

## GENERALIZED DIMENSION DISTORTION UNDER PLANAR SOBOLEV HOMEOMORPHISMS

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ABSTRACT. We prove essentially sharp dimension distortion estimates for planar Sobolev-Orlicz homeomorphisms.

### 1. INTRODUCTION

Let  $\Omega, \Omega' \subset \mathbb{R}^2$  be open and connected. We consider homeomorphisms  $f: \Omega \rightarrow \Omega'$  that belong to the Sobolev class  $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^2)$ , which means that both component functions of  $f$  have locally integrable distributional partial derivatives. It is by now well-known that the Luzin condition  $(N)$ , which requires that  $f$  map Lebesgue null sets to Lebesgue null sets, holds if we additionally assume that  $|Df| \in L_{\text{loc}}^2(\Omega)$  [18, 15, 14], but may fail if  $|Df| \in L_{\text{loc}}^p(\Omega)$  for some  $p < 2$  [16, 17]. On the other hand, if  $|Df| \in L_{\text{loc}}^p(\Omega)$  for some  $p > 2$ , then the image of any set of Hausdorff dimension strictly less than two is also of Hausdorff dimension strictly less than two [4, 10]. Recently it was proven [11] that local integrability of  $|Df|^2 \log^{-1}(e + |Df|)$  already suffices for the Luzin condition  $(N)$  to hold. The motivation for this result and our results below arises in part from the theory of mappings with finite distortion, where the natural regularity assumption is that  $|Df|^2 \log^{\lambda-1}(e + |Df|) \in L_{\text{loc}}^1$  for some  $\lambda > 0$  [2, 1, 7, 8, 3].

Analogously to the  $L^p$ -scale setting, one expects some kind of dimension distortion estimate to hold when  $\lambda$  as above is strictly positive. However, it is rather easy to map, for example, a subset of the real line onto a set of Hausdorff dimension two [6, 19], and thus we have to work with a refined scale. Towards this end, we consider the gauge functions  $h_\lambda(t) = t^2 \log^\lambda \frac{1}{t}$ ,  $\lambda > 0$ . In Section 2, we describe a homeomorphism  $f$  that maps a Cantor set  $E$  of Minkowski (and so also Hausdorff) dimension strictly less than two to a set of positive  $\mathcal{H}^{h_\lambda}$ -measure, with  $|Df|^2 \log^{t-1}(e + |Df|) \in L_{\text{loc}}^1$  for all  $t < \lambda$ .

Our main result shows that this homeomorphism is critical for our generalized dimension distortion.

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**Theorem 1.1.** *Let  $\Omega$  and  $\Omega'$  be open sets in  $\mathbb{R}^2$  and  $f: \Omega \rightarrow \Omega'$  a homeomorphism of class  $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^2)$  with*

$$|Df|^2 \log^{\lambda-1}(e + |Df|) \in L_{\text{loc}}^1(\Omega)$$

for some  $\lambda > 0$ . Then

$$\mathcal{H}^{h\lambda}(f(E)) = 0$$

for every set  $E \subset \Omega$  of lower Minkowski dimension  $\dim_{\mathcal{M}}(E)$  strictly less than two.

We conjecture that one may replace the Minkowski dimension in Theorem 1.1 with the Hausdorff dimension. For a related, weaker result in this direction, see [12].

This note is organized as follows. In Section 2 we recall the necessary definitions and describe the construction of the homeomorphism referred to above. Section 3 contains the proof of Theorem 1.1.

## 2. PRELIMINARIES

Let  $U \subset \mathbb{R}^2$  be open and connected. We say that a mapping  $f \in L^1(U; \mathbb{R}^2)$  has *bounded variation*,  $f \in BV(U)$ , if the component functions  $f_1$  and  $f_2$  of  $f$  are of bounded variation, that is,

$$\sup \left\{ \int_U f_i \operatorname{div} \varphi \, dx \mid \varphi \in C_0^1(U; \mathbb{R}^2), |\varphi| \leq 1 \right\} < \infty, \quad i = 1, 2.$$

We write  $f \in BV_{\text{loc}}(U)$  if  $f \in BV(G)$  for each open and connected  $G$  compactly contained in  $U$ . For each function  $g \in BV(U; \mathbb{R})$  of bounded variation we can define a Radon measure  $\|Dg\|$  in the following way: for an open set  $V \subset U$  we put

$$\|Dg\|(V) = \sup \left\{ \int_V g \operatorname{div} \varphi \, dx \mid \varphi \in C_0^1(V; \mathbb{R}^2), |\varphi| \leq 1 \right\},$$

and for  $A \subset U$  not necessarily open,

$$\|Dg\|(A) = \inf \{ \|Dg\|(V) \mid A \subset V \subset U, V \text{ is open} \}.$$

For a set  $V$  and a number  $\delta > 0$ ,  $V + \delta$  denotes the set  $\{y \mid \operatorname{dist}(y, V) < \delta\}$ .

We write  $\mathcal{H}^h(A)$  for the *generalized Hausdorff measure* of a set  $A$ , given by

$$\mathcal{H}^h(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^h(A) = \lim_{\delta \rightarrow 0} \left[ \inf \left\{ \sum_{i=1}^{\infty} h(\operatorname{diam} U_i) : A \subset \bigcup_{i=1}^{\infty} U_i, \operatorname{diam} U_i \leq \delta \right\} \right],$$

where  $h$  is a dimension gauge (non-decreasing, with  $h(0) = 0$ ). If  $h(t) = t^\alpha$  for some  $\alpha \geq 0$ , we write simply  $\mathcal{H}^\alpha$  for  $\mathcal{H}^{t^\alpha}$  and call it the *Hausdorff  $\alpha$ -dimensional measure*; the *Hausdorff dimension*  $\dim_{\mathcal{H}}(A)$  of the set  $A$  is the smallest  $\alpha_0 \geq 0$  such that  $\mathcal{H}^\alpha(A) = 0$  for any  $\alpha > \alpha_0$ . The *lower Minkowski dimension*  $\dim_{\mathcal{M}}(A)$  of a bounded set  $A \subset \mathbb{R}^2$  is defined as

$$\dim_{\mathcal{M}}(A) = \inf \{ s : \liminf_{\varepsilon \rightarrow 0^+} N(A, \varepsilon) \varepsilon^s = 0 \},$$

where  $N(A, \varepsilon)$ ,  $\varepsilon > 0$ , denotes the smallest number of balls of radius  $\varepsilon$  needed to cover  $A$ :

$$N(A, \varepsilon) = \min \{ k : A \subset \bigcup_{i=1}^k B(x_i, \varepsilon) \text{ for some } x_i \in \mathbb{R}^2 \}.$$

Finally, let  $a \lesssim b$  mean that there exists some constant  $C > 0$  such that  $a \leq Cb$ .

In [6] a homeomorphism  $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$  was constructed which maps a set  $\mathcal{C}$  of Minkowski and Hausdorff dimension  $n \log 2 / \log(1/\sigma)$  for some  $0 < \sigma < 1/2$  onto a set  $\mathcal{C}'$  of positive  $\mathcal{H}^h$ -measure with  $h(t) = t^n (\log(1/t))^{pn}$  for given  $p > 0$ . This mapping is the identity outside the cube  $[0, 1]^n$  and satisfies  $|Dh(x)| \leq \frac{\tau_1 \cdots \tau_k}{\sigma^k}$  in  $A_{ki}$ . Here  $A_{ki}$ ,  $k = 1, 2, \dots$  and  $i = 1, \dots, 2^{kn}$ , are the open ‘‘cubical frames’’ needed to construct the Cantor set  $\mathcal{C}$ . They are pairwise disjoint with respect to both  $i$  and  $k$ ; that is,  $\text{int}(A_{ki}) \cap \text{int}(A_{lj}) = \emptyset$  when  $(k, i) \neq (l, j)$ , they cover the set  $[0, 1]^n$  up to a set of zero  $n$ -Lebesgue measure, and each  $A_{ki}$  is contained in a cube of edge length  $(1/2)\sigma^{k-1}$ . The numbers  $\tau_k$ ,  $k = 1, 2, \dots$ , used to construct the image Cantor-type set are defined as follows:

$$\tau_1 = \frac{1}{2} \frac{1}{\log^p 4} \quad \text{and} \quad \tau_k = \frac{1}{2} \left(1 - \frac{1}{k}\right)^p \quad \text{for } k = 2, 3, \dots$$

Note that

$$\tau_1 \cdots \tau_k = \frac{1}{2^k} \frac{1}{\log^p 4} \frac{1}{k^p},$$

so in the case  $n = 2$ , we have

$$\begin{aligned} \int_{[0,1]^2} |Dh|^2 \log^s(e + |Dh|) &= \sum_{k=1}^{\infty} \sum_{i=1}^{4^k} \int_{A_{ki}} |Dh|^2 \log^s(e + |Dh|) \\ &\leq \sum_{k=1}^{\infty} 4^k \frac{1}{4} \sigma^{2k-2} \frac{(\tau_1 \cdots \tau_k)^2}{\sigma^{2k}} \log^s\left(e + \frac{\tau_1 \cdots \tau_k}{\sigma^k}\right) \\ &= \sum_{k=1}^{\infty} \frac{1}{4\sigma^2 k^{2p} \log^{2p} 4} \log^s\left(e + \frac{1}{(2\sigma)^k k^p \log^p 4}\right) \\ &\lesssim \sum_{k=1}^{\infty} k^{s-2p} < \infty \end{aligned}$$

when  $s + 1 < 2p$ .

### 3. PROOFS

Clearly, we may assume in the rest of this paper that  $\Omega$  is an open and connected subset of  $\mathbb{R}^2$ . We begin with the following lemma.

**Lemma 3.1.** *Let  $f: \Omega \rightarrow f(\Omega) \subset \mathbb{R}^2$  be a homeomorphism in  $W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^2)$ . Then there exists a set  $F \subset f(\Omega)$  with  $\mathcal{H}^{3/2}(F) = 0$  such that for all  $y \in f(\Omega) \setminus F$  there exist constants  $C_y > 0$  and  $r_y > 0$  such that*

$$(3.1) \quad \text{diam}(f^{-1}(B(y, r))) \leq C_y r^{1/2}$$

for all  $0 < r < r_y$ .

*Proof.* First, note that by Theorem 1.2 in [5],  $f^{-1}$  is in  $BV_{\text{loc}}(f(\Omega))$ . Next, fix  $y \in f(\Omega)$  and  $r > 0$  such that  $B(y, 3r) \subset f(\Omega)$ . Let  $Q(y, t)$  be the square centered at  $y$  and having edge length  $2t$ . As  $f^{-1}$  is a homeomorphism, for  $t \in (r, 2r)$  we have

$$\begin{aligned} \text{diam } f^{-1}(B(y, r)) &< \text{diam } f^{-1}(Q(y, t)) \leq \text{diam } f^{-1}(\partial Q(y, t)) \\ &\leq \text{diam } f_1^{-1}(\partial Q(y, t)) + \text{diam } f_2^{-1}(\partial Q(y, t)), \end{aligned}$$

where  $f_i^{-1}$ ,  $i = 1, 2$ , denotes the  $i$ -th component function of  $f^{-1}$ . Integrating this inequality over the interval  $[r, 2r]$  with respect to  $t$ , we obtain

$$(3.2) \quad r \operatorname{diam} f^{-1}(B(y, r)) < \sum_{i=1}^2 \int_{[r, 2r]} \operatorname{diam} f_i^{-1}(\partial Q(y, t)) dt.$$

Let us consider the smooth approximation  $g_i^\varepsilon = \eta_\varepsilon * f_i^{-1}$  of  $f_i^{-1}$ ,  $i = 1, 2$ , on the cube  $Q(y, 2r)$ . Here  $\eta_\varepsilon$  is a standard bump function. As  $f^{-1}$  is continuous, the convergence  $g_i^\varepsilon \rightarrow f_i^{-1}$  is pointwise and uniform on each compact set  $K \subset Q(y, 2r)$ . So, for  $t \in (r, 2r)$  and  $i = 1, 2$  we have  $\operatorname{diam} f_i^{-1}(\partial Q(y, t)) = \lim_{\varepsilon \rightarrow 0} \operatorname{diam} g_i^\varepsilon(\partial Q(y, t))$ . Put  $a_i = y_i - 2r$  and  $b_i = y_i + 2r$ ,  $i = 1, 2$ , where  $y = (y_1, y_2)$ . Fatou's Lemma implies that

$$(3.3) \quad \begin{aligned} \int_{[r, 2r]} \operatorname{diam} f_i^{-1}(\partial Q(y, t)) dt &= \int_{[r, 2r]} \lim_{\varepsilon \rightarrow 0} \operatorname{diam} g_i^\varepsilon(\partial Q(y, t)) dt \\ &\leq \liminf_{\varepsilon \rightarrow 0} \int_{[r, 2r]} \operatorname{diam} g_i^\varepsilon(\partial Q(y, t)) dt. \end{aligned}$$

We use the fundamental theorem of calculus and Fubini's theorem to obtain

$$(3.4) \quad \begin{aligned} \int_{[r, 2r]} \operatorname{diam} g_i^\varepsilon(\partial Q(y, t)) dt &\leq \int_{[r, 2r]} \left\{ \int_{[a_2, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial \xi}(y_1 - t, \xi) \right| d\xi \right. \\ &+ \int_{[a_2, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial \xi}(y_1 + t, \xi) \right| d\xi + \int_{[a_1, b_1]} \left| \frac{\partial g_i^\varepsilon}{\partial \xi}(\xi, y_2 - t) \right| d\xi \\ &+ \left. \int_{[a_1, b_1]} \left| \frac{\partial g_i^\varepsilon}{\partial \xi}(\xi, y_2 + t) \right| d\xi \right\} dt = \int_{[a_1, y_1 - r] \times [a_2, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial x_2}(x) \right| dx \\ &+ \int_{[y_1 + r, b_1] \times [a_2, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial x_2}(x) \right| dx + \int_{[a_1, b_1] \times [a_2, y_2 - r]} \left| \frac{\partial g_i^\varepsilon}{\partial x_1}(x) \right| dx \\ &+ \int_{[a_1, b_1] \times [y_2 + r, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial x_1}(x) \right| dx \leq \sum_{j=1}^2 \int_{Q(y, 2r)} \left| \frac{\partial g_i^\varepsilon}{\partial x_j}(x) \right| dx \end{aligned}$$

for  $i = 1, 2$ . Let us show that

$$(3.5) \quad \int_{Q(y, 2r)} \left| \frac{\partial g_i^\varepsilon}{\partial x_j}(x) \right| dx \leq \|Df_i^{-1}\|(Q(y, 2r))$$

for  $i, j = 1, 2$ . Given  $\varphi \in C_0^1(Q(y, 2r))$ ,  $|\varphi| \leq 1$ , we may write

$$\begin{aligned} \int_{Q(y, 2r)} \frac{\partial g_i^\varepsilon}{\partial x_j} \varphi dx &= - \int_{Q(y, 2r)} g_i^\varepsilon \frac{\partial \varphi}{\partial x_j} dx = - \int_{Q(y, 2r)} (\eta_\varepsilon * f_i^{-1}) \frac{\partial \varphi}{\partial x_j} dx \\ &= - \int_{Q(y, 2r)} f_i^{-1} \frac{\partial \eta_\varepsilon * \varphi}{\partial x_j} dx \leq \|Df_i^{-1}\|(Q(y, 2r)). \end{aligned}$$

This implies (3.5), and combining it with (3.2), (3.3) and (3.4), we finally obtain

$$\operatorname{diam} f^{-1}(B(y, r)) < \frac{2}{r} (\|Df_1^{-1}\|(Q(y, 2r)) + \|Df_2^{-1}\|(Q(y, 2r)))$$

for all  $y \in f(\Omega)$  and  $r > 0$  such that  $B(y, 3r) \subset f(\Omega)$ . That is, the inequality (3.1) holds for all  $y \in f(\Omega)$  such that

$$(3.6) \quad \frac{\|Df_i^{-1}\|(Q(y, 2r))}{r^{3/2}} < M_y$$

is valid for  $i = 1, 2$ , all small enough  $r > 0$  and some constant  $M_y$  depending on  $y$ . Let  $F_1$  be the set of those  $y$  for which (3.6) does not hold for  $i = 1$ . Let  $K \subset f(\Omega)$  be a compact set and fix some  $\delta > 0$  such that  $\text{dist}(K, \partial f(\Omega)) > \delta$ . For every  $k \in \mathbb{N}$  and every  $y \in F_1 \cap K$  there exists  $r_{k,y} < \delta\sqrt{2}/20$  such that  $\|Df_1^{-1}\|(Q(y, 2r_{k,y})) \geq k(r_{k,y})^{3/2}$ . Consider the collection of all balls

$$\mathcal{B}_k = \{B(y, 2\sqrt{2}r_{k,y}) : y \in F_1 \cap K\}$$

for  $k \in \mathbb{N}$ . Using Vitali’s covering theorem, we obtain for every  $k \in \mathbb{N}$  a countable subcollection of disjoint balls  $B_{k,j}$ ,  $j = 1, 2, \dots$ , centered in  $F_1 \cap K$ , having radii  $2\sqrt{2}r_j^k < \delta/5$  and with  $5B_{k,j}$  covering  $F_1 \cap K$ . As  $Q(y, 2r_j^k) \subset B_{k,j}$ , we have

$$\begin{aligned} \mathcal{H}_\delta^{3/2}(F_1 \cap K) &\leq \sum_{j=1}^\infty (10\sqrt{2}r_j^k)^{3/2} \leq \frac{(10\sqrt{2})^{3/2}}{k} \sum_{j=1}^\infty \|Df_1^{-1}\|(Q(y, 2r_j^k)) \\ &\leq \frac{(10\sqrt{2})^{3/2}}{k} \sum_{j=1}^\infty \|Df_1^{-1}\|(B_{k,j}) \leq \frac{(10\sqrt{2})^{3/2} \|Df_1^{-1}\|(K + \delta/5)}{k} \end{aligned}$$

for all  $k \in \mathbb{N}$ . Letting  $k \rightarrow \infty$  and  $\delta \rightarrow 0$ , we obtain  $\mathcal{H}^{3/2}(F_1 \cap K) = 0$ . □

The previous lemma implies the following result.

**Lemma 3.2.** *Let  $E \subset \Omega$  and let  $f : \Omega \rightarrow f(\Omega) \subset \mathbb{R}^2$  be a homeomorphism of class  $W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^2)$ . Then there exists a decomposition  $f(E) = \bigcup_{i=0}^\infty F_i$  where  $\mathcal{H}^{3/2}(F_0) = 0$  and for each  $F_i$ ,  $i = 1, 2, \dots$ , there exist constants  $C_i < \infty$  and  $r_i > 0$  such that*

$$f^{-1}(F_i + r) \subset E + C_i r^{1/2}$$

for every  $r \in (0, r_i)$ .

*Proof.* We choose  $F_0 = F$ , where  $F$  is the set from the previous lemma. Moreover, by this lemma we may represent the set  $f(E)$  as

$$f(E) = F_0 \cup \bigcup_{j=1}^\infty \bigcup_{k=1}^\infty \left\{ y \in f(E) \mid \text{diam}(f^{-1}(B(y, r))) \leq kr^{\frac{1}{2}} \text{ for all } r \in (0, \frac{1}{j}) \right\}.$$

So, putting  $C_i = C_{i(j,k)} = k$  and  $r_i = r_{i(j,k)} = \frac{1}{j}$ , we complete the proof. □

*Proof of Theorem 1.1.* As  $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^2)$  is a homeomorphism, its Jacobian  $J_f$  is either non-negative almost everywhere in  $\Omega$  or non-positive almost everywhere in  $\Omega$  [13]. We may assume that  $J_f \geq 0$  almost everywhere in  $\Omega$ . Recalling that

$$|Df|^2 \log^{\lambda-1}(e + |Df|) \in L_{\text{loc}}^1(\Omega),$$

by Corollary 9.1 in [9], we have  $J_f \log^\lambda(e + J_f) \in L_{\text{loc}}^1$ . Next, as  $\dim_{\mathcal{M}}(E) < 2$ , there exist constants  $C, \varepsilon > 0$  and a sequence of numbers  $r_j$ ,  $j = 1, 2, \dots$ , tending to zero as  $j \rightarrow \infty$ , such that  $\mathcal{L}^2(E + r_j) \leq Cr_j^\varepsilon$  for all  $j = 1, 2, \dots$ . By Lemma 3.2, we have  $f(E) = \bigcup_{i=0}^\infty F_i$  where  $\mathcal{H}^{3/2}(F_0) = 0$  and  $f^{-1}(F_i + R_{i,j}) \subset E + r_j$  for all large enough  $j$  ( $j \geq j_i$  for some  $j_i \in \mathbb{N}$ ). Here  $R_{i,j} = (r_j/C_i)^2$  and  $C_i$  are the constants from Lemma 3.2. It suffices to show that  $\mathcal{H}^h(F_i) = 0$  for all  $i \in \mathbb{N}$ . We use the fact

that  $\mathcal{L}^2(f(A)) \leq \int_A J_f$  for each open  $A \subset \Omega$  [11, Lemma 3.2]. Thus, for a fixed  $i \in \mathbb{N}$ , we have

$$\begin{aligned}
 \mathcal{L}^2(F_i + R_{i,j}) &\leq \int_{f^{-1}(F_i + R_{i,j})} J_f(x) dx \leq \int_{E+r_j} J_f(x) dx \\
 &\leq \int_{\{x \in E+r_j : J_f(x) < r_j^{-\varepsilon/2}\}} J_f + \int_{\{x \in E+r_j : J_f(x) \geq r_j^{-\varepsilon/2}\}} J_f \\
 &\leq r_j^{-\varepsilon/2} \mathcal{L}^2(E + r_j) + \log^{-\lambda}(e + r_j^{-\varepsilon/2}) \int_{E+r_j} J_f \log^\lambda(e + J_f) \\
 (3.7) \qquad &\leq Cr_j^{\varepsilon/2} + M(r_j) \log^{-\lambda} \frac{1}{r_j}
 \end{aligned}$$

for big enough  $j$ , where  $M(r) \rightarrow 0$  as  $r \rightarrow 0$ . In other words,

$$\mathcal{L}^2(F_i + R_{i,j}) = o(\log^{-\lambda} \frac{1}{r_j})$$

as  $j \rightarrow \infty$ . Using the Besicovitch covering theorem, for each large enough  $j \in \mathbb{N}$ , we can cover the set  $F_i$  with  $N$  countable families of pairwise disjoint balls centered in  $F_i$  and of radius  $R_{i,j}$  ( $N$  is independent of both  $i$  and  $j$ ). It is obvious that each of these families is finite. Let  $l_{i,j}$  denote the total number of covering balls. We have  $\mathcal{L}^2(F_i + R_{i,j}) \geq Cl_{i,j}R_{i,j}^2$ , where  $C$  is a constant independent of  $i$  and  $j$ . So, for each fixed  $i \in \mathbb{N}$  and all big enough  $j \geq j_i$ , we have

$$\mathcal{H}_{R_{i,j}}^h(F_i) \leq l_{i,j}R_{i,j}^2 \log^\lambda(1/R_{i,j}) \leq \frac{2^\lambda}{C} \mathcal{L}^2(F_i + R_{i,j}) \log^\lambda(C_i/r_j),$$

and thus (3.7) shows that  $\mathcal{H}^h(F_i) = 0$ . It follows that  $\mathcal{H}^h(f(E)) = 0$ .  $\square$

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