

GEOMETRIC RIGIDITY FOR CLASS  $\mathcal{S}$  OF  
TRANSCENDENTAL MEROMORPHIC FUNCTIONS  
WHOSE JULIA SETS ARE JORDAN CURVES

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ABSTRACT. We consider any transcendental meromorphic function  $f$  of Class  $\mathcal{S}$  whose Julia set is a Jordan curve. We show that the Julia set of  $f$  either is an extended straight line or has Hausdorff dimension strictly greater than 1. The proof uses conformal iterated function systems and extends many earlier results of this type.

1. INTRODUCTION

We suppose that  $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$  is a meromorphic function, where  $\hat{\mathbb{C}}$  denotes the Riemann sphere, and that the Julia set of  $f$ , denoted by  $J(f)$ , is a Jordan curve. In 1919 Fatou proved in [3] that if  $f$  is rational, then either  $J(f)$  is a circle in  $\hat{\mathbb{C}}$  or for a dense set of points in  $J(f)$  no tangent lines exist. It was later proved that if  $f$  is hyperbolic, then under the hypotheses above, either  $J(f)$  is a circle or the Hausdorff dimension of  $J(f)$  is strictly greater than 1 (see [2], [11], and also [9]). By the topological exactness of  $f$ , this implies that every non-empty open subset of  $J(f)$  has Hausdorff dimension greater than 1; in particular, the no-tangent-line alternative is much stronger. The hypothesis of hyperbolicity was weakened to allow the existence of parabolic points in [12], and other extensions of the result with weaker assumptions were obtained in [9], [12], [13] and [4], all in the setting of rational functions. Recall that a meromorphic function is in Class  $\mathcal{S}$  provided that the set of its singularities is finite. In the landscape of transcendental functions  $f$ , a result analogous to the one above for rational functions was obtained in [6]. For Class  $\mathcal{S}$  it yields our theorem under the additional assumption that the map  $f$  has no rationally indifferent periodic points. In the current paper we complete the rigidity picture for Class  $\mathcal{S}$ , and we prove the circle/fractal dichotomy in its maximal strength. In the proofs we do not consider cases or use any knowledge about the dynamics of inner functions. Instead, we associate to our transcendental map a conformal iterated function system in the sense of [7] and [8] and apply the results proved in these papers. For the convenience of the reader we provide the formal

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definition of conformal iterated function systems in the Appendix. We also collect there some (basic) results about these systems that we need in the proof of Theorem 2.1. An interested reader is, however, strongly encouraged to consult [8] and/or [5] (especially Chapter 11) for a deeper understanding of the theory of conformal iterated function systems or, more generally, of conformal graph directed Markov systems, the theme of the book [8], where one can also find some applications of this theory. An updated list of its selected applications to the theory of iteration of transcendental meromorphic functions, number theory, Kleinian groups, and other areas can be found in [10]. One can find in [1] concrete examples of meromorphic functions in Class  $\mathcal{S}$  whose Julia sets are Jordan curves with rationally indifferent periodic points.

Beyond Class  $\mathcal{S}$  our theorem in general fails. D. Hamilton has constructed in [4] meromorphic functions that are not in Class  $\mathcal{S}$  and whose Julia sets are rectifiable Jordan curves, but which do not form geometric circles in  $\hat{\mathbb{C}}$ . However, Hamilton's functions are not even in Class  $\mathcal{B}$ , the one consisting of all meromorphic functions with bounded sets of singularities.

## 2. THE THEOREM

In order to fix terminology, in what follows, by an extended straight line we mean any Euclidean straight line in the complex plane including the point at infinity, and by a Euclidean circle we mean the set of points in the complex plane  $\mathbb{C}$  whose Euclidean distance to some (fixed) point in  $\mathbb{C}$  is constant. The result of our paper is the following.

**Theorem 2.1.** *Suppose  $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$  is a Class  $\mathcal{S}$  transcendental meromorphic function. If the Julia set  $J(f)$  of  $f$  is a Jordan curve, then either it is an extended straight line or its Hausdorff dimension is strictly larger than 1.*

*Proof.* Note that because of the Theorem in [1] (where the hypothesis of having two completely invariant domains can be weakened by requiring that the second iterate has two completely invariant domains), our assumption that the Julia set  $J(f)$  is a Jordan curve is equivalent to requiring that  $f^2$  has two completely invariant domains. Let  $A_0$  and  $A_1$  be the two connected components of  $\hat{\mathbb{C}} \setminus J(f)$ . Since our function is in Class  $\mathcal{S}$ , for each  $i = 0, 1$  there exists  $a_i \in \overline{A_i} \setminus \{\infty\}$  such that  $f^2(a_i) = a_i$  and either  $a_i$  is an attracting fixed point for  $f^2$  or  $a_i$  is a rationally indifferent fixed point for  $f^2$  and  $A_i$  is its immediate basin of attraction. In order to work with Euclidean derivatives, we change the system of coordinates by a Möbius transformation so that  $\infty$  is sent to a finite point and, moreover, the image of the whole Julia set is contained in the complex plane  $\mathbb{C}$ . If one of the points  $a_0$  or  $a_1$  is a parabolic point, denote it by  $\omega$ . Otherwise, let  $\omega$  be an arbitrary periodic point in  $J(f)$ . Let  $A$  be either of the sets  $A_0$  or  $A_1$ . Replacing  $f$  by a sufficiently high iterate of it, we may assume without loss of generality that  $f(\omega) = \omega$  and  $f(A) = A$ . For each  $k \geq 1$  denote by  $\hat{\gamma}_k$  the only open arc in  $J(f)$  such that  $\hat{\gamma}_k \cap f^{-k}(\omega) = \{\omega\}$  and  $\hat{\gamma}_k$  has endpoints in  $f^{-k}(\omega) \setminus \{\omega\}$ . Fix  $k \geq 1$  so large that

$$(2.1) \quad \bigcap_{j=0}^{\infty} f^{-j}(\overline{\hat{\gamma}_k}) = \{\omega\}.$$

Set

$$\gamma = J(f) \setminus \hat{\gamma}_k.$$

It follows from our assumptions and part (ii) of the Theorem in [1] that there exists a closed topological disk  $X$  contained in  $\mathbb{C}$  with the following properties:

- (a)  $\gamma \subset X$ .
- (b) The boundary of  $X$  is a piecewise smooth Jordan curve without cusps that contains both endpoints of  $\gamma$ .
- (c) There exists an open simply connected subset  $V$  of  $\mathbb{C}$  that contains  $X$  and is disjoint from the postcritical set of  $f$ .
- (d) If  $f_*^{-n}$  is a holomorphic inverse branch of  $f^n$  defined on  $V$  such that  $f_*^{-n}(X) \cap \text{Int}X \neq \emptyset$ , then  $f_*^{-n}(X) \subset X$  and  $f_*^{-n}(V) \subset V$ .

We now form an iterated function system in the sense of [8]; its definition is provided in the Appendix. This system is defined to consist of all holomorphic inverse branches  $f_*^{-n} : V \rightarrow \mathbb{C}$  such that  $f_*^{-n}(X) \cap \text{Int}X \neq \emptyset$  and  $f^k(f_*^{-n}(X)) \cap \text{Int}X \neq \emptyset$  for all  $k = 1, 2, \dots, n-1$ . We parameterize all such inverse branches by a countable alphabet  $I$  and denote them by  $\phi_i, i \in I$ . It follows immediately from this definition that  $\phi_i(\text{Int}X) \cap \phi_j(\text{Int}X) = \emptyset$  whenever  $i \neq j$ . Along with properties (a)–(d) this implies that  $S = \{\phi_i : X \rightarrow X\}_{i \in I}$  is a conformal iterated function system as defined in the Appendix. Let  $J_S$  be the limit set of  $S$ . Using (2.1), note that

$$J_S = \gamma \setminus \bigcup_{n=0}^{\infty} f^{-n}(\{\omega\} \cup E),$$

where  $E$  is the countable set of essential singularities of  $f$  (for the map  $f$  the set  $E$  is a singleton, say  $e$ , but for an iterate  $f^k$  the entire set  $\{e\} \cup f^{-1}(\{e\}) \cup f^{-2}(\{e\}) \cup \dots \cup f^{-(k-1)}(\{e\})$  consists of essential singularities of  $f^k$ ). In particular,

$$\text{HD}(J_S) = \text{HD}(\gamma) = \text{HD}(J(f)) := h,$$

where HD stands for the Hausdorff dimension of a set. Now suppose  $\text{HD}(J(f)) = 1$ . So  $\text{HD}(J_S) = 1$ , and, by Theorem 3.2 in the Appendix below,  $\text{H}^1(J_S) < +\infty$ . Clearly,  $\text{H}^1(J_S) = \text{H}^1(\gamma) > 0$ ,

$$\text{H}^1(J_S)(\phi_i(A)) = \int_A |\phi'_i| d\text{H}^1(J_S)$$

for every Borel set  $A \subset \gamma$ , and, as the intersections  $\phi_i(\gamma) \cap \phi_j(\gamma)$  are at most singletons (with  $i \neq j$ ),

$$\text{H}^1(J_S)(\phi_i(\gamma) \cap \phi_j(\gamma)) = 0$$

whenever  $i \neq j$ . Thus, the uniqueness part of Remark 3.1 implies that  $m := (\text{H}^1(\gamma))^{-1} \text{H}^1|_{\gamma}$  is the unique 1-conformal measure for the system  $S$ . Let  $\mu$  be the corresponding unique invariant measure equivalent on  $J_S$  to  $m$  that results from Theorem 3.3.

Now consider two Riemann mappings  $R_0 : \overline{\mathbb{D}^1} \rightarrow \overline{A_0}$  and  $R_1 : \hat{\mathbb{C}} \setminus \mathbb{D}^1 \rightarrow \overline{A_1}$  such that  $R_0(1) = R_1(1) = \omega$  (since  $J(f)$  is a Jordan curve,  $R_0$  and  $R_1$  are uniquely defined on the closed disks  $\overline{\mathbb{D}^1}$  and  $\hat{\mathbb{C}} \setminus \mathbb{D}^1$  respectively due to Caratheodory's theorem). Define two continuous maps

$$g_0 := R_0^{-1} \circ f \circ R_0 : \overline{\mathbb{D}^1} \setminus R_0^{-1}(E) \rightarrow \overline{\mathbb{D}^1} \text{ and}$$

$$g_1 := R_1^{-1} \circ f \circ R_1 : (\hat{\mathbb{C}} \setminus \mathbb{D}^1) \setminus R_1^{-1}(E) \rightarrow \hat{\mathbb{C}} \setminus \mathbb{D}^1.$$

Thus, the Schwartz Reflection Principle allows us to extend  $g_0$  and  $g_1$  respectively to  $\mathbb{C} \setminus R_0^{-1}(E)$  and  $\mathbb{C} \setminus R_1^{-1}(E)$ . Then the iterated function system  $S$  lifts to the two systems  $S_0 = \{\phi_i^0\}_{i \in I}$  and  $S_1 = \{\phi_i^1\}_{i \in I}$  formed by respective inverse branches of

iterates of  $g_0$  and  $g_1$ . Fix  $j \in \{0, 1\}$ . Note that the normalized Lebesgue measure  $l_j$  on  $R_j^{-1}(\gamma)$  is a conformal measure for the system  $S_j$ . Again, by Theorem 3.3 in the Appendix, this system has a unique invariant measure  $\mu_j$  equivalent to  $l_j$ . But  $\mu \circ R_j$  is also  $S_j$ -invariant and, by Riesz's Theorem, is equivalent to  $l_j$ . Thus,  $\mu_j = \mu \circ R_j$ . Hence,

$$(2.2) \quad \mu_1 = \mu \circ R_1 = \mu \circ R_0 \circ (R_0^{-1} \circ R_1) = \mu_0 \circ R_0^{-1} \circ R_1 = \mu_0 \circ (R_1^{-1} \circ R_0)^{-1}.$$

For every  $z \in S^1$  put

$$D_j(z) = \frac{d\mu_j}{d\lambda}(z).$$

As Theorem 3.4 states, the function  $z \mapsto D_j(z)$  has a real-analytic extension onto a neighborhood of  $R_j^{-1}(\gamma)$  in  $\mathbb{C}$ . Let

$$F_j(z) = \int_1^z D_j(t) d\lambda(t),$$

where the integration is taken along the unit circle arc from 1 to  $z$  with respect to Lebesgue measure  $\lambda$  on  $S^1$ . Formula (2.2) and  $R_1^{-1} \circ R_0(1) = 1$  then give for every  $z \in R_1^{-1}(\gamma)$  that

$$F_0(z) = F_1(R_1^{-1} \circ R_0(z)).$$

Since both functions  $F_1$  and  $F_0$  are invertible (as  $D_j$  is positive on  $R_j^{-1}(\gamma)$ ), we conclude that  $R_1^{-1} \circ R_0 = F_1^{-1} \circ F_0$  is real-analytic on  $R_0^{-1}(\gamma)$ . Thus,  $R_1^{-1} \circ R_0$  has a holomorphic extension  $\psi$  to an open neighborhood  $U$  of  $R_0^{-1}(\gamma)$  in  $\mathbb{C}$ . The formula

$$T(z) = \begin{cases} R_0(z) & \text{if } z \in \overline{\mathbb{D}^1} \cap U \\ R_1 \circ \psi(z) & \text{if } z \in (\hat{\mathbb{C}} \setminus \mathbb{D}^1) \cap U \end{cases}$$

thus defines a holomorphic map from  $U$  into  $\mathbb{C}$  mapping  $R_0^{-1}(\gamma)$  onto  $\gamma$ . Therefore,  $\gamma$  is a real-analytic curve, and topological exactness of  $f : J(f) \rightarrow J(f)$  implies that  $J(f)$  itself is a real-analytic curve. So, by the Schwartz Reflection Principle,  $R_0$  extends to an entire bijective map of  $\mathbb{C}$  onto  $\mathbb{C}$ . Thus,  $R_0$  is an affine map ( $z \mapsto az + b$ ), and  $J(f) = R_0(S^1)$  is a geometric circle. We are done.  $\square$

### 3. APPENDIX: CONFORMAL INFINITE ITERATED FUNCTION SYSTEMS

Suppose that  $X$  is a subset of a Euclidean space  $\mathbb{R}^d$ . Let  $I$  be a countable set, either finite or infinite. It is called an alphabet. Suppose that a family  $S = \{\phi_i : X \rightarrow X\}_{i \in I}$  of injective self-maps of  $X$  is given. This family  $S$  is said to be a conformal (contracting) iterated function system provided that the following conditions are satisfied.

- (a) There exists  $s \in (0, 1)$  such that all the maps  $\phi_i$ ,  $i \in I$ , are Lipschitz continuous with the Lipschitz constant bounded above by  $s$  (in short: they are uniform contractions).
- (b)  $X$  is compact connected and  $X = \overline{\text{Int}(X)}$ , where the closure and interior are taken with respect to the Euclidean space  $\mathbb{R}^d$ .
- (c) (Open set condition) For all  $i, j \in I$  with  $i \neq j$ ,

$$\phi_i(\text{Int}(X)) \cap \phi_j(\text{Int}(X)) = \emptyset.$$

- (d) There exists an open connected set  $V \supset X$  such that for every  $i \in I$  the map  $\phi_i$  extends to a  $C^1$  conformal diffeomorphism of  $V$  into  $V$ .

- (e) (Cone property) There exist  $\gamma, l > 0, \gamma < \pi/2$ , such that for every  $x \in X \subset \mathbb{R}^d$  there exists an open cone  $\text{Con}(x, \gamma, l) \subset \text{Int}(X)$  with vertex  $x$ , central angle of measure  $\gamma$ , and altitude  $l$ .
- (f) There are two constants  $L \geq 1$  and  $\alpha > 0$  such that

$$|\phi'_i(y)| - |\phi'_j(x)| \leq L \|(\phi'_i)^{-1}\|^{-1} \|y - x\|^\alpha$$

for every  $i \in I$  and every pair of points  $x, y \in X$ , where  $|\phi'_i(x)|$  denotes the norm (equivalently, the scaling factor) of the derivative  $\phi'_i(x)$ .

For every  $\omega = \omega_1 \omega_2 \dots \omega_n \in I^n, n \geq 1$ , define the composition  $\phi_\omega := \phi_{\omega_1} \circ \phi_{\omega_2} \circ \dots \circ \phi_{\omega_n} : X \rightarrow X$ . It is proved in [8] that if  $d \geq 2$  and a family  $S = \{\phi_i\}_{i \in I}$  satisfies conditions (b) and (d), then it also satisfies condition (f) with  $\alpha = 1$ . Condition (f) in turn implies the so-called bounded distortion property, which says that

$$\frac{\|\phi'_\omega(y)\|}{\|\phi'_\omega(x)\|} \leq K$$

with some constant  $K > 0$ , for all  $\omega \in \bigcup_{n \geq 1} I^n$  and all  $x, y \in X$ . The limit set  $J_S$  of the system  $S$  is defined to be

$$\bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \phi_\omega(X)$$

Note that usually, if the alphabet  $I$  is infinite, the limit set  $J_S$  is not closed. The mutual relations of  $J_S$  and its closure are quite interesting and are investigated in detail in [8].

Fix  $t \geq 0$ . A Borel probability measure  $m$  on  $X$  is called  $t$ -conformal with respect to the iterated function system  $S$  provided that  $m(J_S) = 1$  and the following two conditions are satisfied:

$$m(\phi_i(A)) = \int_A |\phi'_i|^t dm$$

for every Borel set  $A \subset X$  and

$$m(\phi_i(X) \cap \phi_j(X)) = 0$$

whenever  $i \neq j$ .

*Remark 3.1.* It is proved in [8] that there exists at most one  $t$ -conformal measure, and if such a measure does exist, then  $t = h = \text{HD}(J_S)$ , the Hausdorff dimension of the limit set  $J_S$ . The system  $S$  is then called regular; otherwise it is called irregular. Theorem 4.5.1 in [8] says that if the system  $S$  is regular, then the  $h$ -dimensional Hausdorff measure  $H^h$  on  $J_S$  is absolutely continuous with respect to the  $h$ -conformal measure  $m$  with the Radon-Nikodym derivative uniformly bounded above; in particular the Hausdorff measure  $H^h(J_S)$  is finite. One of the assertions of Theorem 4.5.11 in [8] is that if the system  $S$  is irregular, then  $H^h(J_S) = 0$ . Thus, we get the following.

**Theorem 3.2.** *If  $S$  is a conformal iterated function system and  $h$  is the Hausdorff dimension of its limit set  $J_S$ , then  $H^h(J_S) < +\infty$ .*

Having a conformal measure, in order to be able to use the full power of ergodic theory one would like to know about the existence (and uniqueness) of invariant measures equivalent to the conformal measure. The answer is positive, and Theorem 4.4.7 in [8] yields this.

**Theorem 3.3.** *If  $S$  is a regular conformal iterated function system,  $h$  is the Hausdorff dimension of its limit set  $J_S$ , and  $m$  is the unique corresponding conformal measure, then there exists a unique Borel probability measure  $\mu$  on  $J_S$  with the following properties:*

- (i)  $\mu(J_S) = 1$ .
- (ii)  $\mu$  and  $m$  are equivalent and have positive and continuous Radon-Nikodym derivatives.
- (iii) (Invariance) For every Borel set  $A \subset X$ ,  $\sum_{i \in I} \mu(\phi_i(A)) = \mu(A)$ .

Having the invariant measure  $\mu$ , one can ask about regularity of the Radon-Nikodym derivative  $d\mu/dm$ . This derivative is strictly speaking defined only on the limit set  $J_S$ , which is typically nowhere dense and porous. Surprisingly, the Radon-Nikodym derivative  $d\mu/dm$  has a version that extends to all of  $X$  and is real-analytic. Indeed, Theorem 6.1.3 from [8] states this.

**Theorem 3.4.** *If  $S$  is a regular conformal iterated function system,  $m$  is the unique corresponding conformal measure on  $J_S$ , and  $\mu$  is the unique invariant measure equivalent to  $m$ , then the Radon-Nikodym derivative  $d\mu/dm$  has a version that extends to all of  $X$  and is real-analytic.*

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