

## FREE ARAKI-WOODS FACTORS AND CONNES' BICENTRALIZER PROBLEM

CYRIL HOUDAYER

(Communicated by Marius Junge)

ABSTRACT. We show that for any type III<sub>1</sub> free Araki-Woods factor  $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ , the bicentralizer of the free quasi-free state  $\varphi_U$  is trivial. Using Haagerup's Theorem, it follows that there always exists a faithful normal state  $\psi$  on  $\mathcal{M}$  such that  $(\mathcal{M}^\psi)' \cap \mathcal{M} = \mathbf{C}$ .

### 1. INTRODUCTION

Let  $\mathcal{M}$  be a separable type III<sub>1</sub> factor and let  $\varphi$  be a faithful normal (f.n.) state on  $\mathcal{M}$ . For any  $x, y \in \mathcal{M}$ , set  $[x, y] = xy - yx$  and  $[x, \varphi] = x\varphi - \varphi x$ . The *asymptotic centralizer* of  $\varphi$  is defined by

$$\text{AC}(\varphi) := \{(x_n) \in \ell^\infty(\mathbf{N}, \mathcal{M}) : \|[x_n, \varphi]\| \rightarrow 0\}.$$

Note that  $\text{AC}(\varphi)$  is a unital  $C^*$ -subalgebra of  $\ell^\infty(\mathbf{N}, \mathcal{M})$ . The *bicentralizer* of  $\varphi$  is defined by

$$\text{AB}(\varphi) := \{a \in \mathcal{M} : [a, x_n] \rightarrow 0 \text{ ultrastrongly}, \forall (x_n) \in \text{AC}(\varphi)\}.$$

It is well-known that  $\text{AB}(\varphi)$  is a von Neumann subalgebra of  $\mathcal{M}$ , globally invariant under the modular group  $(\sigma_t^\varphi)$ . Moreover,  $\text{AB}(\varphi) \subset (\mathcal{M}^\varphi)' \cap \mathcal{M}$ . If  $\text{AB}(\varphi) = \mathbf{C}$ , it follows from the Connes-Størmer Transitivity Theorem ([5]) that  $\text{AB}(\psi) = \mathbf{C}$  for any faithful normal state  $\psi$  on  $\mathcal{M}$ . We shall say in this case that  $\mathcal{M}$  has *trivial bicentralizer*. Connes conjectured that *any* separable type III<sub>1</sub> factor should have trivial bicentralizer. If there exists a faithful normal state  $\varphi$  on  $\mathcal{M}$  such that  $(\mathcal{M}^\varphi)' \cap \mathcal{M} = \mathbf{C}$ , then  $\mathcal{M}$  has trivial bicentralizer. Haagerup proved in [7] that the converse holds true. Haagerup's Theorem leads to the uniqueness of the amenable type III<sub>1</sub> factor (see [4]). The following type III<sub>1</sub> factors are known to have trivial bicentralizer:

- (1) The unique amenable III<sub>1</sub> factor (Haagerup, [7]).
- (2) Full factors that have almost periodic states (Connes, [3]).
- (3) Free products  $(\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2)$  such that the centralizers  $\mathcal{M}_i^{\varphi_i}$  have enough unitaries (Barnett, [1]).

In this paper, we show that the bicentralizer is trivial for a large class of type III<sub>1</sub> factors, namely the *free Araki-Woods factors* of Shlyakhtenko ([14]). We briefly recall the construction here; see Section 2 for more details. To each real separable

---

Received by the editors October 7, 2008, and, in revised form, February 16, 2009.

2000 *Mathematics Subject Classification*. Primary 46L10, 46L54.

*Key words and phrases*. Free Araki-Woods factors, Connes' bicentralizer problem.

Hilbert space  $H_{\mathbf{R}}$  together with an orthogonal representation  $(U_t)$  of  $\mathbf{R}$  on  $H_{\mathbf{R}}$ , one can associate a von Neumann algebra denoted by  $\Gamma(H_{\mathbf{R}}, U_t)''$ , called the free Araki-Woods von Neumann algebra. This is the free analog of the factors coming from the CAR relations. The von Neumann algebra  $\Gamma(H_{\mathbf{R}}, U_t)''$  comes equipped with a unique *free quasi-free state* denoted by  $\varphi_U$ , which is always normal and faithful on  $\Gamma(H_{\mathbf{R}}, U_t)''$ . If  $\dim H_{\mathbf{R}} \geq 2$ , then  $\Gamma(H_{\mathbf{R}}, U_t)''$  is a full factor. It is of type III<sub>1</sub> when  $(U_t)$  is non-periodic and non-trivial. If the representation  $(U_t)$  is almost periodic, then  $\varphi_U$  is an almost periodic state and it follows from [14] that the relative commutant of the centralizer of the free quasi-free state is trivial, i.e. if  $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$ , then  $(\mathcal{M}^{\varphi_U})' \cap \mathcal{M} = \mathbf{C}$ . In the almost periodic case, results in [6] yield  $\mathcal{M}^{\varphi_U} \simeq L(\mathbf{F}_{\infty})$ .

When the representation  $(U_t)$  has no eigenvectors (e.g.  $U_t = \lambda_t$ , the left regular representation of  $\mathbf{R}$  on  $L^2(\mathbf{R}, \mathbf{R})$ ), then the centralizer  $\mathcal{M}^{\varphi_U}$  is trivial. It was unknown in general whether or not  $\Gamma(H_{\mathbf{R}}, U_t)''$  has trivial bicentralizer. Even though the centralizer of the free quasi-free state  $\varphi_U$  may be trivial, we will show that the bicentralizer of  $\varphi_U$  is always trivial. The main result of this paper is the following:

**Theorem.** *Let  $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$  be a free Araki-Woods factor of type III<sub>1</sub>. Denote by  $\varphi_U$  the free quasi-free state. Then  $\text{AB}(\varphi_U) = \mathbf{C}$ . Consequently, there always exists a faithful normal state  $\psi$  on  $\mathcal{M}$  such that  $(\mathcal{M}^{\psi})' \cap \mathcal{M} = \mathbf{C}$ .*

## 2. PRELIMINARIES

**2.1. Preliminaries on spectral analysis.** We shall need a few definitions and results from the spectral theory of abelian automorphism groups. Let  $(\alpha_t)$  be an ultraweakly continuous one-parameter automorphism group on a von Neumann algebra  $\mathcal{M}$ . For  $f \in L^1(\mathbf{R})$  and  $x \in \mathcal{M}$ , set

$$\alpha_f(x) = \int_{-\infty}^{+\infty} f(t)\alpha_t(x) dt.$$

The  $\alpha$ -spectrum  $\text{Sp}_{\alpha}(x)$  of  $x \in \mathcal{M}$  is defined as the set of characters  $\gamma \in \widehat{\mathbf{R}}$  for which  $\widehat{f}(\gamma) = 0$ , for all  $f \in L^1(\mathbf{R})$  satisfying  $\alpha_f(x) = 0$ . We shall identify  $\widehat{\mathbf{R}}$  with  $\mathbf{R}$  in the usual way such that

$$\widehat{f}(\gamma) = \int_{-\infty}^{+\infty} e^{i\gamma t} f(t) dt, \quad \forall \gamma \in \mathbf{R}, \forall f \in L^1(\mathbf{R}).$$

For  $z \in \mathbf{C}$ , denote by  $\Im(z)$  its imaginary part.

**Lemma 2.1** ([7]). *Let  $\mathcal{M}$  and  $(\alpha_t)$  be as above. Let  $x \in \mathcal{M}$  and  $\delta > 0$ . If the function  $t \mapsto \alpha_t(x)$  can be extended to an entire (analytic)  $\mathcal{M}$ -valued function such that*

$$\|\alpha_z(x)\| \leq C e^{\delta|\Im(z)|}, \quad \forall z \in \mathbf{C},$$

*for some constant  $C > 0$ , then  $\text{Sp}_{\alpha}(x) \subset [-\delta, \delta]$ .*

Let  $\varphi$  be a f.n. state on a von Neumann algebra  $\mathcal{M}$ . Denote by  $(\sigma_t^{\varphi})$  the modular group on  $\mathcal{M}$  of the state  $\varphi$ . Denote by  $L^2(\mathcal{M}, \varphi)$  the  $L^2$ -space associated with  $\varphi$  and by  $\xi_{\varphi}$  the canonical cyclic separating vector. We shall write  $\|x\|_{\varphi} = \varphi(x^*x)^{1/2}$ , for any  $x \in \mathcal{M}$ . On bounded subsets of  $\mathcal{M}$ , the topology given by the norm  $\|\cdot\|_{\varphi}$  coincides with the strong operator topology. Recall that  $S_{\varphi}^0 : x\xi_{\varphi} \mapsto x^*\xi_{\varphi}$  is a closable (densely defined) operator on  $L^2(\mathcal{M}, \varphi)$ . Denote by  $S_{\varphi}$  its closure and

write  $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$  for its polar decomposition. Note that  $L^2(\mathcal{M}, \varphi)$  is naturally endowed with an  $\mathcal{M}$ - $\mathcal{M}$  bimodule structure defined as follows:

$$\begin{aligned} x \cdot \xi &:= x\xi, \\ \xi \cdot x &:= J_\varphi x^* J_\varphi \xi, \quad \forall x \in \mathcal{M}, \forall \xi \in L^2(\mathcal{M}, \varphi). \end{aligned}$$

We shall denote  $x \cdot \xi$  and  $\xi \cdot x$  simply by  $x\xi$  and  $\xi x$ . The next lemma is well-known, but we give a proof for the reader's convenience.

**Lemma 2.2.** *Let  $\mathcal{M}$  and  $\varphi$  be as above. Let  $x \in \mathcal{M}$  and  $0 < \delta < 1$ . Assume that  $\text{Sp}_{\sigma^\varphi}(x) \subset [-\delta, \delta]$ . Then  $\|x\xi_\varphi - \xi_\varphi x\| \leq \delta \|x\|_\varphi$ .*

*Proof.* Let  $x \in \mathcal{M}$  and  $0 < \delta < 1$  be such that  $\text{Sp}_{\sigma^\varphi}(x) \subset [-\delta, \delta]$ . Let  $f \in L^1(\mathbf{R})$  be such that the Fourier transform  $\widehat{f}$  vanishes on  $[-\delta, \delta]$ . Since  $\text{Sp}_{\sigma^\varphi}(\sigma_f^\varphi(x)) \subset \text{Sp}_{\sigma^\varphi}(x) \cap \text{support}(\widehat{f}) = \emptyset$  (see [2]), it follows that  $\sigma_f^\varphi(x) = 0$ . We have

$$\begin{aligned} \widehat{f}(\log \Delta_\varphi) x \xi_\varphi &= \int_{-\infty}^{+\infty} f(t) \Delta_\varphi^{it} x \xi_\varphi dt \\ &= \int_{-\infty}^{+\infty} f(t) \sigma_t^\varphi(x) \xi_\varphi dt \\ &= \sigma_f^\varphi(x) \xi_\varphi \\ &= 0. \end{aligned}$$

Thus, by approximating  $\mathbf{1}_{\mathbf{R} \setminus [-\delta, \delta]}$  by such functions  $\widehat{f}$ , we get

$$\mathbf{1}_{\mathbf{R} \setminus [-\delta, \delta]}(\log \Delta_\varphi) x \xi_\varphi = 0;$$

i.e.  $x\xi_\varphi$  is in the spectral subspace of  $\log \Delta_\varphi$  corresponding to the interval  $[-\delta, \delta]$ . Notice that

$$\xi_\varphi x = J_\varphi x^* J_\varphi \xi_\varphi = J_\varphi x^* \xi_\varphi = J_\varphi S_\varphi x \xi_\varphi = \Delta_\varphi^{1/2} x \xi_\varphi.$$

Clearly,  $\sup\{|e^{t/2} - 1| : t \in [-\delta, \delta]\} = e^{\delta/2} - 1$ . Moreover, one can see that the operator  $(1 - \Delta_\varphi^{1/2})\mathbf{1}_{[-\delta, \delta]}(\log \Delta_\varphi)$  is bounded and, to be precise,

$$\|(1 - \Delta_\varphi^{1/2})\mathbf{1}_{[-\delta, \delta]}(\log \Delta_\varphi)\| \leq e^{\delta/2} - 1 \leq \delta,$$

since  $0 < \delta < 1$ . Thus, we get

$$\begin{aligned} \|x\xi_\varphi - \xi_\varphi x\| &= \|(1 - \Delta_\varphi^{1/2})x\xi_\varphi\| \\ &= \|(1 - \Delta_\varphi^{1/2})\mathbf{1}_{[-\delta, \delta]}(\log \Delta_\varphi)x\xi_\varphi\| \\ &\leq \|(1 - \Delta_\varphi^{1/2})\mathbf{1}_{[-\delta, \delta]}(\log \Delta_\varphi)\| \|x\xi_\varphi\| \\ &\leq \delta \|x\|_\varphi. \end{aligned}$$

□

**Lemma 2.3** ([7]). *Let  $\mathcal{M}$  and  $\varphi$  be as above. Let  $(x_n) \in \ell^\infty(\mathbf{N}, \mathcal{M})$ . Then*

$$\lim_n \|x_n \xi_\varphi - \xi_\varphi x_n\| = 0 \iff \lim_n \|x_n \varphi - \varphi x_n\| = 0.$$

**2.2. Preliminaries on Shlyakhtenko’s free Araki-Woods factors.** Recall now the construction of the free Araki-Woods factors due to Shlyakhtenko ([14]). Let  $H_{\mathbf{R}}$  be a real separable Hilbert space and let  $(U_t)$  be an orthogonal representation of  $\mathbf{R}$  on  $H_{\mathbf{R}}$ . Let  $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$  be the complexified Hilbert space. Let  $J$  be the canonical anti-unitary involution on  $H$  defined by

$$J(\xi + i\eta) = \xi - i\eta, \quad \forall \xi, \eta \in H_{\mathbf{R}}.$$

If  $A$  is the infinitesimal generator of  $(U_t)$  on  $H$ , we recall that  $j : H_{\mathbf{R}} \rightarrow H$  defined by  $j(\zeta) = (\frac{2}{A^{-1}+1})^{1/2}\zeta$  is an isometric embedding of  $H_{\mathbf{R}}$  into  $H$ . Moreover  $JAJ = A^{-1}$  and  $JA^{it} = A^{it}J$ , for every  $t \in \mathbf{R}$ . Let  $K_{\mathbf{R}} = j(H_{\mathbf{R}})$ . It is easy to see that  $K_{\mathbf{R}} \cap iK_{\mathbf{R}} = \{0\}$  and that  $K_{\mathbf{R}} + iK_{\mathbf{R}}$  is dense in  $H$ . Write  $T = JA^{-1/2}$ . Then  $T$  is an anti-linear closed invertible operator on  $H$  satisfying  $T = T^{-1}$ . Such an operator is called an *involution* on  $H$ . Moreover,  $K_{\mathbf{R}} = \{\xi \in \text{dom}(T) : T\xi = \xi\}$ .

We introduce the *full Fock space* of  $H$ :

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

The unit vector  $\Omega$  is called the *vacuum vector*. For any  $\xi \in H$ , define the *left creation operator*

$$\ell(\xi) : \mathcal{F}(H) \rightarrow \mathcal{F}(H) : \begin{cases} \ell(\xi)\Omega = \xi, \\ \ell(\xi)(\xi_1 \otimes \dots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \dots \otimes \xi_n. \end{cases}$$

We have  $\|\ell(\xi)\| = \|\xi\|$  and that  $\ell(\xi)$  is an isometry if  $\|\xi\| = 1$ . For any  $\xi \in H$ , we denote by  $s(\xi)$  the real part of  $\ell(\xi)$  given by

$$s(\xi) = \frac{\ell(\xi) + \ell(\xi)^*}{2}.$$

A crucial result of Voiculescu [16] claims that the distribution of the operator  $s(\xi)$  with respect to the vacuum vector state  $\varphi(x) = \langle x\Omega, \Omega \rangle$  is the semicircular law of Wigner supported on the interval  $[-\|\xi\|, \|\xi\|]$ .

**Definition 2.4** (Shlyakhtenko, [14]). Let  $(U_t)$  be an orthogonal representation of  $\mathbf{R}$  on the real Hilbert space  $H_{\mathbf{R}}$ . The *free Araki-Woods von Neumann algebra* associated with  $(H_{\mathbf{R}}, U_t)$ , denoted by  $\Gamma(H_{\mathbf{R}}, U_t)''$ , is defined by

$$\Gamma(H_{\mathbf{R}}, U_t)'' := \{s(\xi) : \xi \in K_{\mathbf{R}}\}''.$$

The vector state  $\varphi_U(x) = \langle x\Omega, \Omega \rangle$  is called the *free quasi-free state* and is faithful on  $\Gamma(H_{\mathbf{R}}, U_t)''$ . Let  $\xi, \eta \in K_{\mathbf{R}}$  and write  $\zeta = \xi + i\eta$ . We have

$$2s(\xi) + 2is(\eta) = \ell(\zeta) + \ell(T\zeta)^*.$$

Thus,  $\Gamma(H_{\mathbf{R}}, U_t)''$  is generated as a von Neumann algebra by the operators of the form  $\ell(\zeta) + \ell(T\zeta)^*$  where  $\zeta \in \text{dom}(T)$ . Note that the modular group  $(\sigma_t^{\varphi_U})$  of the free quasi-free state  $\varphi_U$  is given by  $\sigma_{-t}^{\varphi_U} = \text{Ad}(\mathcal{F}(U_t))$ , where  $\mathcal{F}(U_t) = \text{id} \oplus \bigoplus_{n \geq 1} U_t^{\otimes n}$ . In particular, it satisfies

$$\sigma_{-t}^{\varphi_U} (\ell(\zeta) + \ell(T\zeta)^*) = \ell(U_t\zeta) + \ell(TU_t\zeta)^*, \quad \forall \zeta \in \text{dom}(T), \forall t \in \mathbf{R}.$$

The free Araki-Woods factors provided many new examples of full factors of type III [1, 2, 11]. We can summarize the general properties of the free Araki-Woods factors in the following theorem (see also [15]):

**Theorem 2.5** (Shlyakhtenko, [11, 12, 13, 14]). *Let  $(U_t)$  be an orthogonal representation of  $\mathbf{R}$  on the real Hilbert space  $H_{\mathbf{R}}$  with  $\dim H_{\mathbf{R}} \geq 2$ . Write  $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$ .*

- (1)  $\mathcal{M}$  is a full factor and Connes' invariant  $\tau(\mathcal{M})$  is the weakest topology on  $\mathbf{R}$  that makes the map  $t \mapsto U_t$  strongly continuous.
- (2)  $\mathcal{M}$  is of type  $\text{II}_1$  iff  $U_t = \text{id}$ , for every  $t \in \mathbf{R}$ .
- (3)  $\mathcal{M}$  is of type  $\text{III}_\lambda$  ( $0 < \lambda < 1$ ) iff  $(U_t)$  is periodic of period  $\frac{2\pi}{|\log \lambda|}$ .
- (4)  $\mathcal{M}$  is of type  $\text{III}_1$  in the other cases.
- (5) The factor  $\mathcal{M}$  has almost periodic states iff  $(U_t)$  is almost periodic.

Let  $H_{\mathbf{R}} = \mathbf{R}^2$  and  $0 < \lambda < 1$ . Let

$$(1) \quad U_t^\lambda = \begin{pmatrix} \cos(t \log \lambda) & -\sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{pmatrix}.$$

*Notation 2.6* ([14]). Write  $(T_\lambda, \varphi_\lambda) := (\Gamma(H_{\mathbf{R}}, U_t)'', \varphi_U)$  where  $H_{\mathbf{R}} = \mathbf{R}^2$  and  $(U_t)$  is given by equation (1).

Using a powerful tool called the *matricial model*, Shlyakhtenko was able to prove the following isomorphism:

$$(T_\lambda, \varphi_\lambda) \cong (\mathbf{B}(\ell^2(\mathbf{N})), \psi_\lambda) * (L^\infty[-1, 1], \mu),$$

where  $\psi_\lambda(e_{ij}) = \delta_{ij} \lambda^j (1 - \lambda)$ ,  $i, j \in \mathbf{N}$ , and  $\mu$  is a non-atomic measure on  $[-1, 1]$ . The notation  $\cong$  means a state-preserving isomorphism. He also proved that  $(T_\lambda, \varphi_\lambda)$  has the *free absorption property*, namely, that

$$(T_\lambda, \varphi_\lambda) * L(\mathbf{F}_\infty) \cong (T_\lambda, \varphi_\lambda).$$

### 3. THE MAIN RESULT

**3.1. Technical lemmas.** As we said before, the centralizer of the free quasi-free state may be trivial; this is the case for instance when the orthogonal representation  $(U_t)$  on  $H_{\mathbf{R}}$  has no eigenvectors. Nevertheless, the following lemma shows that for any free Araki-Woods von Neumann algebra, there exists a non-trivial sequence of unitaries  $(u_n)$  in the asymptotic centralizer of the free quasi-free state  $\varphi_U$ .

**Lemma 3.1** (Vaes, [15]). *Let  $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$  be a free Araki-Woods von Neumann algebra. Denote by  $\varphi$  the free quasi-free state and by  $(\sigma_t)$  the modular group of the state  $\varphi$ . Then there exists a sequence of unitaries  $(u_n)$  in  $\mathcal{M}$ , entire (analytic) w.r.t.  $(\sigma_t)$ , such that*

- (1)  $\|\sigma_z(u_n) - u_n\| \rightarrow 0$  uniformly on compact sets of  $\mathbf{C}$ ,
- (2)  $\varphi(u_n) \rightarrow 0$ ,
- (3)  $(u_n) \in \text{AC}(\varphi)$ .

*Proof.* This lemma, with the exception of item (3), is Vaes' result (see Lemma 4.3 in [15]). Item (3) was not observed by Vaes but is immediate from the construction using Lemmas 2.1, 2.2 and 2.3. □

The following lemma is a generalization of Barnett's lemma (see [1]), which was itself a generalization of Murray and Neumann's  $14\epsilon$  lemma.

**Lemma 3.2** (Vaes, [15]). *For  $i = 1, 2$ , let  $(\mathcal{M}_i, \varphi_i)$  be a von Neumann algebra endowed with an f.n. state. Denote by  $(\mathcal{M}, \varphi) = (\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2)$  the free*

product. Let  $a \in \mathcal{M}_1$  and  $b, c \in \mathcal{M}_2$ . Assume that  $a, b, c$  belong to the domain of  $\sigma_{i/2}^\varphi$ . Then, for every  $x \in \mathcal{M}$ ,

$$\|x - \varphi(x)1\|_\varphi \leq \mathcal{E}(a, b, c) \max \{ \| [x, a] \|_\varphi, \| [x, b] \|_\varphi, \| [x, c] \|_\varphi \} + \mathcal{F}(a, b, c) \|x\|_\varphi$$

where

$$\begin{aligned} \mathcal{E}(a, b, c) &= 6\|a\|^3 + 4\|b\|^3 + 4\|c\|^3, \\ \mathcal{F}(a, b, c) &= 3\mathcal{C}(a) + 2\mathcal{C}(b) + 2\mathcal{C}(c) + 12|\varphi(cb^*)| \|cb^*\|, \\ \mathcal{C}(a) &= 2\|a\|^3 \|\sigma_{i/2}^\varphi(a) - a\| + 2\|a\|^2 \|a^*a - 1\| \\ &\quad + 3(1 + \|a\|^2) \|aa^* - 1\| + 6|\varphi(a)| \|a\|. \end{aligned}$$

**3.2. Proof of the theorem.** Let  $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$  be a free Araki-Woods factor of type III<sub>1</sub> and denote by  $\varphi$  the free quasi-free state. We recall that such a factor can always be written as the free product of three free Araki-Woods von Neumann algebras (see the proof of Theorem 2.7 in [15]):

$$(\mathcal{M}, \varphi) \cong (\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2) * (\mathcal{M}_3, \varphi_3).$$

Notice that  $\sigma_t^\varphi = \sigma_t^{\varphi_1} * \sigma_t^{\varphi_2} * \sigma_t^{\varphi_3}, \forall t \in \mathbf{R}$ .

Thanks to Lemma 3.1, we may choose three sequences of unitaries  $(u_n^j)$ , for  $j \in \{1, 2, 3\}$ , such that  $u_n^j \in \mathcal{U}(\mathcal{M}_j)$  is analytic w.r.t.  $(\sigma_t^{\varphi_j})$  and satisfies conditions (1) – (3) of Lemma 3.1, for all  $j \in \{1, 2, 3\}$ . The way the sequence of unitaries  $(u_n^j)$  is constructed in Lemma 3.1 (see Lemma 4.3 in [15]) shows that conditions (1) – (3) are satisfied for the state  $\varphi$ ; i.e. the sequence of unitaries  $(u_n^j)$  in  $\mathcal{M}_j$  satisfies, for every  $j \in \{1, 2, 3\}$ ,

- (1)  $\|\sigma_{\frac{1}{2}}^\varphi(u_n^j) - u_n^j\| \rightarrow 0$  uniformly on compact sets of  $\mathbf{C}$ ,
- (2)  $\varphi(u_n^j) \rightarrow 0$ ,
- (3)  $\|[u_n^j, \varphi]\| \rightarrow 0$ .

Moreover, by freeness,  $\varphi(u_n^3(u_n^2)^*) = \varphi(u_n^3)\overline{\varphi(u_n^2)} \rightarrow 0$ .

Assume that  $a \in \text{AB}(\varphi)$ . Fix  $\varepsilon > 0$ . Since  $(u_n^j) \in \text{AC}(\varphi)$ , it follows that  $[a, u_n^j] \rightarrow 0$  ultrastrongly for any  $j \in \{1, 2, 3\}$ , and thus we may choose  $n \in \mathbf{N}$  large enough such that

$$\begin{aligned} \|[a, u_n^j]\|_\varphi &\leq \varepsilon/28, \quad \forall j \in \{1, 2, 3\}, \\ \mathcal{F}(u_n^1, u_n^2, u_n^3) \|a\|_\varphi &\leq \varepsilon/2. \end{aligned}$$

Thus, thanks to Lemma 3.2, we get  $\|a - \varphi(a)1\|_\varphi \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary,  $a = \varphi(a)1$ . Thus  $\text{AB}(\varphi) = \mathbf{C}$ , and we are done.

**3.3. Final remark.** Set  $\mathcal{M} := \Gamma(L^2(\mathbf{R}, \mathbf{R}), \lambda_t)''$ , the free Araki-Woods factor associated with the left regular representation  $(\lambda_t)$  of  $\mathbf{R}$  on the real Hilbert space  $L^2(\mathbf{R}, \mathbf{R})$ . Shlyakhtenko showed in [13] that the continuous core of  $\mathcal{M}$  is isomorphic to  $L(\mathbf{F}_\infty) \otimes \mathbf{B}(\ell^2)$  and that the dual action is precisely the one constructed by Rădulescu in [9]. As observed in [10], for any f.n. state  $\varphi$  on  $\mathcal{M}$ , the centralizer  $\mathcal{M}^\varphi$  is amenable. Indeed, first we have

$$(2) \quad \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R} \simeq L(\mathbf{F}_\infty) \otimes \mathbf{B}(\ell^2).$$

Choose on the left-hand side of (2) a non-zero projection  $p \in L(\mathbf{R})$  such that  $\text{Tr}(p) < +\infty$ . We know that  $p(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R})p \simeq L(\mathbf{F}_\infty)$  is *solid* by Ozawa’s result ([8]). Since  $L(\mathbf{R})p$  is diffuse in  $p(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R})p$ , its relative commutant must be amenable.

In particular  $\mathcal{M}^\varphi \otimes L(\mathbf{R})p$  is amenable. Thus,  $\mathcal{M}^\varphi$  is amenable. Consequently, we obtain

**Corollary 3.3.** *Let  $\mathcal{M} := \Gamma(L^2(\mathbf{R}, \mathbf{R}), \lambda_t)''$ . Then there exists an f.n. state  $\psi$  on  $\mathcal{M}$  such that  $(\mathcal{M}^\psi)' \cap \mathcal{M} = \mathbf{C}$ . Moreover,  $\mathcal{M}^\psi$  is isomorphic to the unique hyperfinite  $\text{II}_1$  factor.*

#### ACKNOWLEDGEMENTS

Part of this work was done while the author was visiting the University of Tokyo. He gratefully thanks Professors Y. Kawahigashi and N. Ozawa for their kind invitation.

#### REFERENCES

- [1] L. Barnett, *Free product von Neumann algebras of type III*. Proc. Amer. Math. Soc. **123** (1995), 543–553. MR1224611 (95c:46096)
- [2] A. Connes, *Une classification des facteurs de type III*. Ann. Sci. École Norm. Sup. **6** (1973), 133–252. MR0341115 (49:5865)
- [3] A. Connes, *Almost periodic states and factors of type  $\text{III}_1$* . J. Funct. Anal. **16** (1974), 415–445. MR0358374 (50:10840)
- [4] A. Connes, *Factors of type  $\text{III}_1$ , property  $L'_\lambda$  and closure of inner automorphisms*. J. Operator Theory **14** (1985), 189–211. MR789385 (88b:46088)
- [5] A. Connes and E. Størmer, *Homogeneity of the state space of factors of type  $\text{III}_1$* . J. Funct. Anal. **28** (1978), 187–196. MR0470689 (57:10435)
- [6] K. Dykema, *Free products of finite-dimensional and other von Neumann algebras with respect to non-tracial states*. Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun. **12**. Amer. Math. Soc., Providence, RI, 1997, pp. 41–88. MR1426835 (98c:46131)
- [7] U. Haagerup, *Connes' bicentralizer problem and uniqueness of the injective factor of type  $\text{III}_1$* . Acta Math. **69** (1986), 95–148. MR880070 (88f:46117)
- [8] N. Ozawa, *Solid von Neumann algebras*. Acta Math. **192** (2004), 111–117. MR2079600 (2005e:46115)
- [9] F. Rădulescu, *A one-parameter group of automorphisms of  $L(\mathbf{F}_\infty) \otimes \mathbf{B}(H)$  scaling the trace*. C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), 1027–1032. MR1168529 (93i:46111)
- [10] D. Shlyakhtenko, *On multiplicity and free absorption for free Araki-Woods factors*. arXiv:math.OA/0302217
- [11] D. Shlyakhtenko, *On the classification of full factors of type III*. Trans. Amer. Math. Soc. **356** (2004), 4143–4159. MR2058841 (2005a:46124)
- [12] D. Shlyakhtenko, *A-valued semicircular systems*. J. Funct. Anal. **166** (1999), 1–47. MR1704661 (2000j:46124)
- [13] D. Shlyakhtenko, *Some applications of freeness with amalgamation*. J. Reine Angew. Math. **500** (1998), 191–212. MR1637501 (99j:46079)
- [14] D. Shlyakhtenko, *Free quasi-free states*. Pacific J. Math. **177** (1997), 329–368. MR1444786 (98b:46086)
- [15] S. Vaes, *États quasi-libres libres et facteurs de type III (d'après D. Shlyakhtenko)*. Séminaire Bourbaki, exposé 937, Astérisque **299** (2005), 329–350. MR2167212 (2007d:46056)
- [16] D.-V. Voiculescu, K.J. Dykema and A. Nica, *Free random variables*. CRM Monograph Series **1**. Amer. Math. Soc., Providence, RI, 1992. MR1217253 (94c:46133)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, 520 PORTOLA PLAZA, LOS ANGELES, CALIFORNIA 90095

*E-mail address:* [cyril@math.ucla.edu](mailto:cyril@math.ucla.edu)

*Current address:* CNRS-ENS Lyon, UMPA UMR 5669, 69364 Lyon cedex 7, France

*E-mail address:* [cyril.houdayer@umpa.ens-lyon.fr](mailto:cyril.houdayer@umpa.ens-lyon.fr)