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A STABLY ELEMENTARY HOMOTOPY

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Abstract. If R is an affine algebra of dimension d over a perfect C_1 field and $\sigma \in SL_{d+1}(R)$ is a stably elementary matrix, we show that there is a stably elementary matrix $\sigma(X) \in SL_{d+1}(R[X])$ with $\sigma(1) = \sigma$ and $\sigma(0) = I_{d+1}$.

1. Introduction

In [8], Theorem 1, it is shown that if A is a non-singular algebra over a perfect C_1 field of dimension $d \geq 2$, then a stably elementary matrix $\sigma \in \mathrm{SL}_{d+1}(A)$ is an elementary matrix. This was obtained by showing that σ is homotopic to the identity and then applying T. Vorst's K_1 analogue in [19] of H. Lindel's theorem in [4]. The homotopy was constructed even in the case where A is singular, but in this case the homotopy is not necessarily stably elementary.

The result in this article may be regarded as a K_1 analogue of theorems of A.A. Suslin in [12, 13]. The result is obtained by proving a relative version of theorems in [12, 13]. Note that A.A. Suslin's theorems do not need the hypothesis that the algebra is non-singular (see the argument of P. Raman in [8], Proposition 3.1).

In this paper we prove that one can always construct a homotopy which is stably elementary. More precisely, we generalize the above result and show the following:

Theorem. *Let A be an affine algebra of dimension d over a perfect C_1 field. Let $\sigma \in \mathrm{SL}_{d+1}(A)$ be a stably elementary matrix. Then there is a stably elementary $\sigma(X) \in \mathrm{SL}_{d+1}(A[X])$ with $\sigma(1) = \sigma$ and $\sigma(0) = I_{d+1}$.*

An interesting consequence is that, for such A, if the injective stability estimate for $K_1(A_{\mathfrak{p}}[X])$ falls to $(d+1)$ for every prime ideal $\mathfrak{p}$ of A, then the injective stability dimension of $K_1(A)$ will also fall to $(d+1)$.

The definition of the Suslin matrix is over a commutative ring. We will recall this matrix and its properties in §3. Since we use this matrix, we will assume throughout that the rings under consideration are commutative with 1.

2. An example

In this section we begin with an example of why some condition is needed on the base field. Here we give an example of a stably elementary matrix of size $(d+1)$ over a non-singular affine algebra of dimension d over a function field of one variable

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$\mathbb{R}(X)$ over the real number field. This matrix is not an elementary matrix and is not even homotopic to the identity.

There is a symbiotic relationship between 1-stably elementary matrices of size r over the ring A_{st} and stably free A-projective modules of rank r and type 1.

How does this occur in practice? Let A be a ring and let $s, t \in A$ with $As + At = 1$. Given a 1-stably elementary matrix $\alpha \in \mathrm{SL}_r(A_{st})$ we can associate with it a 1-stably free projective module P of rank r, viz. the "fibre product over α":

$$P = A_s^r \times_\alpha A_t^r = \{(x, y) \in A_s^r \times A_t^r \mid x_t \alpha = y_s\}.$$

Conversely, if P is a stably free module of rank r and type 1, then P is the "solution space" of a unimodular row $(a_0, \ldots, a_r) \in \mathrm{Um}_{r+1}(A)$. (Recall that a row $(a_0, \ldots, a_r) \in A^{r+1}$ is a unimodular row of length $r+1$ if there are elements $b_0, \ldots, b_r \in A$ with $\sum_{i=0}^r a_i b_i = 1$. $\mathrm{Um}_{r+1}(A)$ denotes the set of all unimodular rows over A of length $r+1$.)

We show below that

$$P \simeq A_{b_0}^r \times_\alpha A_{1+b_0A}^r,$$

for some 1-stably elementary $\alpha \in \mathrm{SL}_r(A_{(1+b_0A)b_0})$.

Let $\varepsilon_1 \in \mathrm{E}_{r+1}(A_{b_0})$ with $\varepsilon_1 e_1^t = (b_0, \ldots, b_r)^t$. Then, as $\sum_{i=0}^r a_i b_i = 1$, we have $(a_0, \ldots, a_r)\varepsilon_1 = (1, *, \ldots, *)$. We can suitably modify ε_1, retaining its first column, and further assume that $(a_0, \ldots, a_r)\varepsilon_1 = e_1$. Since b_0 is in the Jacobson radical of A_{1+b_0A}, there is an $\varepsilon_2 \in \mathrm{E}_{r+1}(A_{1+b_0A})$ such that $\varepsilon_2 e_1^t$ is the image of $(b_0, \ldots, b_r)^t$ in A_{1+b_0A}. After a suitable modification, if necessary, we may also ensure that $(a_0, \ldots, a_r)\varepsilon_2 = e_1$. Now ${\varepsilon_1}^{-1}\varepsilon_2 = 1 \perp \alpha$, for some $\alpha \in \mathrm{SL}_r(A_{(1+b_0A)b_0}) \cap \mathrm{E}_{r+1}(A_{(1+b_0A)b_0})$. Clearly, $P \simeq A_{b_0}^r \times_\alpha A_{1+b_0A}^r$.

One can similarly show that $P \simeq A_{a_0}^r \times_\alpha A_{1+a_0A}^r$.

Let P_v be the projective module corresponding to the row v. Then $P_v \simeq A_{v_0}^r \times_{\sigma_v} A_{1+v_0A}^r$, for some 1-stably elementary σ_v.

If σ_v is homotopic to the identity, then by D. Quillen's splitting lemma ([6], Lemma 1) $\sigma_v = {\sigma_1}_{v_0} {\sigma_2}_{1+v_0A}$, for some $\sigma_1 \in \mathrm{SL}_r(A_{v_0})$ and $\sigma_2 \in \mathrm{SL}_r(A_{(1+v_0A)})$.

But then it is easy to conclude that P_v is free. Thus, if P_v is not free, then σ_v is not elementary or homotopic to the identity. So to construct such σ_v it suffices to give an example of a unimodular row v which is not completable. Here is a standard example. We refer the reader to [16] for an instructive proof.

Let $A = \Gamma(S^n)$ be the coordinate ring of the real n-sphere, that is, let $A = \mathbb{R}[x_0, \ldots, x_n]/(\sum_i x_i^2 - 1)$. Then it is known by using results from topology that the unimodular row $v = (\overline{x_0}, \ldots, \overline{x_n})$ is not completable if $n \neq 1, 3, 7$.

3. Suslin's completion

In [12], Proposition 1.6, A.A. Suslin proved that a unimodular row of the form $(a_0, a_1, a_2^2, \ldots, a_r^r)$ can be completed to an invertible matrix.

Given two rows $v, w \in R^{r+1}$ with $\langle v, w \rangle = 1$, A.A. Suslin described an inductive procedure in [12], §5, for constructing a matrix $S_r(v, w) \in \mathrm{SL}_{2^r}(R)$ whose size can be reduced by elementary row and column operations to $r+1$, giving a matrix $\beta_r(v, w) \in \mathrm{SL}_{r+1}(R)$ whose first row is $(a_0, a_1, a_2^2, \ldots, a_r^r)$ if $v = (a_0, \ldots, a_r)$.

We call $S_r(v, w)$ the Suslin matrix w.r.t. the pair (v, w).

We begin by describing the inductive process by which the Suslin matrix $S_r(v, w)$ of size 2^r and determinant $v \cdot w^t$ is constructed from two rows v and w of size $r+1$. We recall this process.

Let $v = (a_0, v_1)$ and $w = (b_0, w_1)$. Set $S_0(v, w) = a_0$ and

$$S_r(v, w) \quad = \quad \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1, w_1) \\ -S_{r-1}(w_1, v_1)^t & b_0 I_{2^{r-1}} \end{pmatrix}.$$

In [12], Lemma 5.1, it is noted that

(i) $S_r(v, w) S_r(w, v)^t = (v \cdot w^t) I_{2^r} = S_r(w, v)^t S_r(v, w)$, and

(ii) $\det S_r(v, w) = (v \cdot w^t)^{2^{r-1}}$, for $r \geq 1$.

Thus, $S_r(v, w)$ is of determinant 1 if $\langle v, w \rangle = v \cdot w^t = 1$.

To justify the claim that if v is a unimodular row and $\langle v, w \rangle = 1$, then the Suslin matrix $S_r(v, w)$ and the matrix $\beta_r(v, w)$ are elementarily equivalent in $\mathrm{E}_{2^r}(R)$, we refer the reader to [15], Proposition 2.2 and Corollary 2.5.

Lemma 3.1. *Let v, w, w^* be rows of length $r + 1 \geq 3$, with $\langle v, w \rangle = 1 = \langle v, w^* \rangle$.*

(i) *Assume that $v = e_1 \varepsilon$ is the first row of an elementary matrix $\varepsilon \in \mathrm{E}_{r+1}(R)$. Then $S_r(e_1 \varepsilon, e_1 (\varepsilon^t)^{-1}) \in \mathrm{E}_{2^r}(R)$.*

(ii) *Let $\varepsilon \in \mathrm{E}_{r+1}(R)$. Then $S_r(v\varepsilon, w(\varepsilon^t)^{-1}) \in S_r(v, w)\mathrm{E}_{2^r}(R)$.*

(iii) *$S_r(v, w^*) \in S_r(v, w)\mathrm{E}_{2^r}(R)$.*

Proof. Though (i) can be deduced from (ii), we give an independent proof, which is the essence of proving (ii).

(i) Let $\varepsilon = \prod_{k=1}^{t} \mathrm{E}_{i_k j_k}(\lambda_k)$, and let

$$\varepsilon(X_1, \ldots, X_t) = \prod_{k=1}^{t} \mathrm{E}_{i_k j_k}(X_k) \in \mathrm{E}_{r+1}(\mathbb{Z}[X_1, \ldots, X_t]).$$

Then $\varepsilon = \varepsilon(\lambda_1, \ldots, \lambda_t)$. Hence,

$$S_r(e_1\varepsilon, e_1(\varepsilon^t)^{-1}) = S_r(e_1\varepsilon(\lambda_1, \ldots, \lambda_t), e_1(\varepsilon(\lambda_1, \ldots, \lambda_t)^t)^{-1}).$$

But

$$\begin{aligned} S_r(e_1\varepsilon(X_1, \ldots, X_t), e_1(\varepsilon(X_1, \ldots, X_t)^t)^{-1}) &\in \mathrm{SL}_{r+1}(\mathbb{Z}[X_1, \ldots, X_t]) \\ &= \mathrm{E}_{r+1}(\mathbb{Z}[X_1, \ldots, X_t]) \end{aligned}$$

by [13], Corollary 6.6.

(iii) By [13], Corollary 2.7, the matrix $(\varepsilon^t)^{-1} = I_{r+1} + v^t(w^* - w) \in \mathrm{E}_{r+1}(R)$. Note that $w(\varepsilon^t)^{-1} = w^*$ and $v\varepsilon = v$. Hence, $S_r(v\varepsilon, w(\varepsilon^t)^{-1}) = S_r(v, w^*)$. Therefore (iii) will follow once (ii) is proved.

(ii) It suffices to show that $S_r(v\mathrm{E}_{ij}(\lambda), w\mathrm{E}_{ji}(-\lambda)) \in S_r(v, w)\mathrm{E}_{2^r}(R)$, for $1 \leq i \neq j \leq r + 1$, $\lambda \in R$.

Consider $S_r(v\mathrm{E}_{ij}(T), w\mathrm{E}_{ji}(-T)) \in \mathrm{SL}_{2^r}(R[T])$. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Note that v can be completed to an elementary matrix over $R_\mathfrak{p}$. Using the argument in the previous paragraph, one can show that there is an $\varepsilon \in \mathrm{E}_{r+1}(R_\mathfrak{p})$ such that $v = e_1\varepsilon$ and $w = (\varepsilon^t)^{-1}$. Now argue as in (i) to show that the matrix $S_r(v_\mathfrak{p}\mathrm{E}_{ij}(T), w_\mathfrak{p}\mathrm{E}_{ji}(-T))$ is obtained by a specialization of an elementary polynomial matrix over $\mathbb{Z}$ and hence is elementary. By the Local Global Principle of A.A. Suslin in [13], Theorem 3.1, it follows that $S_r(v\mathrm{E}_{ij}(T), w\mathrm{E}_{ji}(-T)) \in S_r(v, w)\mathrm{E}_{2^r}(R[T])$. Set $T = \lambda$ to get the desired result. □

Remark. (ii) can alternatively be shown directly using the Key Lemma of [2], Lemma 3.2.

Corollary 3.2. *If $v,\, w \in R^{r+1}$, $r \geq 2$, with $\langle v, w\rangle = 1$, and if v can be completed to an elementary matrix, then $\beta_r(v,w)$ is stably elementary.*

4. Excision

In this section we obtain a relative version of Corollary 3.2. For this we recall the excision theorem of W. van der Kallen in [3], Theorem 3.21. First we recall the relative groups, the excision ring, etc., as defined in [3], §2.1.

The relative groups $\mathrm{E}_n(R,I)$. Let R be a ring with 1, and let I be an ideal of R. The relative group $\mathrm{E}_n(R,I)$ denotes the smallest normal subgroup of $\mathrm{E}_n(R)$ containing the element $\mathrm{E}_{ij}(x)$, $1 \leq i \neq j \leq n$, $x \in I$. Equivalently, $\mathrm{E}_n(R,I)$ is generated by $\mathrm{E}_{ij}(a)\mathrm{E}_{ji}(x)\mathrm{E}_{ij}(-a)$, with $a \in R$, $x \in I$, $1 \leq i \neq j \leq n$, provided $n \geq 3$.

The excision ring $(\mathbb{Z} \oplus I)$. If I is an ideal in the ring R, one can construct the ring $\mathbb{Z} \oplus I$ with multiplication $(n \oplus i)(m \oplus j) = (nm \oplus nj + mi + ij)$, for $m, n \in \mathbb{Z}$, $i, j \in I$. The maximal spectrum of the ring $\mathbb{Z} \oplus I$ is noetherian, being the union of finitely many subspaces of dimension $\leq \dim(R)$.

There is a natural homomorphism $\varphi : \mathbb{Z} \oplus I \longrightarrow R$ given by $(m \oplus i) \to m + i \in R$.

We denote by $\mathrm{Um}_n(R,I)$ the set of all unimodular rows of length n which are congruent to $e_1 = (1, 0, \ldots, 0)$ modulo I.

Theorem 4.1 (W. van der Kallen, [3], Theorem 3.21)**.** *Let $n \geq 3$, and let I be an ideal in the ring R. Then the natural maps*

$$\begin{aligned} \mathrm{Um}_n(\mathbb{Z} \oplus I, I)/\mathrm{E}_n(\mathbb{Z} \oplus I) &\longrightarrow \mathrm{Um}_n(R,I)/\mathrm{E}_n(R,I), \\ \mathrm{Um}_n(\mathbb{Z} \oplus I, I)/\mathrm{E}_n(\mathbb{Z} \oplus I) &\longrightarrow \mathrm{Um}_n(\mathbb{Z} \oplus I)/\mathrm{E}_n(\mathbb{Z} \oplus I) \end{aligned}$$

are bijective.

The excision theorem enables one to transform a problem from the relative case $(R \neq I)$ to the absolute case.

Lemma 4.2. *Let R be a ring, and let J be an ideal of R. Let $\alpha \in \mathrm{SL}_n(R)$ with $\alpha \equiv I_n$ modulo J. Then there is a unique matrix $S \in \mathrm{SL}_n(\mathbb{Z} \oplus J)$ such that $S \equiv I_n$ modulo $(0 \oplus J)$ and $\varphi(S) = \alpha$, where $\varphi : \mathbb{Z} \oplus J \longrightarrow R$ is the natural homomorphism.*

Moreover, if $n = 2^m$ for some m and α is a Suslin matrix of determinant one, then S is also a Suslin matrix of determinant one.

Proof. Let $\alpha = (\delta_{ij} + a_{ij})$, for some (uniquely defined) $a_{ij} \in J$. (Here δ_{ij} denotes the Kronecker delta symbol.) Similarly, let $\alpha^{-1} = (\delta_{ij} + b_{ij})$, for some $b_{ij} \in J$. Let $S = (\delta_{ij} \oplus a_{ij}) \in \mathrm{M}_n(\mathbb{Z} \oplus J)$, $T = (\delta_{ij} \oplus b_{ij}) \in \mathrm{M}_n(\mathbb{Z} \oplus J)$. Then $ST = I_n$. In fact one can check that S and T are in $\mathrm{SL}_n(\mathbb{Z} \oplus J)$. The last assertion is easy to check in view of the above construction. □

Corollary 4.3. *Let R be a ring, and let J be an ideal of R. If $v,\, w \in \mathrm{Um}_{r+1}(R,J)$, $r \geq 2$, with $\langle v, w\rangle = 1$, then there exists a $\beta_r(v,w)$ such that*

- (i) *the first row of $\beta_r(v,w)$ is $(a_0, a_1, a_2^2, \ldots, a_r^r)$ if $v = (a_0, a_1, \ldots, a_r)$,*
- (ii) *$\beta_r(v,w)$ is stably elementarily equivalent to $S_r(v,w)$,*
- (iii) *$\beta_r(v,w) \equiv I_{r+1}$ modulo J,*
- (iv) *if v is the first row of an elementary matrix, then $\beta_r(v,w)$ is stably elementary.*

Proof. Since $v, w \in \mathrm{Um}_{r+1}(R,J)$, $S_r(v,w) \equiv I_{2^r}$ modulo J. Take $\alpha = S_r(v,w)$ in Lemma 4.2 and consider the corresponding $S = S_r(\overline{v},\overline{w}) \in \mathrm{SL}_{2^r}(\mathbb{Z}\oplus J)$.

By [15], Proposition 2.2 and Corollary 2.5, there exist $\varepsilon \in \mathrm{E}_{2^r}(\mathbb{Z}\oplus J)$ and $\beta_r(\overline{v},\overline{w}) \in \mathrm{SL}_{r+1}(\mathbb{Z}\oplus J)$ such that $\beta_r(\overline{v},\overline{w})$ satisfies (i) and (iii). Since $\mathrm{SL}_{r+1}(\mathbb{Z}) = \mathrm{E}_{r+1}(\mathbb{Z})$, $r \geq 2$, we may modify $\beta_r(\overline{v},\overline{w})$ by an elementary matrix (over $\mathbb{Z}$) so that (i), (ii), and (iii) all hold. Now transform the relations over R by using the homomorphism $\varphi : \mathbb{Z}\oplus J \longrightarrow R$. The last assertion follows from Corollary 3.2. □

5. Proof of Theorem

We first recall the notion of a C_1 field.

By a form f over a field k, we mean a homogeneous polynomial of degree ≥ 1 in one or more variables with coefficients in k. A field k is said to be a C_i field if every form $f(X_1,\dots,X_n)$ in n variables and of degree d, with $n > d^i$, has a non-trivial zero in k; i.e. there exist $a_1,\dots,a_n \in k$ not all zero such that $f(a_1,\dots,a_n) = 0$.

A theorem of Chevalley asserts that a finite field is C_1. A theorem of Tsen asserts that a function field in one variable $\overline{k}(X)$ over an algebraically closed field $\overline{k}$ is a C_1 field. More generally, S. Lang has shown that if k is a C_i field, then $k(X)$ is a C_{i+1} field. If k is a C_1 field, then its Brauer group is trivial. It is known that the Brauer group of the field of real numbers is cyclic of order 2 and is generated by the class of the quaternion algebra $\mathbb{H}$.

The results in this paper are proved for affine algebras over a perfect C_1 field. They can also be proved for fields that satisfy the more technical conditions in [8], Proposition 3.1, which subsumes the case of a perfect C_1 field in view of [8], Remark 3.2. The main result used is Suslin's theorem in [14], as stated in [8], Proposition 3.1, which asserts that $\mathrm{Um}_{d+1}(A) = e_1\mathrm{E}_{d+1}(A)$ if A is an affine algebra of dimension d over such fields.

Before proving the main theorem, we recall a few useful lemmas. We begin with a variant of a lemma of M. Roitman; for an alternative proof see [1], Lemma 3.2.

Lemma 5.1 (M. Roitman, [9], Theorem 2)**.** *Let R be a ring with 1, let $(x_1,\dots,x_r) \in \mathrm{Um}_r(R)$ for $r \geq 3$ and let $n \in R$ be a nilpotent element. For $1 \leq i \leq r$, $(x_1,\dots,x_r)$ and $(x_1,\dots,x_i+n,x_{i+1},\dots,x_r)$ belong to the same elementary orbit.*

Lemma 5.2 (M. Roitman, [10], Lemma 1)**.** *Let $(x_0,x_1,\dots,x_r) \in \mathrm{Um}_{r+1}(R)$, $r \geq 2$. Let t be an element of R which is invertible modulo $(x_0,x_1,\dots,x_{r-2})$. Then, $(x_0,\dots,x_r)$ can be elementarily transformed to $(x_0,x_1,\dots,x_{r-1},t^2x_r)$.*

Lemma 5.3. *For a unit u in R,*

$$\begin{aligned}[u] \perp [u^{-1}] &= \mathrm{E}_{21}(u^{-1})\mathrm{E}_{12}(1-u)\mathrm{E}_{21}(-1)\mathrm{E}_{12}(1-u^{-1})\\ &= \mathrm{E}_{21}(u^{-1}-1)\mathrm{E}_{12}(1)\mathrm{E}_{21}(u-1)\mathrm{E}_{12}(-1)\mathrm{E}_{12}(1-u^{-1}).\end{aligned}$$

One has the following relative version of A.A. Suslin's theorems:

Proposition 5.4. *Let A be an affine algebra of dimension d over a perfect C_1 field k which is infinite. Assume that $2mA = A$. Let $t \in A$ be a non-nilpotent element and let*

$$v \;=\; (v_0,v_1,\dots,v_d) \in \mathrm{Um}_{d+1}(A,(t)).$$

Then there exist

$$\begin{aligned}\varepsilon &\in \mathrm{E}_{d+1}(A,(t)),\\ \beta &\in \mathrm{SL}_{d+1}(A,(t)) \cap S_d(p,q)\mathrm{E}(A),\end{aligned}$$

for some $p, q \in A^{d+1}$ *with* $\langle p, q\rangle = 1$*, such that*

(i) $v\varepsilon = (w_0, w_1, w_2, \ldots, w_d^m)$ $(= \chi_m(w))$,
(ii) *if* $d!|m$*, then* $\chi_m(w)\beta = e_1$.

Proof. The proof below is a relative version of the proof given in [8], Proposition 3.1.

Proof of (i): If A is not regular, let J be the ideal defining the singular locus of A. Clearly, $\dim(A/J) \le d-1$. By Theorem 4.1, $\mathrm{Um}_{d+1}(A/J, (\overline{t})) = e_1\ \mathrm{E}_{d+1}(A/J, (\overline{t}))$. Hence, we may modify v and assume that $v = e_1$ modulo J. If A is a regular ring, we may proceed directly to the next paragraph.

In view of Lemma 5.1 and Theorem 4.1, we may assume that A is a reduced ring.

We first assume that k is a perfect field.

By R.G. Swan's version in [17] of Bertini's theorem, as stated in [5], a general linear combination $u_0 = v_0 + \sum_{j\ge 1} t\lambda_j v_j$, for suitable $\lambda_j \in A$, will have the properties that $u_0 = 1$ modulo J and $A/(u_0)$ is non-singular outside the singular locus of A. But then $A/(u_0)$ is a regular ring. Clearly, by Theorem 4.1,

$$v \sim_{\mathrm{E}_{d+1}(A,(t))} (u_0, v_1, \ldots, v_d).$$

Let an overline denote modulo (u_0). Consider

$$(\overline{v}_1, \overline{v}_2, \ldots, t\overline{v}_{d-1}, t\overline{v}_d) \in \mathrm{Um}_d(\overline{A}).$$

By the standard Bertini theorem we can add multiples of $t\overline{v}_{d-1}$ and $t\overline{v}_d$ to $\overline{v}_1, \ldots, \overline{v}_{d-2}$ so that $v_i \mapsto v_i'$, $1 \le i \le d-2$, are such that

(1) $(u_0, v_1, \ldots, v_d)$ and $(u_0, v_1', \ldots, v_d')$ are in the same $\mathrm{E}_{d+1}(A, (t))$ orbit,
(2) $\dim(A/(u_0, v_1', \ldots, v_{d-3}')$ has dimension ≤ 2, and
(3) $\mathfrak{C}$ =Spec$(A/(u_0, v_1', \ldots, v_{d-2}'))$ is a non-singular curve.

Let $\Gamma(\mathfrak{C})$ denote the coordinate ring of $\mathfrak{C}$. $\mathrm{E}(\Gamma(\mathfrak{C}))$ will denote the infinite elementary group over $\Gamma(\mathfrak{C})$, and $\mathrm{ESp}(\overline{A})$ will denote the infinite elementary symplectic group over $\overline{A}$.

By [14], Proposition 1.7, if $\mathrm{char}(k) \ne 2$, then the canonical homomorphism $\mathrm{K}_1\mathrm{Sp}(\Gamma(\mathfrak{C})) \longrightarrow \mathrm{SK}_1(\Gamma(\mathfrak{C}))$ is an isomorphism. Moreover, by [14], Proposition 1.4, $\mathrm{SK}_1(\Gamma(\mathfrak{C}))$ is $2m$-divisible.

Hence, there exists $\overline{\alpha} \in \mathrm{SL}_2(\Gamma(\mathfrak{C})) \cap \mathrm{E}(\Gamma(\mathfrak{C}))$ such that $(\overline{v}_{d-1}, \overline{v}_d)\overline{\alpha} = (\overline{b}_{d-1}, \overline{b}_d^m)$, for some b_{d-1}, $b_d \in A$. By [14], $\overline{\alpha} \in \mathrm{ESp}(\overline{A})$. By [14], Lemma 2.1, or [18], Chapter III, $\overline{\alpha}$ has a lift $\alpha \in \mathrm{SL}_2(A) \cap \mathrm{ESp}_4(A)$.

We would like to ensure that this α satisfies $e_1\alpha = e_1$ modulo (t^2). In fact, we need $\alpha \equiv I_2$ modulo (t^2). For this, as in [8], Proposition 3.3, we work with the excision algebra introduced by W. van der Kallen, viz. $B = A[T]/(T^2 - t^2T)$, instead of A above, and with the corresponding row over it; in other words, let $v = e_1 + t^2 w$ and consider $u(T) = e_1 + Tw \in \mathrm{Um}_{d+2}(B)$. Apply the above reasoning to this row. Let $\beta = \alpha(t^2)\alpha(0)^{-1}$. Then $e_1\beta = e_1$ modulo (t^2). Moreover, $\beta_{22} = 1$ modulo (t^2). Hence $\beta\mathrm{E}_{21}(-\beta_{21}) \equiv I_2$ modulo (t^2). Rename $\beta\mathrm{E}_{21}(-\beta_{21})$ as β.

Now in the relative group $\mathrm{Um}_{d+1}(A, (t))/\mathrm{E}_{d+1}(A, (t))$,

$$[(u_0, v_1', \ldots, v_{d-1}', v_d)] \quad = \quad [((u_0, v_1')\beta, \ldots, v_{d-1}', v_d)]$$

via [3], Theorem 3.25 (iii), and Theorem 4.1. By Lemma 5.2 and Theorem 4.1,

$$[((u_0, v_1')\beta, \ldots, v_{d-1}', v_d)] \quad = \quad [((u_0, v_1')\beta, \ldots, v_{d-2}'(X), t^m(v_{d-1}, v_d))].$$

By Lemma 5.3 the right hand side above equals

$$[(u_0, v'_1, \ldots, v'_{d-2}, (t^m v_{d-1}, t^m v_d)\beta)].$$

This equals

$$[(u_0, v'_1, \ldots, v'_{d-2}, t^m(x, y^m)\alpha(0)^{-1})].$$

Obviously, this equals

$$[(u_0, v'_1, v'_2, \ldots, v'_{d-2}, (t^m x, (ty)^m)\alpha(0)^{-1})].$$

By Lemma 5.2 and Theorem 4.1, this equals

$$[(u_0, v'_1, v'_2, \ldots, v'_{d-2}, t^m(t^m x, (ty)^m)\alpha(0)^{-1})].$$

Via [3], Theorem 3.25 (iii), and Theorem 4.1 we get that this equals

$$[(u_0, v'_1, v'_2, \ldots, v'_{d-2}, t^m(t^m x, (ty)^m)].$$

Again via [3], Theorem 3.25 (iii), and Theorem 4.1 we get that this equals

$$[(u_0, v'_1, v'_2, \ldots, v'_{d-2}, t^m x, (ty)^m)].$$

If $mA = A$ (resp. $d!A = A$), then the row has an "m-th coordinate power" (resp. "factorial") row in its relative orbit.

If k is not perfect, then by Bertini's theorem as stated in [5] the curve $\mathfrak{C}$ above is geometrically smooth by nilpotent. If one tensors with the perfect field $L^\infty = \bigcup_n k^{1/p^n}$, the curve $\mathfrak{C} \otimes L^\infty$ is smooth by nilpotent. Hence there is a finite purely inseparable extension L over which one can find a factorial row in the relative orbit. One has the composite map

$$\frac{\mathrm{SK}_1(\Gamma(\mathfrak{C}), (\mathrm{t}))}{m\mathrm{SK}_1(\Gamma(\mathfrak{C}), (\mathrm{t}))} \longrightarrow \frac{\mathrm{SK}_1(\Gamma(\mathfrak{C}) \otimes L), (\mathrm{t}))}{m\mathrm{SK}_1(\Gamma(\mathfrak{C}) \otimes L, (\mathrm{t}))} \xrightarrow{\mathrm{Tr}} \frac{\mathrm{SK}_1(\Gamma(\mathfrak{C}), (\mathrm{t}))}{(d+1)!\mathrm{SK}_1(\Gamma(\mathfrak{C}), (\mathrm{t}))},$$

where the first map is the natural map and the second map is the induced transfer homomorphism.

The composite map is multiplication by $(L : k)$. Since g.c.d.$((L : k), \mathrm{char}(k)) = 1$, the triviality of the middle group implies the triviality of $\frac{\mathrm{SK}_1(\Gamma(\mathfrak{C}), (\mathrm{t}))}{m\mathrm{SK}_1(\Gamma(\mathfrak{C}), (\mathrm{t}))}$. The rest of the argument is now exactly as in the case where the base field is perfect.

Proof of (ii). Let $w' = (w_0, \ldots, w_d)$. Choose a v' with $\langle w', v' \rangle = 1$. By Corollary 4.3 there is a $\beta_r(w', v')$ which has $w = e_1\beta_r(w', v')$ and which is stably elementarily equivalent to $S_d(w', v')$. □

Remark 5.5. Suppose that we have $A = R[X]$, with R an affine algebra of dimension $d - 1$ as above. In the notation of the above proof, in this case we observe that β (chosen above) is stably elementary.

Proof. For any $\mathfrak{p} \in \mathrm{Spec}(A)$, the stable dimension of $A_\mathfrak{p}[X]$ is $d - 1$. Hence, $w'_\mathfrak{p}(X)$ and $w'_\mathfrak{p}(0)$ are in the same elementary orbit. By the Local Global Principle for elementary action, as enunciated in [7], it follows that $w'(X)$ and $w'(0)$ are in the same elementary orbit. As $w'(0)$ is elementarily completable, by Corollary 3.2 (iv) $\beta_r(w'(X), v'(X))$ is stably elementary. □

Proof of the main theorem. Let $\sigma \in \mathrm{SL}_{d+1}(A)$ be a stably elementary matrix. By classical stability estimates, σ is 1-stably elementary. Let $\rho(X) \in \mathrm{E}_{d+2}(A[X])$ be a homotopy of $(1 \perp \sigma)$ and I_{d+2}. Let $v(X) = e_1\rho(X)$.

Let k be a finite field. By L.N. Vaserstein's theorem (see [18], Chapter III) the ring $B = A[X,Z]/(Z^2 - (X^2 - X)Z)$ has stable dimension $\leq d$. In particular, it follows that $\mathrm{Um}_{d+2}(B) = e_1 \mathrm{E}_{d+2}(B)$. Arguing as in [8], Proposition 3.3, there is an

$$\varepsilon(X) \in \mathrm{E}_{d+2}(A) \cap \mathrm{SL}_{d+2}(A[X], (X^2 - X)),$$

with $v(X)\varepsilon(X) = e_1$. Now argue as in the proof of [8], Theorem 3.4, to get a stably elementary homotopy of σ.

Let k be an infinite perfect C_1 field. Apply Proposition 5.4 to get

$$\varepsilon(X) \in \mathrm{E}_{d+2}(A[X], (X^2 - X)),$$
$$\beta(X) \in \mathrm{SL}_{d+2}(A[X], (X^2 - X)) \cap \mathrm{E}_{d+2}(A[X]),$$

with $v(X)\varepsilon(X)\beta(X) = e_1$. Now argue as in the proof of [8], Theorem 3.4, to get a stably elementary homotopy of σ. □

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