PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 11, November 2009, Pages 3749–3755
S 0002-9939(09)09923-7
Article electronically published on May 21, 2009

FREE ARAKI-WOODS FACTORS AND CONNES' BICENTRALIZER PROBLEM

CYRIL HOUDAYER

(Communicated by Marius Junge)

ABSTRACT. We show that for any type III_1 free Araki-Woods factor $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$, the bicentralizer of the free quasi-free state φ_U is trivial. Using Haagerup's Theorem, it follows that there always exists a faithful normal state ψ on $\mathcal{M}$ such that $(\mathcal{M}^\psi)' \cap \mathcal{M} = \mathbf{C}$.

1. INTRODUCTION

Let $\mathcal{M}$ be a separable type III_1 factor and let φ be a faithful normal (f.n.) state on $\mathcal{M}$. For any $x, y \in \mathcal{M}$, set $[x, y] = xy - yx$ and $[x, \varphi] = x\varphi - \varphi x$. The *asymptotic centralizer* of φ is defined by

$$\mathrm{AC}(\varphi) := \{(x_n) \in \ell^\infty(\mathbf{N}, \mathcal{M}) : \|[x_n, \varphi]\| \to 0\}.$$

Note that $\mathrm{AC}(\varphi)$ is a unital C*-subalgebra of $\ell^\infty(\mathbf{N}, \mathcal{M})$. The *bicentralizer* of φ is defined by

$$\mathrm{AB}(\varphi) := \{a \in \mathcal{M} : [a, x_n] \to 0 \text{ ultrastrongly}, \forall (x_n) \in \mathrm{AC}(\varphi)\}.$$

It is well-known that $\mathrm{AB}(\varphi)$ is a von Neumann subalgebra of $\mathcal{M}$, globally invariant under the modular group (σ_t^φ). Moreover, $\mathrm{AB}(\varphi) \subset (\mathcal{M}^\varphi)' \cap \mathcal{M}$. If $\mathrm{AB}(\varphi) = \mathbf{C}$, it follows from the Connes-Størmer Transitivity Theorem ([5]) that $\mathrm{AB}(\psi) = \mathbf{C}$ for any faithful normal state ψ on $\mathcal{M}$. We shall say in this case that $\mathcal{M}$ has *trivial bicentralizer*. Connes conjectured that *any* separable type III_1 factor should have trivial bicentralizer. If there exists a faithful normal state φ on $\mathcal{M}$ such that $(\mathcal{M}^\varphi)' \cap \mathcal{M} = \mathbf{C}$, then $\mathcal{M}$ has trivial bicentralizer. Haagerup proved in [7] that the converse holds true. Haagerup's Theorem leads to the uniqueness of the amenable type III_1 factor (see [4]). The following type III_1 factors are known to have trivial bicentralizer:

(1) The unique amenable III_1 factor (Haagerup, [7]).
(2) Full factors that have almost periodic states (Connes, [3]).
(3) Free products $(\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2)$ such that the centralizers $\mathcal{M}_i^{\varphi_i}$ have enough unitaries (Barnett, [1]).

In this paper, we show that the bicentralizer is trivial for a large class of type III_1 factors, namely the *free Araki-Woods factors* of Shlyakhtenko ([14]). We briefly recall the construction here; see Section 2 for more details. To each real separable

Received by the editors October 7, 2008, and, in revised form, February 16, 2009.
2000 *Mathematics Subject Classification.* Primary 46L10, 46L54.
Key words and phrases. Free Araki-Woods factors, Connes' bicentralizer problem.

Hilbert space $H_{\mathbf{R}}$ together with an orthogonal representation (U_t) of $\mathbf{R}$ on $H_{\mathbf{R}}$, one can associate a von Neumann algebra denoted by $\Gamma(H_{\mathbf{R}}, U_t)''$, called the free Araki-Woods von Neumann algebra. This is the free analog of the factors coming from the CAR relations. The von Neumann algebra $\Gamma(H_{\mathbf{R}}, U_t)''$ comes equipped with a unique *free quasi-free state* denoted by φ_U, which is always normal and faithful on $\Gamma(H_{\mathbf{R}}, U_t)''$. If $\dim H_{\mathbf{R}} \geq 2$, then $\Gamma(H_{\mathbf{R}}, U_t)''$ is a full factor. It is of type III_1 when (U_t) is non-periodic and non-trivial. If the representation (U_t) is almost periodic, then φ_U is an almost periodic state and it follows from [14] that the relative commutant of the centralizer of the free quasi-free state is trivial, i.e. if $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$, then $(\mathcal{M}^{\varphi_U})' \cap \mathcal{M} = \mathbf{C}$. In the almost periodic case, results in [6] yield $\mathcal{M}^{\varphi_U} \simeq L(\mathbf{F}_\infty)$.

When the representation (U_t) has no eigenvectors (e.g. $U_t = \lambda_t$, the left regular representation of $\mathbf{R}$ on $L^2(\mathbf{R}, \mathbf{R})$), then the centralizer $\mathcal{M}^{\varphi_U}$ is trivial. It was unknown in general whether or not $\Gamma(H_{\mathbf{R}}, U_t)''$ has trivial bicentralizer. Even though the centralizer of the free quasi-free state φ_U may be trivial, we will show that the bicentralizer of φ_U is always trivial. The main result of this paper is the following:

Theorem. *Let $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor of type* III_1. *Denote by φ_U the free quasi-free state. Then $\mathrm{AB}(\varphi_U) = \mathbf{C}$. Consequently, there always exists a faithful normal state ψ on $\mathcal{M}$ such that $(\mathcal{M}^\psi)' \cap \mathcal{M} = \mathbf{C}$.*

2. Preliminaries

2.1. **Preliminaries on spectral analysis.** We shall need a few definitions and results from the spectral theory of abelian automorphism groups. Let (α_t) be an ultraweakly continuous one-parameter automorphism group on a von Neumann algebra $\mathcal{M}$. For $f \in L^1(\mathbf{R})$ and $x \in \mathcal{M}$, set

$$\alpha_f(x) = \int_{-\infty}^{+\infty} f(t)\alpha_t(x)\, dt.$$

The α-spectrum $\mathrm{Sp}_\alpha(x)$ of $x \in \mathcal{M}$ is defined as the set of characters $\gamma \in \widehat{\mathbf{R}}$ for which $\widehat{f}(\gamma) = 0$, for all $f \in L^1(\mathbf{R})$ satisfying $\alpha_f(x) = 0$. We shall identify $\widehat{\mathbf{R}}$ with $\mathbf{R}$ in the usual way such that

$$\widehat{f}(\gamma) = \int_{-\infty}^{+\infty} e^{i\gamma t} f(t)\, dt, \quad \forall \gamma \in \mathbf{R}, \forall f \in L^1(\mathbf{R}).$$

For $z \in \mathbf{C}$, denote by $\Im(z)$ its imaginary part.

Lemma 2.1 ([7])**.** *Let $\mathcal{M}$ and (α_t) be as above. Let $x \in \mathcal{M}$ and $\delta > 0$. If the function $t \mapsto \alpha_t(x)$ can be extended to an entire (analytic) $\mathcal{M}$-valued function such that*

$$\|\alpha_z(x)\| \leq C e^{\delta |\Im(z)|}, \quad \forall z \in \mathbf{C},$$

for some constant $C > 0$, then $\mathrm{Sp}_\alpha(x) \subset [-\delta, \delta]$.

Let φ be a f.n. state on a von Neumann algebra $\mathcal{M}$. Denote by (σ_t^φ) the modular group on $\mathcal{M}$ of the state φ. Denote by $L^2(\mathcal{M}, \varphi)$ the L^2-space associated with φ and by ξ_φ the canonical cyclic separating vector. We shall write $\|x\|_\varphi = \varphi(x^*x)^{1/2}$, for any $x \in \mathcal{M}$. On bounded subsets of $\mathcal{M}$, the topology given by the norm $\|\cdot\|_\varphi$ coincides with the strong operator topology. Recall that $S_\varphi^0 : x\xi_\varphi \mapsto x^*\xi_\varphi$ is a closable (densely defined) operator on $L^2(\mathcal{M}, \varphi)$. Denote by S_φ its closure and write $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ for its polar decomposition. Note that $L^2(\mathcal{M}, \varphi)$ is naturally endowed with an $\mathcal{M}$-$\mathcal{M}$ bimodule structure defined as follows:

$$\begin{aligned} x \cdot \xi &:= x\xi, \\ \xi \cdot x &:= J_\varphi x^* J_\varphi \xi, \quad \forall x \in \mathcal{M}, \forall \xi \in L^2(\mathcal{M}, \varphi). \end{aligned}$$

We shall denote $x \cdot \xi$ and $\xi \cdot x$ simply by $x\xi$ and ξx. The next lemma is well-known, but we give a proof for the reader's convenience.

Lemma 2.2. *Let $\mathcal{M}$ and φ be as above. Let $x \in \mathcal{M}$ and $0 < \delta < 1$. Assume that $\mathrm{Sp}_{\sigma^\varphi}(x) \subset [-\delta, \delta]$. Then $\|x\xi_\varphi - \xi_\varphi x\| \leq \delta \|x\|_\varphi$.*

Proof. Let $x \in \mathcal{M}$ and $0 < \delta < 1$ be such that $\mathrm{Sp}_{\sigma^\varphi}(x) \subset [-\delta, \delta]$. Let $f \in L^1(\mathbf{R})$ be such that the Fourier transform $\widehat{f}$ vanishes on $[-\delta, \delta]$. Since $\mathrm{Sp}_{\sigma^\varphi}(\sigma_f^\varphi(x)) \subset \mathrm{Sp}_{\sigma^\varphi}(x) \cap \mathrm{support}(\widehat{f}) = \emptyset$ (see [2]), it follows that $\sigma_f^\varphi(x) = 0$. We have

$$\begin{aligned} \widehat{f}(\log \Delta_\varphi) x \xi_\varphi &= \int_{-\infty}^{+\infty} f(t) \Delta_\varphi^{it} x \xi_\varphi \, dt \\ &= \int_{-\infty}^{+\infty} f(t) \sigma_t^\varphi(x) \xi_\varphi \, dt \\ &= \sigma_f^\varphi(x) \xi_\varphi \\ &= 0. \end{aligned}$$

Thus, by approximating $\mathbf{1}_{\mathbf{R} \setminus [-\delta, \delta]}$ by such functions $\widehat{f}$, we get

$$\mathbf{1}_{\mathbf{R} \setminus [-\delta, \delta]}(\log \Delta_\varphi) x \xi_\varphi = 0;$$

i.e. $x\xi_\varphi$ is in the spectral subspace of $\log \Delta_\varphi$ corresponding to the interval $[-\delta, \delta]$. Notice that

$$\xi_\varphi x = J_\varphi x^* J_\varphi \xi_\varphi = J_\varphi x^* \xi_\varphi = J_\varphi S_\varphi x \xi_\varphi = \Delta_\varphi^{1/2} x \xi_\varphi.$$

Clearly, $\sup\{|e^{t/2} - 1| : t \in [-\delta, \delta]\} = e^{\delta/2} - 1$. Moreover, one can see that the operator $(1 - \Delta_\varphi^{1/2}) \mathbf{1}_{[-\delta, \delta]}(\log \Delta_\varphi)$ is bounded and, to be precise,

$$\|(1 - \Delta_\varphi^{1/2}) \mathbf{1}_{[-\delta, \delta]}(\log \Delta_\varphi)\| \leq e^{\delta/2} - 1 \leq \delta,$$

since $0 < \delta < 1$. Thus, we get

$$\begin{aligned} \|x\xi_\varphi - \xi_\varphi x\| &= \|(1 - \Delta_\varphi^{1/2}) x \xi_\varphi\| \\ &= \|(1 - \Delta_\varphi^{1/2}) \mathbf{1}_{[-\delta, \delta]}(\log \Delta_\varphi) x \xi_\varphi\| \\ &\leq \|(1 - \Delta_\varphi^{1/2}) \mathbf{1}_{[-\delta, \delta]}(\log \Delta_\varphi)\| \|x \xi_\varphi\| \\ &\leq \delta \|x\|_\varphi. \end{aligned}$$

□

Lemma 2.3 ([7])**.** *Let $\mathcal{M}$ and φ be as above. Let $(x_n) \in \ell^\infty(\mathbf{N}, \mathcal{M})$. Then*

$$\lim_n \|x_n \xi_\varphi - \xi_\varphi x_n\| = 0 \Longleftrightarrow \lim_n \|x_n \varphi - \varphi x_n\| = 0.$$

2.2. **Preliminaries on Shlyakhtenko's free Araki-Woods factors.** Recall now the construction of the free Araki-Woods factors due to Shlyakhtenko ([14]). Let $H_{\mathbf{R}}$ be a real separable Hilbert space and let (U_t) be an orthogonal representation of $\mathbf{R}$ on $H_{\mathbf{R}}$. Let $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexified Hilbert space. Let J be the canonical anti-unitary involution on H defined by

$$J(\xi + i\eta) = \xi - i\eta, \quad \forall \xi, \eta \in H_{\mathbf{R}}.$$

If A is the infinitesimal generator of (U_t) on H, we recall that $j : H_{\mathbf{R}} \to H$ defined by $j(\zeta) = (\frac{2}{A^{-1}+1})^{1/2}\zeta$ is an isometric embedding of $H_{\mathbf{R}}$ into H. Moreover $JAJ = A^{-1}$ and $JA^{it} = A^{it}J$, for every $t \in \mathbf{R}$. Let $K_{\mathbf{R}} = j(H_{\mathbf{R}})$. It is easy to see that $K_{\mathbf{R}} \cap iK_{\mathbf{R}} = \{0\}$ and that $K_{\mathbf{R}} + iK_{\mathbf{R}}$ is dense in H. Write $T = JA^{-1/2}$. Then T is an anti-linear closed invertible operator on H satisfying $T = T^{-1}$. Such an operator is called an *involution* on H. Moreover, $K_{\mathbf{R}} = \{\xi \in \mathrm{dom}(T) : T\xi = \xi\}$.

We introduce the *full Fock space* of H:

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

The unit vector Ω is called the *vacuum vector*. For any $\xi \in H$, define the *left creation* operator

$$\ell(\xi) : \mathcal{F}(H) \to \mathcal{F}(H) : \begin{cases} \ell(\xi)\Omega = \xi, \\ \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{cases}$$

We have $\|\ell(\xi)\| = \|\xi\|$ and that $\ell(\xi)$ is an isometry if $\|\xi\| = 1$. For any $\xi \in H$, we denote by $s(\xi)$ the real part of $\ell(\xi)$ given by

$$s(\xi) = \frac{\ell(\xi) + \ell(\xi)^*}{2}.$$

A crucial result of Voiculescu [16] claims that the distribution of the operator $s(\xi)$ with respect to the vacuum vector state $\varphi(x) = \langle x\Omega, \Omega\rangle$ is the semicircular law of Wigner supported on the interval $[-\|\xi\|, \|\xi\|]$.

Definition 2.4 (Shlyakhtenko, [14])**.** Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$. The *free Araki-Woods* von Neumann algebra associated with $(H_{\mathbf{R}}, U_t)$, denoted by $\Gamma(H_{\mathbf{R}}, U_t)''$, is defined by

$$\Gamma(H_{\mathbf{R}}, U_t)'' := \{s(\xi) : \xi \in K_{\mathbf{R}}\}''.$$

The vector state $\varphi_U(x) = \langle x\Omega, \Omega\rangle$ is called the *free quasi-free state* and is faithful on $\Gamma(H_{\mathbf{R}}, U_t)''$. Let $\xi, \eta \in K_{\mathbf{R}}$ and write $\zeta = \xi + i\eta$. We have

$$2s(\xi) + 2is(\eta) = \ell(\zeta) + \ell(T\zeta)^*.$$

Thus, $\Gamma(H_{\mathbf{R}}, U_t)''$ is generated as a von Neumann algebra by the operators of the form $\ell(\zeta)+\ell(T\zeta)^*$ where $\zeta \in \mathrm{dom}(T)$. Note that the modular group $(\sigma_t^{\varphi_U})$ of the free quasi-free state φ_U is given by $\sigma_{-t}^{\varphi_U} = \mathrm{Ad}(\mathcal{F}(U_t))$, where $\mathcal{F}(U_t) = \mathrm{id} \oplus \bigoplus_{n\geq 1} U_t^{\otimes n}$. In particular, it satisfies

$$\sigma_{-t}^{\varphi_U}(\ell(\zeta) + \ell(T\zeta)^*) = \ell(U_t\zeta) + \ell(TU_t\zeta)^*, \quad \forall \zeta \in \mathrm{dom}(T), \forall t \in \mathbf{R}.$$

The free Araki-Woods factors provided many new examples of full factors of type III [1, 2, 11]. We can summarize the general properties of the free Araki-Woods factors in the following theorem (see also [15]):

Theorem 2.5 (Shlyakhtenko, [11, 12, 13, 14]). *Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$ with $\dim H_{\mathbf{R}} \geq 2$. Write $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$.*

(1) *$\mathcal{M}$ is a full factor and Connes' invariant $\tau(\mathcal{M})$ is the weakest topology on $\mathbf{R}$ that makes the map $t \mapsto U_t$ strongly continuous.*
(2) *$\mathcal{M}$ is of type II_1 iff $U_t = \mathrm{id}$, for every $t \in \mathbf{R}$.*
(3) *$\mathcal{M}$ is of type III_λ $(0 < \lambda < 1)$ iff (U_t) is periodic of period $\frac{2\pi}{|\log \lambda|}$.*
(4) *$\mathcal{M}$ is of type III_1 in the other cases.*
(5) *The factor $\mathcal{M}$ has almost periodic states iff (U_t) is almost periodic.*

Let $H_{\mathbf{R}} = \mathbf{R}^2$ and $0 < \lambda < 1$. Let

$$U_t^\lambda = \begin{pmatrix} \cos(t \log \lambda) & -\sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{pmatrix}. \tag{1}$$

Notation 2.6 ([14]). *Write $(T_\lambda, \varphi_\lambda) := (\Gamma(H_{\mathbf{R}}, U_t)'', \varphi_U)$ where $H_{\mathbf{R}} = \mathbf{R}^2$ and (U_t) is given by equation (1).*

Using a powerful tool called the *matricial model*, Shlyakhtenko was able to prove the following isomorphism:

$$(T_\lambda, \varphi_\lambda) \cong (\mathbf{B}(\ell^2(\mathbf{N})), \psi_\lambda) * (L^\infty[-1,1], \mu),$$

where $\psi_\lambda(e_{ij}) = \delta_{ij}\lambda^j(1-\lambda)$, $i, j \in \mathbf{N}$, and μ is a non-atomic measure on $[-1,1]$. The notation $\cong$ means a state-preserving isomorphism. He also proved that $(T_\lambda, \varphi_\lambda)$ has the *free absorption property*, namely, that

$$(T_\lambda, \varphi_\lambda) * L(\mathbf{F}_\infty) \cong (T_\lambda, \varphi_\lambda).$$

3. The main result

3.1. **Technical lemmas.** As we said before, the centralizer of the free quasi-free state may be trivial; this is the case for instance when the orthogonal representation (U_t) on $H_{\mathbf{R}}$ has no eigenvectors. Nevertheless, the following lemma shows that for any free Araki-Woods von Neumann algebra, there exists a non-trivial sequence of unitaries (u_n) in the asymptotic centralizer of the free quasi-free state φ_U.

Lemma 3.1 (Vaes, [15]). *Let $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods von Neumann algebra. Denote by φ the free quasi-free state and by (σ_t) the modular group of the state φ. Then there exists a sequence of unitaries (u_n) in $\mathcal{M}$, entire (analytic) w.r.t. (σ_t), such that*

(1) *$\|\sigma_z(u_n) - u_n\| \to 0$ uniformly on compact sets of $\mathbf{C}$,*
(2) *$\varphi(u_n) \to 0$,*
(3) *$(u_n) \in \mathrm{AC}(\varphi)$.*

Proof. This lemma, with the exception of item (3), is Vaes' result (see Lemma 4.3 in [15]). Item (3) was not observed by Vaes but is immediate from the construction using Lemmas 2.1, 2.2 and 2.3. □

The following lemma is a generalization of Barnett's lemma (see [1]), which was itself a generalization of Murray and Neumann's 14ε lemma.

Lemma 3.2 (Vaes, [15]). *For $i = 1, 2$, let $(\mathcal{M}_i, \varphi_i)$ be a von Neumann algebra endowed with an f.n. state. Denote by $(\mathcal{M}, \varphi) = (\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2)$ the free* *product. Let $a \in \mathcal{M}_1$ and $b, c \in \mathcal{M}_2$. Assume that a, b, c belong to the domain of $\sigma^\varphi_{i/2}$. Then, for every $x \in \mathcal{M}$,*

$$\|x - \varphi(x)1\|_\varphi \leq \mathcal{E}(a,b,c) \max\{\|[x,a]\|_\varphi, \|[x,b]\|_\varphi, \|[x,c]\|_\varphi\} + \mathcal{F}(a,b,c)\|x\|_\varphi$$

where

$$\begin{aligned}
\mathcal{E}(a,b,c) &= 6\|a\|^3 + 4\|b\|^3 + 4\|c\|^3, \\
\mathcal{F}(a,b,c) &= 3\mathcal{C}(a) + 2\mathcal{C}(b) + 2\mathcal{C}(c) + 12|\varphi(cb^*)|\|cb^*\|, \\
\mathcal{C}(a) &= 2\|a\|^3\|\sigma^\varphi_{i/2}(a) - a\| + 2\|a\|^2\|a^*a - 1\| \\
&\quad + 3(1 + \|a\|^2)\|aa^* - 1\| + 6|\varphi(a)|\|a\|.
\end{aligned}$$

3.2. **Proof of the theorem.** Let $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor of type III_1 and denote by φ the free quasi-free state. We recall that such a factor can always be written as the free product of three free Araki-Woods von Neumann algebras (see the proof of Theorem 2.7 in [15]):

$$(\mathcal{M}, \varphi) \cong (\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2) * (\mathcal{M}_3, \varphi_3).$$

Notice that $\sigma^\varphi_t = \sigma^{\varphi_1}_t * \sigma^{\varphi_2}_t * \sigma^{\varphi_3}_t, \forall t \in \mathbf{R}$.

Thanks to Lemma 3.1, we may choose three sequences of unitaries (u^j_n), for $j \in \{1,2,3\}$, such that $u^j_n \in \mathcal{U}(\mathcal{M}_j)$ is analytic w.r.t. $(\sigma^{\varphi_j}_t)$ and satisfies conditions $(1)-(3)$ of Lemma 3.1, for all $j \in \{1,2,3\}$. The way the sequence of unitaries (u^j_n) is constructed in Lemma 3.1 (see Lemma 4.3 in [15]) shows that conditions $(1)-(3)$ are satisfied for the state φ; i.e. the sequence of unitaries (u^j_n) in $\mathcal{M}_j$ satisfies, for every $j \in \{1,2,3\}$,

(1) $\|\sigma^\varphi_z(u^j_n) - u^j_n\| \to 0$ uniformly on compact sets of $\mathbf{C}$,
(2) $\varphi(u^j_n) \to 0$,
(3) $\|[u^j_n, \varphi]\| \to 0$.

Moreover, by freeness, $\varphi(u^3_n(u^2_n)^*) = \varphi(u^3_n)\overline{\varphi(u^2_n)} \to 0$.

Assume that $a \in \mathrm{AB}(\varphi)$. Fix $\varepsilon > 0$. Since $(u^j_n) \in \mathrm{AC}(\varphi)$, it follows that $[a, u^j_n] \to 0$ ultrastrongly for any $j \in \{1,2,3\}$, and thus we may choose $n \in \mathbf{N}$ large enough such that

$$\begin{aligned}
\|[a, u^j_n]\|_\varphi &\leq \varepsilon/28, \quad \forall j \in \{1,2,3\}, \\
\mathcal{F}(u^1_n, u^2_n, u^3_n)\|a\|_\varphi &\leq \varepsilon/2.
\end{aligned}$$

Thus, thanks to Lemma 3.2, we get $\|a - \varphi(a)1\|_\varphi \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $a = \varphi(a)1$. Thus $\mathrm{AB}(\varphi) = \mathbf{C}$, and we are done.

3.3. **Final remark.** Set $\mathcal{M} := \Gamma(L^2(\mathbf{R}, \mathbf{R}), \lambda_t)''$, the free Araki-Woods factor associated with the left regular representation (λ_t) of $\mathbf{R}$ on the real Hilbert space $L^2(\mathbf{R}, \mathbf{R})$. Shlyakhtenko showed in [13] that the continuous core of $\mathcal{M}$ is isomorphic to $L(\mathbf{F}_\infty) \otimes \mathbf{B}(\ell^2)$ and that the dual action is precisely the one constructed by Rădulescu in [9]. As observed in [10], for any f.n. state φ on $\mathcal{M}$, the centralizer $\mathcal{M}^\varphi$ is amenable. Indeed, first we have

$$\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R} \simeq L(\mathbf{F}_\infty) \otimes \mathbf{B}(\ell^2). \tag{2}$$

Choose on the left-hand side of (2) a non-zero projection $p \in L(\mathbf{R})$ such that $\mathrm{Tr}(p) < +\infty$. We know that $p(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R})p \simeq L(\mathbf{F}_\infty)$ is *solid* by Ozawa's result ([8]). Since $L(\mathbf{R})p$ is diffuse in $p(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R})p$, its relative commutant must be amenable.

In particular $\mathcal{M}^\varphi \otimes L(\mathbf{R})p$ is amenable. Thus, $\mathcal{M}^\varphi$ is amenable. Consequently, we obtain

Corollary 3.3. *Let $\mathcal{M} := \Gamma(L^2(\mathbf{R}, \mathbf{R}), \lambda_t)''$. Then there exists an f.n. state ψ on $\mathcal{M}$ such that $(\mathcal{M}^\psi)' \cap \mathcal{M} = \mathbf{C}$. Moreover, $\mathcal{M}^\psi$ is isomorphic to the unique hyperfinite* II_1 *factor.*

Acknowledgements

Part of this work was done while the author was visiting the University of Tokyo. He gratefully thanks Professors Y. Kawahigashi and N. Ozawa for their kind invitation.

References

[1] L. Barnett, *Free product von Neumann algebras of type* III. Proc. Amer. Math. Soc. **123** (1995), 543–553. MR1224611 (95c:46096)
[2] A. Connes, *Une classification des facteurs de type* III. Ann. Sci. École Norm. Sup. **6** (1973), 133–252. MR0341115 (49:5865)
[3] A. Connes, *Almost periodic states and factors of type* III_1. J. Funct. Anal. **16** (1974), 415–445. MR0358374 (50:10840)
[4] A. Connes, *Factors of type* III_1*, property* L'_λ *and closure of inner automorphisms.* J. Operator Theory **14** (1985), 189–211. MR789385 (88b:46088)
[5] A. Connes and E. Størmer, *Homogeneity of the state space of factors of type* III_1. J. Funct. Anal. **28** (1978), 187–196. MR0470689 (57:10435)
[6] K. Dykema, *Free products of finite-dimensional and other von Neumann algebras with respect to non-tracial states.* Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun. **12**. Amer. Math. Soc., Providence, RI, 1997, pp. 41–88. MR1426835 (98c:46131)
[7] U. Haagerup, *Connes' bicentralizer problem and uniqueness of the injective factor of type* III_1. Acta Math. **69** (1986), 95–148. MR880070 (88f:46117)
[8] N. Ozawa, *Solid von Neumann algebras.* Acta Math. **192** (2004), 111–117. MR2079600 (2005e:46115)
[9] F. Rădulescu, *A one-parameter group of automorphisms of* $L(\mathbf{F}_\infty)\otimes\mathbf{B}(H)$ *scaling the trace.* C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), 1027–1032. MR1168529 (93i:46111)
[10] D. Shlyakhtenko, *On multiplicity and free absorption for free Araki-Woods factors.* arXiv:math.OA/0302217
[11] D. Shlyakhtenko, *On the classification of full factors of type* III. Trans. Amer. Math. Soc. **356** (2004), 4143–4159. MR2058841 (2005a:46124)
[12] D. Shlyakhtenko, *A-valued semicircular systems.* J. Funct. Anal. **166** (1999), 1–47. MR1704661 (2000j:46124)
[13] D. Shlyakhtenko, *Some applications of freeness with amalgamation.* J. Reine Angew. Math. **500** (1998), 191–212. MR1637501 (99j:46079)
[14] D. Shlyakhtenko, *Free quasi-free states.* Pacific J. Math. **177** (1997), 329–368. MR1444786 (98b:46086)
[15] S. Vaes, *États quasi-libres libres et facteurs de type* III *(d'après D. Shlyakhtenko).* Séminaire Bourbaki, exposé 937, Astérisque **299** (2005), 329–350. MR2167212 (2007d:46056)
[16] D.-V. Voiculescu, K.J. Dykema and A. Nica, *Free random variables.* CRM Monograph Series **1**. Amer. Math. Soc., Providence, RI, 1992. MR1217253 (94c:46133)

Department of Mathematics, University of California, Los Angeles, 520 Portola Plaza, Los Angeles, California 90095

E-mail address: cyril@math.ucla.edu

Current address: CNRS-ENS Lyon, UMPA UMR 5669, 69364 Lyon cedex 7, France

E-mail address: cyril.houdayer@umpa.ens-lyon.fr