PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 11, November 2009, Pages 3787–3795
S 0002-9939(09)09943-2
Article electronically published on June 5, 2009

COMMON HYPERCYCLIC FUNCTIONS FOR MULTIPLES OF CONVOLUTION AND NON-CONVOLUTION OPERATORS

LUIS BERNAL-GONZÁLEZ

(Communicated by Nigel J. Kalton)

Dedicated to the memory of Professor Antonio Aizpuru, who died in March 2008

Abstract. We prove the existence of a residual set of entire functions, all of whose members are hypercyclic for every non-zero scalar multiple of T, where T is the differential operator associated to an entire function of order less than $1/2$. The same result holds if T is a finite-order linear differential operator with non-constant coefficients.

1. Introduction

In this paper, we are concerned with the existence of vectors having dense orbit with respect to each member of a non-denumerable family of operators. Specifically, we deal with the problem of the existence of entire functions that are simultaneously hypercyclic with respect to all non-zero scalar multiples either of a convolution operator or of a linear differential operator with non-constant coefficients. Precise definitions are given below, in this section and in the next one. See [22] and [23] for excellent surveys about hypercyclicity.

Assume that X is a Hausdorff topological vector space and that $T : X \to X$ is a (linear, continuous) operator in it. Then T is said to be *hypercyclic* provided that there is a vector $x \in X$, called *hypercyclic for* T, whose orbit $\{T^k x : k = 0, 1, 2, \dots\}$ under T is dense in X. By $HC(T)$ we denote the subset of all hypercyclic vectors for T. Note that the separability of X is a necessary condition in order that $HC(T) \neq \emptyset$.

It is easy to see that $HC(T)$ is dense in X if T is hypercyclic. If in addition, X is an F-space (that is, X is a completely metrizable topological vector space), then $HC(T)$ is a dense G_δ subset of X; in particular, $HC(T)$ is residual in X. Denote $\mathbb{N} = \{1, 2, \dots\}$. By Baire's category theorem, if $\{T_n : n \in \mathbb{N}\}$ is a denumerable family of hypercyclic vectors on an F-space X, then $\bigcap_{n=1}^{\infty} HC(T_n)$ is still residual (hence non-empty) in X. But Baire's theorem is no longer at our disposal when we

Received by the editors July 7, 2008, and, in revised form, February 23, 2009.

2000 *Mathematics Subject Classification.* Primary 47A16; Secondary 30E10, 47B33.

Key words and phrases. Hypercyclic operators, common hypercyclic vectors, entire functions, linear differential operators, Borel transform.

The author has been partially supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127, by MEC Grant MTM2006-13997-C02-01 and by MEC Acción Especial MTM2006-26627-E.

are facing a non-denumerable family of dense open subsets. Hence the problem is: Given a non-denumerable family $\{T_\lambda : \lambda \in \Lambda\}$ of hypercyclic operators satisfying appropriate conditions, is $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ non-empty?

During the last seven years, much effort has been devoted to this matter; see [1], [5], [6], [7], [8], [12], [13], [14], [15], [16], [20], [27]. Several authors have provided criteria for the existence of common hypercyclic vectors for uncountable families $\{T_\lambda : \lambda \in \Lambda\}$ of operators and, especially, some of the mentioned references deal with the interesting case of a family $\{\lambda T : \lambda \in A\}$, where T is a fixed operator and A is a non-denumerable subset of $\mathbb{K} := \mathbb{R}$ (the real line) or $\mathbb{C}$ (the complex field).

For instance, Abakumov and Gordon [1], and independently Peris [27], proved that the set $\bigcap_{|\lambda|>1} HC(\lambda B)$ is residual in l^2, the space of square summable sequences, where B is the backward shift $(x_n) \mapsto (x_{n+1})$. Gallardo and Partington [20] demonstrated the existence of a residual set of common hypercyclic vectors for the scalar multiples λM_ϕ^* ($|\lambda| > \|1/\phi\|_{L^\infty(\mathbb{T})}$) of the adjoint of the multiplier M_ϕ in the Hardy space, where ϕ is a bounded non-outer function and $\mathbb{T}$ is the unit circle. Costakis and Sambarino [16, Corollary 3] (see also [8, Example 4.6]) proved that $\bigcap_{\lambda\in\mathbb{C}\setminus\{0\}} HC(\lambda D)$ is residual in the space $H(\mathbb{C})$ of entire functions (endowed with compact convergence), where $D : f \in H(\mathbb{C}) \mapsto f' \in H(\mathbb{C})$ is the differentiation operator.

We focus our attention on the last finding. Costakis and Mavroudis [15, Theorem 1.1] improved this result by showing that, in fact, the set $\bigcap_{\lambda\in\mathbb{C}\setminus\{0\}} HC(\lambda p(D))$ is residual in $H(\mathbb{C})$ if p is any non-constant polynomial. Observe that $p(D)$ is an operator on $H(\mathbb{C})$ commuting with translations T_a ($a \in \mathbb{C}$) and, in addition, that it is not a scalar multiple of the identity. Each translation operator T_a is defined as $(T_a f)(z) = f(z + a)$ ($f \in H(\mathbb{C})$, $z \in \mathbb{C}$). In 1991, Godefroy and Shapiro [21] prove that every operator T on $H(\mathbb{C})$ that is not a scalar multiple of the identity and that commutes with translations is hypercyclic. Their result unifies and covers both theorems by Birkhoff [10] ($T = T_a$, $a \in \mathbb{C} \setminus \{0\}$) and MacLane [25] ($T = D$). We wonder whether Costakis-Mavroudis' statement extends to some "Godefroy-Shapiro" operators.

In Section 3 of this paper, we prove that the answer is affirmative, at least when T is the differential operator associated to an entire function with not too fast growth. By using a different approach, the statement is also extended to finite-order linear differential operators whose coefficients are entire functions. Observe that the last operators do not commute, in general, with translations. Finally, in Section 4, we will discuss how these results may be extended to a simply connected domain.

2. Preliminary results

In order to settle the class of operators we will deal with, we present the following assertion, which can be found in [21].

Proposition 2.1. *Let T be an operator on $H(\mathbb{C})$. Then the following conditions are equivalent:*

(a) *T commutes with translations, i.e. $TT_a = T_aT$ ($a \in \mathbb{C}$).*
(b) *T commutes with differentiation, i.e. $TD = DT$.*

(c) *T is a convolution operator; that is, there exists a finite complex Borel measure μ on $\mathbb{C}$ with compact support such that*

$$(Tf)(z) = \int_{\mathbb{C}} f(z+w)\, d\mu(w) \quad (z \in \mathbb{C},\ f \in H(\mathbb{C})).$$

(d) *There exists an entire function Φ of exponential type such that $T = \Phi(D)$.*

Let us explain the last property. That Φ is *of exponential type* means that there are constants $A, B \in (0, +\infty)$ such that $|\Phi(z)| \le Ae^{B|z|}$ for all $z \in \mathbb{C}$. Recall that the (growth) *order* of a function $\Phi \in H(\mathbb{C})$ is defined as $\rho(\Phi) := \limsup_{r\to\infty} \frac{\log\log M(r,\Phi)}{\log r}$, where $M(r,\Phi) := \max\{|f(z)| : |z| = r\}$ (see [11]). If $\alpha \in [0, +\infty)$, the *α-type* of Φ is $\tau_\alpha(\Phi) := \limsup_{r\to\infty} \frac{\log M(r,\Phi)}{r^\alpha}$. The *exponential type* of Φ is $\tau(\Phi) := \tau_1(\Phi)$. Then Φ is of exponential type if and only if $\tau(\Phi) < +\infty$ (necessarily, $\rho(\Phi) < +\infty$ in this case). Observe that, in general, we have $\rho(\Phi) < \alpha \Rightarrow \tau_\alpha(\Phi) = 0 \Rightarrow \tau_\alpha(\Phi) < +\infty \Rightarrow \rho(\Phi) \le \alpha$. Finally, Φ is said to be *of subexponential type* if $\tau(\Phi) = 0$.

Assume that Φ is an entire function of exponential type having Taylor expansion $\Phi(z) = \sum_{n=0}^\infty a_n z^n$. Then for each $f \in H(\mathbb{C})$ the series $\Phi(D)f := \sum_{n=0}^\infty a_n D^n f = \sum_{n=0}^\infty a_n f^{(n)}$ converges compactly in $\mathbb{C}$, therefore defining on $H(\mathbb{C})$ a (in general, infinite-order) linear differential operator $\Phi(D) = \sum_{n=0}^\infty a_n D^n$, where $D^0 = I =$ the identity operator (see [9]).

By the Malgrange-Ehrenpreis theorem (see [9], [18] or [26]), any non-zero differential operator $\Phi(D)$ is *surjective*. The following statement, which will be used in the proof of the first of our main results, characterizes the *injective* convolution operators on the space of entire functions.

Lemma 2.2. *Let T be an operator on $H(\mathbb{C})$ that commutes with translations. Then the following properties are equivalent:*

(a) *T is injective or, equivalently, T is an onto isomorphism.*
(b) *T is a non-zero constant multiple of a translation.*
(c) *$T = \Phi(D)$, where Φ is an entire function of exponential type having no zeros in $\mathbb{C}$.*

Proof. By Proposition 2.1, we have $T = \Phi(D)$, where Φ is an entire function of exponential type. It is clear that (b) implies (a). If (c) holds, then $\Phi(z) = e^{az+b}$ for certain constants $a, b \in \mathbb{C}$, since the Weierstrass product corresponding to the zeros of Φ reduces to the constant 1 and, by Hadamard's factorization theorem (see [2]), genus$(\Phi) \le \rho(\Phi) \le 1$. Therefore $T = \Phi(D) = e^{aD+b} = \lambda T_a$ (with $\lambda = e^b$), a multiple of a translation. Hence (c) implies (b). Finally, assume that (a) is satisfied but Φ has some zero $\alpha \in \mathbb{C}$. Define $e_\alpha(z) = e^{\alpha z}$. Then e_α is entire and $Te_\alpha = \Phi(D)e_\alpha = \Phi(\alpha)e_\alpha = 0$, so T is not injective, a contradiction. Thus, (a) implies (c), and the proof is finished. □

Note that in part (b) of Lemma 2.2 the case $T = \lambda I$ is possible.

In the proof of Theorem 1.1 in [15], a variant of a general criterion for simultaneous hypercyclicity due to Bayart and Matheron [8, Proposition 4.2] was needed. We also need it, but in a slightly improved form; see Lemma 2.3. Since its proof is a trivial modification of the one in [8, Proposition 4.2], we omit it.

Lemma 2.3. *Let X be a separable Fréchet space and let $T : X \to X$ be an operator. Assume that the following properties are satisfied:*

(i) *There exists a subset* $A \subset \bigcup_{n=1}^{\infty} \operatorname{Ker} T^n$ *such that* A *is dense in* X *and each iterate* T^n *has a right inverse* $S_n : A \to X$ *(i.e.* $T^n S_n x = x$ *for all* $x \in A$*) satisfying* $T^n S_m = S_{m-n}$ *for all* m, n *with* $m > n$*.*

(ii) *There exists* $\lambda_0 \geq 0$ *such that for each* $\lambda > \lambda_0$ *and each* $u \in A$*, the set* $\{\lambda^{-n} S_n u : n \in \mathbb{N}\}$ *is bounded in* X*.*

Then $\bigcap_{\lambda > \lambda_0} HC(\lambda T)$ *is a dense* G_δ *subset of* X*.*

Remark 2.4. In [8, Proposition 4.2] the set A is the whole generalized kernel $\bigcup_{n=1}^{\infty} \operatorname{Ker} T^n$, and it is assumed that there exists a right inverse $S : A \to X$ for T such that $\{\lambda^{-n} S^n u : n \in \mathbb{N}\}$ is bounded ($\lambda > \lambda_0$, $u \in A$). In Lemma 2.3 we are not supposing that $S_n = S^n$ necessarily.

Assume that f is an entire function with $\tau = \tau(f) < +\infty$. The *Borel transform* of f is the function given by the series

$$(Bf)(z) = \sum_{n=0}^{\infty} \frac{n! a_n}{z^{n+1}}, \tag{1}$$

provided that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $\limsup_{n \to \infty} |f^{(n)}(0)|^{1/n} = \tau$ (see [11, pp. 11–12]), the series in (1) defines an analytic function in $\{z : |z| > \tau\}$. The following property, usually known as Polya's representation, can be found in [11, pp. 73–74].

Lemma 2.5. *Suppose that* f *is an entire function of exponential type. Then, for every* $R > \tau(f)$ *and every* $z \in \mathbb{C}$*, we have*

$$f(z) = \frac{1}{2\pi i} \oint_{|t|=R} e^{zt}\, (Bf)(t)\, dt.$$

The next assertion (see [11, p. 30] for a proof), which will be used in the proof of Theorem 3.1, establishes that in the "first half" of the growth range of functions with exponential type, the minimum modulus is unbounded as a function of the radius. If $f \in H(\mathbb{C})$ and $r > 0$, we denote $m(r, f) = \min\{|f(z)| : |z| = r\}$.

Lemma 2.6. *If* f *is entire and* $\tau_{1/2}(f) = 0$*, then* $\limsup_{r \to \infty} m(r, f) = +\infty$*.*

The next lemma contains a theorem by León and Müller [24] about rotations of hypercyclic operators. As noticed in [15], the theorem in [24] was stated in the setting of complex Banach spaces, but it still works on complex topological vector spaces.

Lemma 2.7. *Let* $T : X \to X$ *be an operator acting on a complex topological vector space* X*. If* T *is hypercyclic, then for every complex number* μ *with* $|\mu| = 1$ *the operator* μT *is hypercyclic and in addition* $HC(T) = HC(\mu T)$*.*

The following deep result, due to Delsarte and Lions [17], will be used in our non-convolution result.

Lemma 2.8. *Assume that* $T : H(\mathbb{C}) \to H(\mathbb{C})$ *is a differential operator of the form* $T = D^N + a_{N-1}(z) D^{N-1} + \cdots + a_0(z) I$*, where* $a_0, a_1, \ldots, a_{N-1}$ *are entire functions. Then there exists an onto isomorphism* $U : H(\mathbb{C}) \to H(\mathbb{C})$ *such that* $UT = D^N U$*.*

The next, well known result states that hypercyclicity remains unaltered under conjugation. We omit its easy proof.

Lemma 2.9. *Let $T, R : X \to X$ be two operators on a topological vector space X. Suppose that there exists an onto homeomorphism $\varphi : X \to X$ such that $T = \varphi^{-1} R \varphi$. Then T is hypercyclic if and only if R is hypercyclic. Moreover, $HC(R) = \varphi(HC(T))$.*

Finally, in this section we present an important statement due to S. Ansari [3] about stability of hypercyclicity under powers. Similarly to Lemma 2.7, the assertion was originally established in the setting of Banach spaces, but it holds on any topological vector spaces (see [28]).

Lemma 2.10. *Let $N \in \mathbb{N}$ and $T : X \to X$ be an operator on a topological vector space X. Then T is hypercyclic if and only if T^N is hypercyclic. Even more, we have $HC(T) = HC(T^N)$.*

3. Common hypercyclic entire functions

We will state here the two promised results about existence of common hypercyclic functions. They concern respectively convolution and non-convolution operators.

Since every polynomial p satisfies $\tau_\alpha(p) = 0$ for all $\alpha > 0$, the following theorem extends Theorem 1.1 in [15].

Theorem 3.1. (a) *Let T be a non-zero non-injective convolution operator on $H(\mathbb{C})$. Then there exists $\lambda_0 \in [0, +\infty)$ such that the set $\bigcap_{|\lambda| > \lambda_0} HC(\lambda T)$ is residual in $H(\mathbb{C})$.*

(b) *If Φ is a non-constant entire function with $\tau_{1/2}(\Phi) = 0$, then the set $\bigcap_{\lambda \in \mathbb{C}\setminus\{0\}} HC(\lambda \Phi(D))$ is residual in $H(\mathbb{C})$. In particular, this holds if $\rho(\Phi) < 1/2$.*

Proof. By using Lemma 2.7, we obtain as in the proof of Theorem 1.1 in [15] that if T is an operator on $H(\mathbb{C})$, then $HC(rT) = HC(r\mu T)$ for all $r > 0$ and all μ with $|\mu| = 1$. Therefore $\bigcap_{\lambda > \lambda_0} HC(\lambda T) = \bigcap_{|\lambda| > \lambda_0} HC(\lambda T)$ for all $\lambda_0 \geq 0$. Then it suffices to prove the residuality in $H(\mathbb{C})$ of $\bigcap_{\lambda > \lambda_0} HC(\lambda T)$ (in (a)) and of $\bigcap_{\lambda > 0} HC(\lambda \Phi(D))$ (in (b)).

Assume that T is an operator as in (a). Then by Proposition 2.1 and Lemma 2.2, there exists $\Phi \in H(\mathbb{C})$ with $T = \Phi(D)$, $\Phi \not\equiv 0$, $\tau(\Phi) < +\infty$ and $Z(\Phi) := \{\text{zeros of } \Phi\} \neq \emptyset$. Define

$$\lambda_0 = \lambda_0(\Phi) := \inf \left\{ \sup_{|t|=r} \frac{1}{|\Phi(t)|} : \ r > \operatorname{dist}(0, Z(\Phi)) \right\}. \tag{2}$$

If we show that $\bigcap_{\lambda > \lambda_0} HC(\lambda T)$ is residual for λ_0 defined by (2), then (b) is derived because, from Lemma 2.6, $\lambda_0 = 0$ if Φ is non-constant with $\tau_{1/2}(\Phi) = 0$ (note that $Z(f) \neq \emptyset$ if $\rho(f) < 1$ and f is non-constant).

Consequently, it is enough to demonstrate the residuality of $\bigcap_{\lambda > \lambda_0} HC(\lambda T)$ provided that T is as in (a) and λ_0 is as in (2).

Put $T = \Phi(D)$ as before. Take $\alpha \in Z(\Phi)$ with minimum modulus. Following the proof of Proposition 2.2 in [15], we will try to apply Lemma 2.3. Since $\Phi(\alpha) = 0$, there is a non-zero entire function Ψ with exponential type such that

$$\Phi(z) = (z - \alpha)\Psi(z) \quad (z \in \mathbb{C}).$$

Observe that $\Phi(D)^n = (D-\alpha I)^n \Psi(D)^n$ for all $n \geq 1$. As in [15], a direct calculation shows that $(D-\alpha I)^n (e^{\alpha z} q(z)) = 0$ for every polynomial q with degree$(q) < n$. Consider the set

$$A := \{e^{\alpha z} q(z) : \, q \text{ polynomial}\}.$$

From the denseness of the set of polynomials in $H(\mathbb{C})$ and from the fact that $e^{\alpha z}$ never vanishes in $\mathbb{C}$, we derive that A is dense in $H(\mathbb{C})$. Moreover,

$$A \subset \bigcup_{n=1}^{\infty} \operatorname{Ker} (D-\alpha I)^n \subset \bigcup_{n=1}^{\infty} \operatorname{Ker} \Phi(D)^n = \bigcup_{n=1}^{\infty} \operatorname{Ker} T^n.$$

Note that every function in A is of exponential type. In fact, $\tau(f) \leq |\alpha|$ for all $f \in A$. Fix $R > |\alpha|$. According to Lemma 2.5, we have

$$f(z) = \frac{1}{2\pi i} \oint_{|t|=R} e^{zt}\, (Bf)(t)\, dt \quad (f \in A, \, z \in \mathbb{C}).$$

Let $h(z) = \sum_{n=0}^{\infty} h_n z^n$ be any entire function with exponential type and $R > |\alpha|$ be such that $h(t) \neq 0$ for all t with $|t| = R$. If $f \in A$, define the entire function

$$g(z) := \frac{1}{2\pi i} \oint_{|t|=R} e^{zt}\, \frac{(Bf)(t)}{h(t)}\, dt \quad (z \in \mathbb{C}).$$

Then we obtain that

$$\begin{aligned} h(D)g(z) &= \sum_{n=0}^{\infty} h_n f^{(n)}(z) = \sum_{n=0}^{\infty} \frac{h_n}{2\pi i} \oint_{|t|=R} t^n e^{zt} \frac{(Bf)(t)}{h(t)}\, dt \\ &= \frac{1}{2\pi i} \oint_{|t|=R} \left(\sum_{n=0}^{\infty} h_n t^n \right) e^{zt} \frac{(Bf)(t)}{h(t)}\, dt \\ &= \frac{1}{2\pi i} \oint_{|t|=R} e^{zt} (Bf)(t)\, dt = f(z); \end{aligned}$$

that is, g is a solution of the equation $h(D)g = f$. Note that in order to get the interchange of integration and summation at the third equality, we have used $\sum_{n=0}^{\infty} \oint_{|t|=R} |\frac{h_n}{2\pi i} t^n e^{zt} \frac{(Bf)(t)}{h(t)}|\, dt < +\infty$, which follows easily from $h_n^{1/n} \to 0$, which in turn holds because h is entire.

For fixed $\mu > \lambda_0$, we can select $R(\mu) > |\alpha|$ such that

$$|\Phi(t)| > 1/\mu \quad \text{if} \quad |t| = R(\mu). \tag{3}$$

We define the maps $S_n : A \to H(\mathbb{C})$ $(n \in \mathbb{N})$ by

$$(S_n f)(z) = \frac{1}{2\pi i} \oint_{|t|=R(\mu)} e^{zt} \frac{(Bf)(t)}{\Phi(t)^n}\, dt. \tag{4}$$

Observe that from the previous calculation and from the fact that each power Φ^n is also of exponential type, we get

$$T^n S_n f = \Phi(D)^n S_n f = \Phi^n(D) S_n f = f \quad (f \in A).$$

Consequently, S_n is a right inverse for T^n. Moreover, if $m > n$, one obtains by an analogous calculation that

$$\begin{aligned} T^n S_m f(z) &= \Phi^n(D) \frac{1}{2\pi i} \oint_{|t|=R(\mu)} e^{zt} \frac{(Bf)(t)}{\Phi(t)^m} \, dt \\ &= \frac{1}{2\pi i} \oint_{|t|=R(\mu)} e^{zt} \frac{(Bf)(t)}{\Phi(t)^{m-n}} \, dt = S_{m-n} f(z) \end{aligned}$$

for all $f \in A$ and $z \in \mathbb{C}$. Consequently, condition (i) of Lemma 2.3 is fulfilled.

Now, fix $\lambda > \mu$ and $f \in A$. Then (3) and (4) drive us to

$$\begin{aligned} |\lambda^{-n} S_n f(z)| &= \left| \lambda^{-n} \frac{1}{2\pi i} \oint_{|t|=R(\mu)} e^{zt} \frac{(Bf)(t)}{\Phi(t)^n} \right| \\ &\leq (\mu/\lambda)^n R(\mu) \cdot \sup_{|t|=R(\mu)} |(Bf)(t)| \cdot e^{R(\mu)|z|} \quad (z \in \mathbb{C}). \end{aligned}$$

The right hand side tends compactly to zero in $\mathbb{C}$. Therefore $\lambda^{-n} S_n f \to 0$ $(n \to \infty)$ for every $f \in A$. Thus, condition (ii) of Lemma 2.3 is satisfied for μ instead of λ_0.

Consequently, the set $\bigcap_{\lambda > \mu} HC(\lambda T)$ is a dense G_δ subset of $H(\mathbb{C})$ for each $\mu > \lambda_0$. But $\bigcap_{\lambda > \lambda_0} HC(\lambda T) = \bigcap_{n=1}^\infty \bigcap_{\lambda > \lambda_0 + \frac{1}{n}} HC(\lambda T)$, a countable intersection of dense G_δ subsets of $H(\mathbb{C})$, so it is also a dense G_δ set. This finishes the proof. $\square$

Remark 3.2. It is well known that $H(\mathbb{C})$ is a non-normable space. By the Malgrange-Ehrenpreis theorem, our operators $\Phi(D)$ in the last theorem are onto. They also satisfy the fact that $\bigcup_{n=1}^\infty \operatorname{Ker} \Phi(D)^n$ is dense in $H(\mathbb{C})$. This should be compared with Corollary 4.5 in [8], where it is stated that if T is an onto operator on a *Banach* space X with $\bigcup_{n=1}^\infty \operatorname{Ker} T^n$ dense in X, then $\bigcap_{\lambda > \lambda_0} HC(\lambda T)$ is dense in X for some $\lambda_0 \in [0, +\infty)$.

Next, we establish our assertion on non-convolution differential operators. It also covers Theorem 1.1 in [15].

Theorem 3.3. *Assume that* $T : H(\mathbb{C}) \to H(\mathbb{C})$ *is a differential operator of the form*

$$T = D^N + a_{N-1}(z) D^{N-1} + \cdots + a_1(z) D + a_0(z) I,$$

where $N \in \mathbb{N}$ *and* $a_0, \ldots, a_{N-1}$ *are entire functions. Then the set* $\bigcap_{\lambda \in \mathbb{C} \setminus \{0\}} HC(\lambda T)$ *is residual in* $H(\mathbb{C})$.

Proof. By Lemma 2.8, there is an onto isomorphism $U : H(\mathbb{C}) \to H(\mathbb{C})$ such that $UT = D^N U$. Then $T = U^{-1} D^N U$, so $\lambda T = U^{-1}(\lambda D^N) U$ for all $\lambda \in \mathbb{C}$. From Lemma 2.9, we obtain $HC(\lambda T) = U^{-1}(HC(\lambda D^N))$. For every $\lambda \in \mathbb{C} \setminus \{0\}$, select an Nth-root μ, i.e. $\mu^N = \lambda$. Note that $\mu \in \mathbb{C} \setminus \{0\}$. Thanks to Lemma 2.10, we get $HC(\lambda D^N) = HC(\mu D)$. Therefore, for every $\lambda \in \mathbb{C} \setminus \{0\}$ there exists $\mu \in \mathbb{C} \setminus \{0\}$ with $HC(\lambda T) = U^{-1}(HC(\mu D))$. Hence $\bigcap_{\lambda \in \mathbb{C} \setminus \{0\}} HC(\lambda T) \supset \bigcap_{\mu \in \mathbb{C} \setminus \{0\}} U^{-1}(HC(\mu D)) = U^{-1}(\bigcap_{\mu \in \mathbb{C} \setminus \{0\}} HC(\mu D))$. But the set $\bigcap_{\mu \in \mathbb{C} \setminus \{0\}} HC(\mu D)$ (hence, its image under U^{-1}) is residual in $H(\mathbb{C})$ by [16, Corollary 3]. Consequently, $\bigcap_{\lambda \in \mathbb{C} \setminus \{0\}} HC(\lambda T)$ is residual, as required. $\square$

4. Extension to simply connected domains and open problems

Assume that Ω is a domain, that is, a non-empty connected open subset of $\mathbb{C}$. Consider the space $H(\Omega)$ of holomorphic functions on Ω, endowed with the compact-open topology. Suppose that Φ is an entire function of subexponential type. Then $\Phi(D)$ is a well-defined operator, not only on $H(\mathbb{C})$ but even on $H(\Omega)$ (see [9]).

Assume that T is an operator defined by either $T = \Phi(D)$ (where Φ is a non-constant and $\tau(\Phi) = 0$) or $T = D^N + a_{N-1}(\cdot)D^{N-1} + \cdots + a_0(\cdot)I$ ($N \in \mathbb{N}$; $a_0, \dots, a_{N-1}$ entire). By Theorems 3.1 and 3.3, there is $C \geq 0$ ($C = 0$ for the second kind of operators, and also for the case $T = \Phi(D)$ provided that $\tau_{1/2}(\Phi) = 0$) and an entire function f (hence $f \in H(\Omega)$) whose orbit under each multiple λT ($|\lambda| > C$) is dense in $H(\mathbb{C})$. If Ω is *simply connected* (i.e. its complement in the extended complex plane is connected), then $H(\mathbb{C})$ is dense in $H(\Omega)$ by Runge's theorem (see [19]). Therefore, $f \in \bigcap_{|\lambda|>C} HC(\lambda T)$. Moreover, since the set of such functions f is residual (so dense) in $H(\mathbb{C})$, we obtain, again by Runge's theorem, that $\bigcap_{|\lambda|>C} HC(\lambda T)$ is *dense* in $H(\Omega)$.

But this does *not* prove the residuality of the last set in $H(\Omega)$, because $H(\mathbb{C})$ is not residual in $H(\Omega)$. In fact, for any pair of domains Ω_1, Ω_2 with $\Omega_1 \subset \Omega_2$, $\Omega_1 \neq \Omega_2$, we have that $H(\Omega_2)$ is of the first category in $H(\Omega_1)$. For the sake of completeness, we provide a proof of this assertion. Take a point $a \in (\partial\Omega_1) \cap \Omega_2$ and consider the sets $A_{n,k} := \{f \in H(\Omega_1) : |f(z)| \leq n \text{ for all } z \in \Omega_1 \cap B(a, 1/k)\}$ ($n, k \in \mathbb{N}$), where $B(z_0, r)$ is the disk $\{z : |z - z_0| < r\}$. It is easy to see that each $A_{n,k}$ is closed in $H(\Omega_1)$ and that $H(\Omega_2) \subset \bigcup_{n,k} A_{n,k}$. It is enough to show that, for fixed n, k, the set $A_{n,k}$ has empty interior. For this, fix any non-empty open subset U of $H(\Omega_1)$. Then there are a function $f \in H(\Omega_1)$, a compact set $K \subset \Omega_1$ and a number $\varepsilon > 0$ for which $\{g \in H(\Omega_1) : |g(z) - f(z)| < \varepsilon \text{ for all } z \in K\} \subset U$. Select a point $b \in \Omega_1 \cap B(a, 1/k) \setminus K$. Consider the compact set $L := K \cup \{b\}$ and define $\widetilde{f} : L \to \mathbb{C}$ as

$$\widetilde{f}(z) = \begin{cases} f(z) & \text{if } z \in K, \\ 2n & \text{if } z = b. \end{cases}$$

By Runge's approximation theorem, there exists a rational function g with poles outside Ω_1 (hence $g \in H(\Omega_1)$) such that $|g(z) - \widetilde{f}(z)| < \min\{\varepsilon, 1\}$ ($z \in L$). Therefore $g \in U$ but $|g(b) - 2n| < 1$, so $|g(b)| > 2n - 1 \geq n$. Consequently, $g \notin A_{n,k}$, which proves that $A_{n,k}$ has empty interior.

The comments of this section, together with the results in this paper, raise the following questions:

(a) If T is any non-scalar convolution operator on $H(\mathbb{C})$, is $\bigcap_{\lambda \in \mathbb{C} \setminus \{0\}} HC(\lambda T)$ non-empty/residual?

(b) Does Theorem 3.3 hold for a simply connected domain Ω of $\mathbb{C}$, assuming only that $a_0, \dots, a_{N-1} \in H(\Omega)$ (i.e., not necessarily entire)?

(c) Is $\bigcap_{|\lambda|>C} HC(\lambda T)$ residual in $H(\Omega)$ for some $C \geq 0$, assuming that Ω is simply connected and T is as described at the beginning of this section?

.

References

[1] E. Abakumov and J. Gordon, Common hypercyclic vectors for multiples of backward shift, *J. Funct. Anal.* **200** (2003), 494–504. MR1979020 (2004g:47012)

[2] L.V. Ahlfors, *Complex analysis*, 3rd ed., MacGraw-Hill, London, 1978. MR510197 (80c:30001)
[3] S.I. Ansari, Hypercyclic and cyclic vectors, *J. Funct. Anal.* **128** (1995), 374–383. MR1319961 (96h:47002)
[4] R. Aron, J. Bès, F. León and A. Peris, Operators with common hypercyclic subspaces, *J. Operator Theory* **54** (2005), 251–260. MR2186352 (2006h:47010)
[5] F. Bayart, Common hypercyclic vectors for composition operators, *J. Operator Theory* **52** (2004), 353–370. MR2119275 (2006a:47014)
[6] F. Bayart, Common hypercyclic subspaces, *Integr. Equ. Oper. Theory* **53** (2005), 467–476. MR2187432 (2006k:47016)
[7] F. Bayart, Topological and algebraic genericity of divergence and universality, *Studia Math.* **167** (2005), 161–181. MR2134382 (2006b:46024)
[8] F. Bayart and É. Matheron, How to get common universal vectors, *Indiana Univ. Math. J.* **56** (2007), 553–580. MR2317538 (2008f:47008)
[9] C.A. Berenstein and R. Gay, *Complex analysis and special topics in harmonic analysis*, Springer, New York, 1995. MR1344448 (96j:30001)
[10] G.D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris* **189** (1929), 473–475.
[11] R.P. Boas, *Entire functions*, Academic Press, New York, 1954. MR0068627 (16:914f)
[12] K. Chan and R. Sanders, Common supercyclic vectors for a path of operators, *J. Math. Anal. Appl.* **337** (2008), 646–658. MR2356099 (2008k:47019)
[13] K. Chan and R. Sanders, Two criteria for a path of operators to have common hypercyclic vectors, *J. Operator Theory*, **61** (2009), 191–223.
[14] G. Costakis, Common Cesàro hypercyclic vectors, preprint.
[15] G. Costakis and P. Mavroudis, Common hypercyclic entire functions for multiples of differential operators, *Colloq. Math.* **111** (2008), 199–203. MR2365797 (2008m:47009)
[16] G. Costakis and M. Sambarino, Genericity of wild holomorphic functions and common hypercyclic vectors, *Adv. Math.* **182** (2004), 278–306. MR2032030 (2004k:47009)
[17] J. Delsarte and J.L. Lions, Transmutations d'opérateurs différentiels dans le domaine complexe, *Comment. Math. Helv.* **32** (1957), 113–128. MR0091386 (19:959c)
[18] L. Ehrenpreis, Mean periodic functions. I, *Amer. J. Math.* **77** (1955), 293–328. MR0070047 (16:1122d)
[19] D. Gaier, *Lectures on complex approximation*, Birkhäuser, Boston, 1987. MR894920 (88i:30059b)
[20] E.A. Gallardo-Gutiérrez and J.R. Partington, Common hypercyclic vectors for families of operators, *Proc. Amer. Math. Soc.* **136** (2008), 119–126. MR2350396 (2008i:47021)
[21] G. Godefroy and J.H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, *J. Funct. Anal.* **98** (1991), 229–269. MR1111569 (92d:47029)
[22] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators, *Bull. Amer. Math. Soc.* (*N.S.*) **36** (1999), 345–381. MR1685272 (2000c:47001)
[23] K.-G. Grosse-Erdmann, Recent developments in hypercyclicity, *RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **97** (2003), 273–286. MR2068180 (2005c:47010)
[24] F. León-Saavedra and V. Müller, Rotations of hypercyclic and supercyclic operators, *Integr. Equ. Oper. Theory* **50** (2004), 385–391. MR2104261 (2005g:47009)
[25] G.R. MacLane, Sequences of derivatives and normal families, *J. Anal. Math.* **2** (1952), 72–87. MR0053231 (14:741d)
[26] B. Malgrange, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, *Ann. Inst. Fourier* (*Grenoble*) **6** (1955/1956), 271–355. MR0086990 (19:280a)
[27] A. Peris, Common hypercyclic vectors for backward shifts, *Operator theory seminar*, Michigan State University, 2000-2001.
[28] J. Wengenroth, Hypercyclic operators on non-locally convex spaces, *Proc. Amer. Math. Soc.* **131** (2003), 1759–1761. MR1955262 (2003j:47007)

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, Apdo. 1160, Avda. Reina Mercedes, Sevilla-41080, Spain
E-mail address: lbernal@us.es