2}(F * G - G * F)(z) = \frac{4}{\pi}\{f, g\}_b(\frac{z}{|z|})|z|^{-2}.$$

Proof. By the general result for the symbol calculus for pseudo-Toeplitz operators (cf. (2.2.5) in [10]), we have $F * G - G * F \in \mathcal{PT}_{rad}(-2\mu, \mu)$ with the principal symbol

$$\sigma_{-2}(F * G - G * F)(z) = \frac{4}{\pi}\{\sigma_0(F)), \sigma_0(G)\}(z),$$

where $\{\cdot, \cdot\}$ is the ordinary Poisson bracket in complex coordinates

$$\{\Psi, \Phi\} := \sum_{j=1}^{d}(\partial_j \Psi \, \overline{\partial}_j \Phi - \partial_j \Phi \, \overline{\partial}_j \Psi).$$

The function $\sigma_{-2}(F*G-G*F)(z)$ is positive homogeneous of degree -2. We need only to compute it for $z \in S$. Defining the Reeb vector field E and the outward normal vector field N in terms of the radial derivative R,

$$E := \frac{1}{2}(\overline{R} - R), \quad N := \overline{R} + R,$$

we can write

$$R = -E + \frac{N}{2}.$$

Note that E is well-defined on S. The vector field $\partial_j^b - \overline{z}_j E$ is thus a well-defined vector field on S, and for any function $\Phi(z) = \phi(\frac{z}{|z|})$ we have

$$\partial_j \Phi(z) = (\partial_j^b + \overline{z}_j R)\Phi(z) = (\partial_j^b - \overline{z}_j E + \frac{\overline{z}_j}{2} N)\Phi(z) = (\partial_j^b - \overline{z}_j E)\phi(z),$$

since $N\Phi(z) = 0$ by homogeneity. Similarly $\overline{\partial}_j \Phi = (\partial_j^b + z_j E)\phi$ on S. From this it follows that for $z \in S$,

$$\begin{aligned}\{\sigma_0(F), \sigma_0(G)\}(z) = \sum_{j=1}^{d} \Big(&(\partial_j^b f(z) - \overline{z}_j E f(z))(\overline{\partial}_j^b g(z) + z_j E g(z)) \\ &- (\overline{\partial}_j^b f(z) - \overline{z}_j E f(z))(\partial_j^b g(z) + z_j E g(z))\Big) \\ = \{f, g\}_b, &\end{aligned}$$

by using (4.1). □

Theorem 4.4. *Let $f_1, g_1, \cdots, f_d, g_d$ be smooth functions on S, $\tilde{f}_1, \tilde{g}_1, \cdots, \tilde{f}_d, \tilde{g}_d$ their smooth extensions to B and $T_{\tilde{f}_1}, T_{\tilde{g}_1}, \cdots, T_{\tilde{f}_d}, T_{\tilde{g}_d}$ the associated Toeplitz operators on $\mathcal{H}_\nu$ for $\nu \geq d$. Then the product $\prod_{j=1}^d [T_{\tilde{f}_j}, T_{\tilde{g}_j}]$ is in the Macaev class and its Dixmier trace is given by*

$$\mathrm{tr}_\omega \prod_{j=1}^{d} [T_{\tilde{f}_j}, T_{\tilde{g}_j}] = \frac{1}{d!} \int_S \prod_{j=1}^{d} \{f_j, g_j\}_b.$$

Proof. The proof is straightforward from the preceding lemma, formula (2.2.5) in [10] and Theorem 4.1. □

We apply our result to Hankel operators with anti-holomorphic symbols. Let f be a holomorphic function in a neighborhood of B and $H_{\bar{f}} g = (I - P)\bar{f} g$, $g \in \mathcal{H}_\nu$ the Hankel operator. Then

$$[T_{\bar{f}}, T_f] = [T_f^*, T_f] = |H_{\bar{f}}|^2 = H_{\bar{f}}^* H_{\bar{f}}.$$

Corollary 4.5. *Let f be as above. Then the Hankel operator is in $\mathcal{L}^{2d,\infty}$; equivalently the commutator $[T_{\bar{f}}, T_f]$ is in $\mathcal{L}^{d,\infty}$ and we have*

$$\mathrm{tr}_\omega\, |H_{\bar{f}}|^{2d} = \mathrm{tr}_\omega([T_{\bar{f}}, T_f]^d) = \frac{1}{d!}\int_S (|\nabla f|^2 - |Rf|^2)^d.$$

Notice that $H_{\bar{f}}$ is in the Schatten class $\mathcal{L}^p$ for $p > 2d$ and that its Schatten norm is

$$\|H_{\bar{f}}\|_p^p \approx \int_B (1-|z|^2)^p (|\nabla f|^2 - |Rf|^2)^{\frac{p}{2}}\, dm(z);$$

see [3] and [19] for the Bergman space case ($\nu = d+1$) and the Hardy space case ($\nu = d$). Our resulting formula thus provides a limiting result of the above estimates, and it is interesting to note that the estimate has an equality as its limit for $p \to 2d$.

References

[1] J. Arazy, S. D. Fisher, and J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. Math. **110** (1988), no. 6, 989–1053. MR970119 (90a:47067)

[2] J. Arazy, S. D. Fisher, S. Janson, and J. Peetre, *An identity for reproducing kernels in a planar domain and Hilbert-Schmidt Hankel operators*, J. Reine Angew. Math. **406** (1990), 179–199. MR1048240 (91b:47050)

[3] ———, *Membership of Hankel operators on the ball in unitary ideals*, J. London Math. Soc. (2) **43** (1991), no. 3, 485–508. MR1113389 (93c:47030)

[4] P. Boggiatto and F. Nicola, *Non-commutative residues for anisotropic pseudo-differential operators in* $\mathbb{R}^n$, J. Funct. Anal. **203** (2003), no. 2, 305–320. MR2003350 (2004g:58036)

[5] M. Feldman and R. Rochberg, *Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group*, Analysis and partial differential equations, Lecture Notes in Pure and Appl. Math., vol. 122, Dekker, New York, 1990, pp. 121–159. MR1044786 (91e:47024)

[6] V. Guillemin, *Toeplitz operators in n dimensions*, Integral Equations Operator Theory **7** (1984), no. 2, 145–205. MR750217 (86i:58130)

[7] A. Connes, *The action functional in noncommutative geometry*, Comm. Math. Phys. **117** (1988), no. 4, 673–683. MR953826 (91b:58246)

[8] A. Connes, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994. MR1303779 (95j:46063)

[9] J. W. Helton and R. E. Howe, *Traces of commutators of integral operators*, Acta Math. **135** (1975), no. 3-4, 271–305. MR0438188 (55:11106)

[10] R. Howe, *Quantum mechanics and partial differential equations*, J. Funct. Anal. **38** (1980), no. 2, 188–254. MR587908 (83b:35166)

[11] S. Janson, J. Peetre, and R. Rochberg, *Hankel forms and the Fock space*, Rev. Mat. Iberoamericana **3** (1987), no. 1, 61–138. MR1008445 (91a:47029)

[12] S. Janson, H. Upmeier, and R. Wallstén, *Schatten-norm identities for Hankel operators*, J. Funct. Anal. **119** (1994), no. 1, 210–216. MR1255278 (95a:47021)

[13] J. Peetre, *On the S_4-norm of a Hankel form*, Rev. Mat. Iberoamericana **8** (1992), no. 1, 121–130. MR1178451 (93f:47027)

[14] V. V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003. MR1949210 (2004e:47040)

[15] R. Ponge, *Noncommutative residue invariants for CR and contact manifolds*, J. Reine Angew. Math. **614** (2008), 117–151. MR2376284 (2008j:58039)

[16] R. Rochberg, *Trace ideal criteria for Hankel operators and commutators*, Indiana Univ. Math. J. **31** (1982), no. 6, 913–925. MR674875 (84d:47036)

[17] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag, New York-Berlin, 1980. MR601594 (82i:32002)

[18] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, with the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR1232192 (95c:42002)

[19] K. H. Zhu, *Schatten class Hankel operators on the Bergman space of the unit ball*, Amer. J. Math. **113** (1991), no. 1, 147–167. MR1087805 (91m:47036)

Mathematics Institute AS ČR, Žitná 25, 11567 Prague 1, Czech Republic – and – Mathematics Institute, Silesian University, Na Rybníčku 1, 74601 Opava, Czech Republic

E-mail address: englis@math.cas.cz

Department of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

E-mail address: kyguo@fudan.edu.cn

Department of Mathematics, Chalmers University of Technology and Göteborg University, Göteborg, Sweden

E-mail address: genkai@chalmers.se