PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 11, November 2009, Pages 3815–3821
S 0002-9939(09)09948-1
Article electronically published on June 9, 2009

GENERALIZED DIMENSION DISTORTION UNDER PLANAR SOBOLEV HOMEOMORPHISMS

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(Communicated by Mario Bonk)

ABSTRACT. We prove essentially sharp dimension distortion estimates for planar Sobolev-Orlicz homeomorphisms.

1. INTRODUCTION

Let Ω, $\Omega' \subset \mathbb{R}^2$ be open and connected. We consider homeomorphisms $f\colon \Omega \to \Omega'$ that belong to the Sobolev class $W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^2)$, which means that both component functions of f have locally integrable distributional partial derivatives. It is by now well-known that the Luzin condition (N), which requires that f map Lebesgue null sets to Lebesgue null sets, holds if we additionally assume that $|Df| \in L^2_{\text{loc}}(\Omega)$ [18, 15, 14], but may fail if $|Df| \in L^p_{\text{loc}}(\Omega)$ for some $p < 2$ [16, 17]. On the other hand, if $|Df| \in L^p_{\text{loc}}(\Omega)$ for some $p > 2$, then the image of any set of Hausdorff dimension strictly less than two is also of Hausdorff dimension strictly less than two [4, 10]. Recently it was proven [11] that local integrability of $|Df|^2 \log^{-1}(e + |Df|)$ already suffices for the Luzin condition (N) to hold. The motivation for this result and our results below arises in part from the theory of mappings with finite distortion, where the natural regularity assumption is that $|Df|^2 \log^{\lambda-1}(e+|Df|) \in L^1_{\text{loc}}$ for some $\lambda > 0$ [2, 1, 7, 8, 3].

Analogously to the L^p–scale setting, one expects some kind of dimension distortion estimate to hold when λ as above is strictly positive. However, it is rather easy to map, for example, a subset of the real line onto a set of Hausdorff dimension two [6, 19], and thus we have to work with a refined scale. Towards this end, we consider the gauge functions $h_\lambda(t) = t^2 \log^\lambda \frac{1}{t}$, $\lambda > 0$. In Section 2, we describe a homeomorphism f that maps a Cantor set E of Minkowski (and so also Hausdorff) dimension strictly less than two to a set of positive $\mathcal{H}^{h_\lambda}$-measure, with $|Df|^2 \log^{t-1}(e + |Df|) \in L^1_{\text{loc}}$ for all $t < \lambda$.

Our main result shows that this homeomorphism is critical for our generalized dimension distortion.

Received by the editors October 17, 2008, and, in revised form, February 26, 2009.
2000 *Mathematics Subject Classification.* Primary 30C65.
The first author was supported partially by the Academy of Finland, grant No. 120972.
The second author was supported partially by the Academy of Finland, grant No. 120972.
The third author was supported by the Swiss National Science Foundation and GALA.

Theorem 1.1. *Let Ω and Ω' be open sets in $\mathbb{R}^2$ and $f\colon \Omega \to \Omega'$ a homeomorphism of class $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ with*

$$|Df|^2 \log^{\lambda-1}(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega)$$

for some $\lambda > 0$. Then

$$\mathcal{H}^{h_\lambda}(f(E)) = 0$$

for every set $E \subset \Omega$ of lower Minkowski dimension $\dim_{\mathcal{M}}(E)$ strictly less than two.

We conjecture that one may replace the Minkowski dimension in Theorem 1.1 with the Hausdorff dimension. For a related, weaker result in this direction, see [12].

This note is organized as follows. In Section 2 we recall the necessary definitions and describe the construction of the homeomorphism referred to above. Section 3 contains the proof of Theorem 1.1.

2. Preliminaries

Let $U \subset \mathbb{R}^2$ be open and connected. We say that a mapping $f \in L^1(U;\mathbb{R}^2)$ has *bounded variation*, $f \in BV(U)$, if the component functions f_1 and f_2 of f are of bounded variation, that is,

$$\sup\Big\{\int_U f_i \operatorname{div}\varphi\, dx \,|\, \varphi \in C^1_0(U;\mathbb{R}^2),\ |\varphi| \le 1\Big\} < \infty, \quad i = 1,2.$$

We write $f \in BV_{\mathrm{loc}}(U)$ if $f \in BV(G)$ for each open and connected G compactly contained in U. For each function $g \in BV(U;\mathbb{R})$ of bounded variation we can define a Radon measure $||Dg||$ in the following way: for an open set $V \subset U$ we put

$$||Dg||(V) = \sup\Big\{\int_V g \operatorname{div}\varphi\, dx \,|\, \varphi \in C^1_0(V;\mathbb{R}^2),\ |\varphi| \le 1\Big\},$$

and for $A \subset U$ not necessarily open,

$$||Dg||(A) = \inf\big\{||Dg||(V) \,|\, A \subset V \subset U,\ V \text{ is open}\big\}.$$

For a set V and a number $\delta > 0$, $V + \delta$ denotes the set $\{y |\operatorname{dist}(y,V) < \delta\}$.

We write $\mathcal{H}^h(A)$ for the *generalized Hausdorff measure* of a set A, given by

$$\mathcal{H}^h(A) = \lim_{\delta\to 0} \mathcal{H}^h_\delta(A) = \lim_{\delta \to 0}\Big[\inf\Big\{\sum_{i=1}^\infty h(\operatorname{diam} U_i)\colon A \subset \bigcup_{i=1}^\infty U_i, \operatorname{diam} U_i \le \delta\Big\}\Big],$$

where h is a dimension gauge (non-decreasing, with $h(0) = 0$). If $h(t) = t^\alpha$ for some $\alpha \ge 0$, we write simply $\mathcal{H}^\alpha$ for $\mathcal{H}^{t^\alpha}$ and call it the *Hausdorff α-dimensional measure*; the *Hausdorff dimension* $\dim_{\mathcal{H}}(A)$ of the set A is the smallest $\alpha_0 \ge 0$ such that $\mathcal{H}^\alpha(A) = 0$ for any $\alpha > \alpha_0$. The *lower Minkowski dimension* $\dim_{\mathcal{M}}(A)$ of a bounded set $A \subset \mathbb{R}^2$ is defined as

$$\dim_{\mathcal{M}}(A) = \inf\{s\colon \liminf_{\varepsilon\to 0+} N(A,\varepsilon)\varepsilon^s = 0\},$$

where $N(A,\varepsilon)$, $\varepsilon > 0$, denotes the smallest number of balls of radius ε needed to cover A:

$$N(A,\varepsilon) = \min\Big\{k\colon A \subset \bigcup_{i=1}^k B(x_i,\varepsilon) \text{ for some } x_i \in \mathbb{R}^2\Big\}.$$

Finally, let $a \lesssim b$ mean that there exists some constant $C > 0$ such that $a \le Cb$.

In [6] a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ was constructed which maps a set $\mathcal{C}$ of Minkowski and Hausdorff dimension $n\log 2/\log(1/\sigma)$ for some $0<\sigma<1/2$ onto a set $\mathcal{C}'$ of positive $\mathcal{H}^h$-measure with $h(t)=t^n(\log(1/t))^{pn}$ for given $p>0$. This mapping is the identity outside the cube $[0,1]^n$ and satisfies $|Dh(x)|\le \frac{\tau_1\cdots\tau_k}{\sigma^k}$ in A_{ki}. Here A_{ki}, $k=1,2,\dots$ and $i=1,\dots,2^{kn}$, are the open "cubical frames" needed to construct the Cantor set $\mathcal{C}$. They are pairwise disjoint with respect to both i and k; that is, $\mathrm{int}(A_{ki})\cap\mathrm{int}(A_{lj})=\emptyset$ when $(k,i)\neq(l,j)$, they cover the set $[0,1]^n$ up to a set of zero n-Lebesgue measure, and each A_{ki} is contained in a cube of edge length $(1/2)\sigma^{k-1}$. The numbers τ_k, $k=1,2,\dots$, used to construct the image Cantor-type set are defined as follows:

$$\tau_1=\frac{1}{2}\frac{1}{\log^p 4}\quad\text{and}\quad \tau_k=\frac{1}{2}\big(1-\frac{1}{k}\big)^p\quad\text{for } k=2,3,\dots.$$

Note that

$$\tau_1\cdots\tau_k=\frac{1}{2^k}\frac{1}{\log^p 4}\frac{1}{k^p},$$

so in the case $n=2$, we have

$$\begin{aligned}\int_{[0,1]^2}|Dh|^2\log^s(e+|Dh|)&=\sum_{k=1}^{\infty}\sum_{i=1}^{4^k}\int_{A_{ki}}|Dh|^2\log^s(e+|Dh|)\\&\le\sum_{k=1}^{\infty}4^k\frac{1}{4}\sigma^{2k-2}\frac{(\tau_1\cdots\tau_k)^2}{\sigma^{2k}}\log^s(e+\frac{\tau_1\cdots\tau_k}{\sigma^k})\\&=\sum_{k=1}^{\infty}\frac{1}{4\sigma^2k^{2p}\log^{2p}4}\log^s\big(e+\frac{1}{(2\sigma)^kk^p\log^p 4}\big)\\&\lesssim\sum_{k=1}^{\infty}k^{s-2p}<\infty\end{aligned}$$

when $s+1<2p$.

3. Proofs

Clearly, we may assume in the rest of this paper that Ω is an open and connected subset of $\mathbb{R}^2$. We begin with the following lemma.

Lemma 3.1. *Let $f\colon\Omega\to f(\Omega)\subset\mathbb{R}^2$ be a homeomorphism in $W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$. Then there exists a set $F\subset f(\Omega)$ with $\mathcal{H}^{3/2}(F)=0$ such that for all $y\in f(\Omega)\setminus F$ there exist constants $C_y>0$ and $r_y>0$ such that*

$$\mathrm{diam}(f^{-1}(B(y,r)))\le C_y r^{1/2}\tag{3.1}$$

for all $0<r<r_y$.

Proof. First, note that by Theorem 1.2 in [5], f^{-1} is in $BV_{\mathrm{loc}}(f(\Omega))$. Next, fix $y\in f(\Omega)$ and $r>0$ such that $B(y,3r)\subset f(\Omega)$. Let $Q(y,t)$ be the square centered at y and having edge length $2t$. As f^{-1} is a homeomorphism, for $t\in(r,2r)$ we have

$$\begin{aligned}\mathrm{diam}\, f^{-1}(B(y,r))&<\mathrm{diam}\, f^{-1}(Q(y,t))\le\mathrm{diam}\, f^{-1}(\partial Q(y,t))\\&\le\mathrm{diam}\, f_1^{-1}(\partial Q(y,t))+\mathrm{diam}\, f_2^{-1}(\partial Q(y,t)),\end{aligned}$$

where f_i^{-1}, $i=1,2$, denotes the i-th component function of f^{-1}. Integrating this inequality over the interval $[r,2r]$ with respect to t, we obtain

$$r\operatorname{diam} f^{-1}(B(y,r)) < \sum_{i=1}^{2}\int_{[r,2r]} \operatorname{diam} f_i^{-1}(\partial Q(y,t))dt. \tag{3.2}$$

Let us consider the smooth approximation $g_i^\varepsilon = \eta_\varepsilon * f_i^{-1}$ of f_i^{-1}, $i=1,2$, on the cube $Q(y,2r)$. Here η_ε is a standard bump function. As f^{-1} is continuous, the convergence $g_i^\varepsilon \to f_i^{-1}$ is pointwise and uniform on each compact set $K\subset Q(y,2r)$. So, for $t\in(r,2r)$ and $i=1,2$ we have $\operatorname{diam} f_i^{-1}(\partial Q(y,t)) = \lim_{\varepsilon\to 0}\operatorname{diam} g_i^\varepsilon(\partial Q(y,t))$. Put $a_i = y_i - 2r$ and $b_i = y_i + 2r$, $i=1,2$, where $y=(y_1,y_2)$. Fatou's Lemma implies that

$$\begin{aligned}\int_{[r,2r]} \operatorname{diam} f_i^{-1}(\partial Q(y,t))dt &= \int_{[r,2r]} \lim_{\varepsilon\to 0}\operatorname{diam} g_i^\varepsilon(\partial Q(y,t))dt \\ &\le \liminf_{\varepsilon\to 0}\int_{[r,2r]} \operatorname{diam} g_i^\varepsilon(\partial Q(y,t))dt.\end{aligned}\tag{3.3}$$

We use the fundamental theorem of calculus and Fubini's theorem to obtain

$$\begin{aligned}&\int_{[r,2r]} \operatorname{diam} g_i^\varepsilon(\partial Q(y,t))dt \le \int_{[r,2r]}\left\{\int_{[a_2,b_2]}\left|\frac{\partial g_i^\varepsilon}{\partial \xi}(y_1-t,\xi)\right|d\xi\right. \\ &+\int_{[a_2,b_2]}\left|\frac{\partial g_i^\varepsilon}{\partial \xi}(y_1+t,\xi)\right|d\xi + \int_{[a_1,b_1]}\left|\frac{\partial g_i^\varepsilon}{\partial \xi}(\xi,y_2-t)\right|d\xi \\ &+\left.\int_{[a_1,b_1]}\left|\frac{\partial g_i^\varepsilon}{\partial \xi}(\xi,y_2+t)\right|d\xi\right\}dt = \int_{[a_1,y_1-r]\times[a_2,b_2]}\left|\frac{\partial g_i^\varepsilon}{\partial x_2}(x)\right|dx \\ &+\int_{[y_1+r,b_1]\times[a_2,b_2]}\left|\frac{\partial g_i^\varepsilon}{\partial x_2}(x)\right|dx + \int_{[a_1,b_1]\times[a_2,y_2-r]}\left|\frac{\partial g_i^\varepsilon}{\partial x_1}(x)\right|dx \\ &+\int_{[a_1,b_1]\times[y_2+r,b_2]}\left|\frac{\partial g_i^\varepsilon}{\partial x_1}(x)\right|dx \le \sum_{j=1}^{2}\int_{Q(y,2r)}\left|\frac{\partial g_i^\varepsilon}{\partial x_j}(x)\right|dx\end{aligned}\tag{3.4}$$

for $i=1,2$. Let us show that

$$\int_{Q(y,2r)}\left|\frac{\partial g_i^\varepsilon}{\partial x_j}(x)\right|dx \le ||Df_i^{-1}||(Q(y,2r)) \tag{3.5}$$

for $i,j=1,2$. Given $\varphi\in C_0^1(Q(y,2r))$, $|\varphi|\le 1$, we may write

$$\begin{aligned}\int_{Q(y,2r)}\frac{\partial g_i^\varepsilon}{\partial x_j}\varphi dx &= -\int_{Q(y,2r)} g_i^\varepsilon\frac{\partial \varphi}{\partial x_j}dx = -\int_{Q(y,2r)}(\eta_\varepsilon * f_i^{-1})\frac{\partial\varphi}{\partial x_j}dx \\ &= -\int_{Q(y,2r)} f_i^{-1}\frac{\partial \eta_\varepsilon*\varphi}{\partial x_j}dx \le ||Df_i^{-1}||(Q(y,2r)).\end{aligned}$$

This implies (3.5), and combining it with (3.2), (3.3) and (3.4), we finally obtain

$$\operatorname{diam} f^{-1}(B(y,r)) < \frac{2}{r}(||Df_1^{-1}||(Q(y,2r)) + ||Df_2^{-1}||(Q(y,2r)))$$

for all $y\in f(\Omega)$ and $r>0$ such that $B(y,3r)\subset f(\Omega)$. That is, the inequality (3.1) holds for all $y\in f(\Omega)$ such that

$$\frac{||Df_i^{-1}||(Q(y,2r))}{r^{3/2}} < M_y \tag{3.6}$$

is valid for $i = 1, 2$, all small enough $r > 0$ and some constant M_y depending on y. Let F_1 be the set of those y for which (3.6) does not hold for $i = 1$. Let $K \subset f(\Omega)$ be a compact set and fix some $\delta > 0$ such that $\operatorname{dist}(K, \partial f(\Omega)) > \delta$. For every $k \in \mathbb{N}$ and every $y \in F_1 \cap K$ there exists $r_{k,y} < \delta\sqrt{2}/20$ such that $||Df_1^{-1}||(Q(y, 2r_{k,y})) \geq k(r_{k,y})^{3/2}$. Consider the collection of all balls

$$\mathcal{B}_k = \{B(y, 2\sqrt{2}r_{k,y})\colon y \in F_1 \cap K\}$$

for $k \in \mathbb{N}$. Using Vitali's covering theorem, we obtain for every $k \in \mathbb{N}$ a countable subcollection of disjoint balls $B_{k,j}$, $j = 1, 2, \ldots$, centered in $F_1 \cap K$, having radii $2\sqrt{2}r_j^k < \delta/5$ and with $5B_{k,j}$ covering $F_1 \cap K$. As $Q(y, 2r_j^k) \subset B_{k,j}$, we have

$$\begin{aligned}\mathcal{H}_\delta^{3/2}(F_1 \cap K) &\leq \sum_{j=1}^{\infty}(10\sqrt{2}r_j^k)^{3/2} \leq \frac{(10\sqrt{2})^{3/2}}{k}\sum_{j=1}^{\infty}||Df_1^{-1}||(Q(y, 2r_j^k)) \\ &\leq \frac{(10\sqrt{2})^{3/2}}{k}\sum_{j=1}^{\infty}||Df_1^{-1}||(B_{k,j}) \leq \frac{(10\sqrt{2})^{3/2}||Df_1^{-1}||(K + \delta/5)}{k}\end{aligned}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ and $\delta \to 0$, we obtain $\mathcal{H}^{3/2}(F_1 \cap K) = 0$. $\square$

The previous lemma implies the following result.

Lemma 3.2. *Let $E \subset \Omega$ and let $f\colon \Omega \to f(\Omega) \subset \mathbb{R}^2$ be a homeomorphism of class $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. Then there exists a decomposition $f(E) = \bigcup_{i=0}^{\infty} F_i$ where $\mathcal{H}^{3/2}(F_0) = 0$ and for each F_i, $i = 1, 2, \ldots$, there exist constants $C_i < \infty$ and $r_i > 0$ such that*

$$f^{-1}(F_i + r) \subset E + C_i r^{1/2}$$

for every $r \in (0, r_i)$.

Proof. We choose $F_0 = F$, where F is the set from the previous lemma. Moreover, by this lemma we may represent the set $f(E)$ as

$$f(E) = F_0 \cup \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty}\{y \in f(E) \mid \operatorname{diam}(f^{-1}(B(y, r))) \leq kr^{\frac{1}{2}} \text{ for all } r \in (0, \frac{1}{j})\}.$$

So, putting $C_i = C_{i(j,k)} = k$ and $r_i = r_{i(j,k)} = \frac{1}{j}$, we complete the proof. $\square$

Proof of Theorem 1.1. As $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ is a homeomorphism, its Jacobian J_f is either non-negative almost everywhere in Ω or non-positive almost everywhere in Ω [13]. We may assume that $J_f \geq 0$ almost everywhere in Ω. Recalling that

$$|Df|^2 \log^{\lambda-1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega),$$

by Corollary 9.1 in [9], we have $J_f \log^{\lambda}(e + J_f) \in L^1_{\mathrm{loc}}$. Next, as $\dim_{\mathcal{M}}(E) < 2$, there exist constants $C, \varepsilon > 0$ and a sequence of numbers r_j, $j = 1, 2, \ldots$, tending to zero as $j \to \infty$, such that $\mathcal{L}^2(E + r_j) \leq Cr_j^{\varepsilon}$ for all $j = 1, 2, \ldots$. By Lemma 3.2, we have $f(E) = \bigcup_{i=0}^{\infty} F_i$ where $\mathcal{H}^{3/2}(F_0) = 0$ and $f^{-1}(F_i + R_{i,j}) \subset E + r_j$ for all large enough j ($j \geq j_i$ for some $j_i \in \mathbb{N}$). Here $R_{i,j} = (r_j/C_i)^2$ and C_i are the constants from Lemma 3.2. It suffices to show that $\mathcal{H}^h(F_i) = 0$ for all $i \in \mathbb{N}$. We use the fact that $\mathcal{L}^2(f(A)) \le \int_A J_f$ for each open $A \subset \Omega$ [11, Lemma 3.2]. Thus, for a fixed $i \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{L}^2(F_i + R_{i,j}) &\le \int_{f^{-1}(F_i+R_{i,j})} J_f(x)dx \le \int_{E+r_j} J_f(x)dx \\ &\le \int_{\{x\in E+r_j\,:\, J_f(x)<r_j^{-\varepsilon/2}\}} J_f + \int_{\{x\in E+r_j\,:\, J_f(x)\ge r_j^{-\varepsilon/2}\}} J_f \\ &\le r_j^{-\varepsilon/2}\mathcal{L}^2(E+r_j) + \log^{-\lambda}(e + r_j^{-\varepsilon/2}) \int_{E+r_j} J_f \log^{\lambda}(e+J_f) \\ &\le C r_j^{\varepsilon/2} + M(r_j)\log^{-\lambda}\frac{1}{r_j} \qquad (3.7)\end{aligned}$$

for big enough j, where $M(r) \to 0$ as $r \to 0$. In other words,

$$\mathcal{L}^2(F_i + R_{i,j}) = o(\log^{-\lambda}\frac{1}{r_j})$$

as $j \to \infty$. Using the Besicovitch covering theorem, for each large enough $j \in \mathbb{N}$, we can cover the set F_i with N countable families of pairwise disjoint balls centered in F_i and of radius $R_{i,j}$ (N is independent of both i and j). It is obvious that each of these families is finite. Let $l_{i,j}$ denote the total number of covering balls. We have $\mathcal{L}^2(F_i + R_{i,j}) \ge C l_{i,j} R_{i,j}^2$, where C is a constant independent of i and j. So, for each fixed $i \in \mathbb{N}$ and all big enough $j \ge j_i$, we have

$$\mathcal{H}^h_{R_{i,j}}(F_i) \le l_{i,j}R_{i,j}^2 \log^{\lambda}(1/R_{i,j}) \le \frac{2^{\lambda}}{C}\mathcal{L}^2(F_i+R_{i,j})\log^{\lambda}(C_i/r_j),$$

and thus (3.7) shows that $\mathcal{H}^h(F_i) = 0$. It follows that $\mathcal{H}^h(f(E)) = 0$. $\square$

References

1. K. Astala, T. Iwaniec, P. Koskela, and G. Martin, *Mappings of BMO-bounded distortion*, Math. Ann. **317** (2000), no. 4, 703–726. MR1777116 (2001i:30016)
2. G. David, *Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$*, Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), no. 1, 25–70. MR975566 (90d:30058)
3. D. Faraco, P. Koskela, and X. Zhong, *Mappings of finite distortion: the degree of regularity*, Adv. Math. **190** (2005), no. 2, 300–318. MR2102659 (2005f:30039)
4. F. W. Gehring and J. Visl, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. (2) **6** (1973), 504–512. MR0324028 (48:2380)
5. S. Hencl, P. Koskela, and J. Onninen, *Homeomorphisms of bounded variation*, Arch. Ration. Mech. Anal. **186** (2007), no. 3, 351–360. MR2350361 (2008g:30009)
6. D. A. Herron and P. Koskela, *Mappings of finite distortion: gauge dimension of generalized quasicircles*, Illinois J. Math. **47** (2003), no. 4, 1243–1259. MR2037001 (2005a:30031)
7. T. Iwaniec, P. Koskela, and G. Martin, *Mappings of BMO-bounded distortion and Beltrami-type operators*, J. Anal. Math. **88** (2002), no. 4, 337–381. MR1979776 (2004f:30015)
8. T. Iwaniec, P. Koskela, G. Martin, and C. Sbordone, *Mappings of finite distortion: $L^n \log^\chi L$-integrability*, J. London Math. Soc. (2) **67** (2003), no. 1, 123–136. MR1942415 (2003h:30031)
9. T. Iwaniec and A. Verde, *A study of Jacobians in Hardy-Orlicz spaces*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1999), no. 3, 539–570. MR1693625 (2000g:42025)
10. R. Kaufman, *Sobolev spaces, dimension, and random series*, Proc. Amer. Math. Soc. **128** (2000), no. 2, 427–431. MR1670383 (2000c:28013)
11. J. Kauhanen, P. Koskela, and J. Malý, *Mappings of finite distortion: condition N*, Michigan Math. J. **49** (2001), no. 1, 169–181. MR1827080 (2002d:30027)
12. P. Koskela, A. Zapadinskaya, and T. Zürcher, *Mappings of finite distortion: generalized Hausdorff dimension distortion*, preprint.

13. O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, New York, 1973. MR0344463 (49:9202)
14. J. Malý and O. Martio, *Lusin's condition* (N) *and mappings of the class* $W^{1,n}$, J. Reine Angew. Math. **458** (1995), 19–36. MR1310951 (95m:26024)
15. O. Martio and W. Ziemer, *Lusin's condition* (N) *and mappings with nonnegative Jacobians*, Michigan Math. J. **39** (1992), no. 3, 495–508. MR1182504 (93h:26021)
16. S. Ponomarev, *Examples of homeomorphisms in the class* $ACTL^p$ *which do not satisfy the absolute continuity condition of Banach*, Dokl. Akad. Nauk USSR (Russian) **201** (1971), 1053–1054. MR0293024 (45:2104)
17. ———, *Property N of homeomorphisms of the class* W_p^1, Sibirsk. Math. Zh. (Russian) **28** (1987), 140–148. MR890732 (88i:26031)
18. Yu. G. Reshetnyak, *Some geometrical properties of functions and mappings with generalized derivatives*, Sibirsk. Math. Zh. (Russian) **7** (1966), 886–919. MR0203013 (34:2872)
19. S. Zakeri, *David maps and Hausdorff dimension*, Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 1, 121–138. MR2041702 (2005a:30034)

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