

## ON ENDOMORPHISM RINGS OF $B_1$ -GROUPS THAT ARE NOT $B_2$ -GROUPS

LUTZ STRÜNGMANN

(Communicated by Julia Knight)

ABSTRACT. Finite rank Butler groups are pure subgroups of completely decomposable groups of finite rank and were defined by M.C.R. Butler. Extending this concept to infinite rank groups, Bican and Salce gave various possible descriptions: A  $B_2$ -group  $G$  is a union of an ascending chain of pure subgroups  $G_\alpha$  such that for every  $\alpha$  we have  $G_{\alpha+1} = G_\alpha + H_\alpha$  for some finite rank Butler group  $H_\alpha$ . A  $B_1$ -group is a torsion-free group  $G$  satisfying  $\text{Bext}_{\mathbb{Z}}^1(G, T) = 0$  for all torsion groups  $T$ . While the class of  $B_2$ -groups is contained in the class of  $B_1$ -groups, it is in general undecidable in ZFC if the two classes coincide. In this paper we study the endomorphism rings of  $B_1$ -groups which are not  $B_2$ -groups working in a model of ZFC that satisfies  $2^{\aleph_0} = \aleph_4$ .

### 1. INTRODUCTION

M.C.R. Butler [3] defined a finite rank Butler group as a pure subgroup or equivalently as an epimorphic image of an abelian completely decomposable group of finite rank. In the theory of Abelian groups the diversity of finite rank Butler groups has played an important role and has attracted much attention (see [1] or [9]). However, an attempt by Bican and Salce [2] to extend Butler's notion to infinite rank torsion-free abelian groups led to competing definitions. Among these the  $B_1$ -groups and  $B_2$ -groups turned out to be of a particular nature and interest. Recall that a torsion-free abelian group  $G$  is called *finitely Butler* if every finite rank pure subgroup  $H$  of  $G$  is a Butler group. A torsion-free group  $G$  is a  $B_2$ -group if it has a filtration  $G = \bigcup_{\alpha < \lambda} G_\alpha$  of pure subgroups  $G_\alpha$  such that for every  $\alpha < \lambda$ ,  $G_{\alpha+1} = G_\alpha + H_\alpha$  for some Butler group  $H_\alpha$  of finite rank. Finally, a torsion-free group  $G$  is called a  $B_1$ -group if  $\text{Bext}_{\mathbb{Z}}^1(G, T) = 0$  for all torsion groups  $T$ . Here  $\text{Bext}_{\mathbb{Z}}^1(-, -)$  denotes the subfunctor of  $\text{Ext}_{\mathbb{Z}}(-, -)$  consisting of all balanced exact extensions. Recall that a short exact sequence  $0 \rightarrow T \rightarrow M \xrightarrow{\varphi} G \rightarrow 0$  is *balanced exact* if for every rational group  $R \subseteq \mathbb{Q}$  and homomorphism  $\delta : R \rightarrow G$  there is a homomorphism  $\psi : R \rightarrow M$  such that  $\psi\varphi = \delta$ . Then  $\text{Bext}_{\mathbb{Z}}^1(G, T)$  consists of equivalence classes of such balanced exact sequences. In particular  $\text{Bext}_{\mathbb{Z}}^1(G, T) = 0$  if every balanced exact sequence as above splits, i.e. if there is  $\psi : G \rightarrow M$  such that  $\psi\varphi = \text{id} \upharpoonright_G$ .

---

Received by the editors November 24, 2008, and, in revised form, March 3, 2009.

2000 *Mathematics Subject Classification*. Primary 20K15, 20K20, 20K35, 20K40; Secondary 18E99, 20J05.

The author was supported by a grant from the German Research Foundation DFG.

©2009 American Mathematical Society  
Reverts to public domain 28 years from publication

It was pointed out in [2] that  $B_2$ -groups are finitely Butler and any  $B_2$ -group is a  $B_1$ -group. However, it had been open for a long time whether the two classes of  $B_1$ -groups and  $B_2$ -groups coincide. Assuming Gödel's axiom of constructibility, Fuchs and Magidor [6] were able to show that this is indeed the case in  $(V = L)$ . On the other hand Shelah and the author [12] proved the consistency with ZFC and  $2^{\aleph_0} = \aleph_4$  of the existence of  $B_1$ -groups which are not  $B_2$ -groups but still finitely Butler. If a  $B_1$ -group has to be finitely Butler is not yet known; this and many other questions on infinite rank Butler groups are still not solved.

In [10], Mader and the author started a systematic study of the groups constructed in [12] and defined the more general class of Hawaiian groups. These groups are built using data that consists of a set of rational groups and some numerical information. Various results on the structure of Hawaiian groups and their endomorphism rings were obtained in [10] without making additional set-theoretic assumptions. The present paper can be seen as a continuation of [10]. We will show that for certain data the Hawaiian groups are  $B_1$ -groups but not  $B_2$ -groups when adding  $\kappa$  Cohen reals to the universe. Moreover, we will prove that we can realize particular rings, e.g. the ring of integers, as the endomorphism ring of such Hawaiian groups and hence obtain in some model of ZFC and  $2^{\aleph_0} = \aleph_4$  the existence of indecomposable  $B_1$ -groups,  $B_1$ -groups with large dual and  $B_1$ -groups having strange decompositions, respectively, that are not  $B_2$ -groups. Moreover, we will show that all these groups are pure subgroups of completely decomposable groups, which answers a question posed in [10].

Our notation is standard and we write maps on the left. All groups under consideration are Abelian and written additively.  $\Pi$  denotes the set of all primes. If  $H$  is a pure subgroup of the Abelian group  $G$ , then we write  $H \subseteq_* G$ . Moreover,  $H_* \subseteq G$  denotes the purification of the subgroup  $H$  of a torsion-free group  $G$ . A reasonable knowledge about Abelian groups as can be found in [5] and also on set-theory as can be found in [8] is assumed. For further details on infinite rank Butler groups we refer to [12] and the references there.

The author would like to thank the referee for helpful comments and for carefully reading the manuscript.

## 2. HAWAIIAN GROUPS AND THEIR ENDOMORPHISM RINGS

In this section we recall the main definitions and results from [10] that will be needed in the sequel. Let  $\kappa$  be an infinite cardinal  $\leq 2^{\aleph_0}$ . We let

$$V := \bigoplus_{n < \omega} \mathbb{Q} x_n \oplus \bigoplus_{\alpha < \kappa} \mathbb{Q} y_\alpha$$

be the vector space with basis  $\{x_n, y_\alpha\}$ . Let

$$\mathcal{R} := (R_n \mid n < \omega) \quad \text{and} \quad \mathcal{S} := (S_\alpha \mid \alpha < \kappa)$$

be sequences of *rational groups*, by which we mean additive subgroups of  $\mathbb{Q}$  that contain  $\mathbb{Z}$ . We define

$$F := F_\kappa(\mathcal{R}, \mathcal{S}) = \left( \bigoplus_{n < \omega} R_n x_n \right) \oplus \left( \bigoplus_{\alpha < \kappa} S_\alpha y_\alpha \right)$$

to be the completely decomposable subgroup of  $V$  generated by the decomposition basis  $\{x_n, y_\alpha\}$ . A Hawaiian group with respect to the given data is defined as a torsion-free group sandwiched between  $F$  and  $V$  in the following way (see [10, Definition 2.1]).

**Definition 2.1.** Let  $\mathcal{Q} := (Q_n \mid n < \omega)$  be a sequence of rational groups, let  $\mathcal{P} := (p_n : n < \omega)$  be a sequence of (distinct) prime numbers, and finally let  $\mathcal{A} = (A_\alpha : \alpha < \kappa)$  be a sequence of subsets of  $\omega$  such that

$$\forall \alpha, \beta \ A_\alpha \cap A_\beta \neq \emptyset, \quad \bigcup_{\alpha < \kappa} A_\alpha = \omega, \quad \text{and} \quad \bigcap_{\alpha < \kappa} A_\alpha = \emptyset.$$

The group

$$B = B_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{Q}, \mathcal{A}, \mathcal{P}) = \langle F_\kappa(\mathcal{R}, \mathcal{S}), p_n^{-1}Q_n(y_\alpha - x_n) : \alpha < \kappa, n \in A_\alpha \rangle \subseteq V$$

is called a  $\kappa$ -Hawaiian group or simply a *Hawaiian group*. The set of  $\kappa$ -Hawaiian groups with the specified data is denoted by  $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{S})$ .

Obviously, the definition of a Hawaiian group is very general, and it is not surprising that additional conditions were imposed on the rational groups involved in the data in [10]. First we recall some notation from [10, Definition 2.2], where the type of a rational group is its isomorphism class.

**Definition 2.2.** Let  $B = B_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{Q}, \mathcal{A}, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R}, \mathcal{S})$ , and  $\alpha < \beta < \kappa$ . Then set

- (1)  $\delta := \text{type}(\mathbb{Z})$ ;
- (2)  $P_\alpha := \langle p_n^{-1} : n \in A_\alpha \rangle$ ;  $\delta_\alpha := \text{type}(P_\alpha)$ ;
- (3)  $P := \langle p_n : n < \omega \rangle = \sum_{\rho < \kappa} P_\rho$ ;  $R := \sum_{n < \omega} R_n$ ;  
 $S := \sum_{\rho < \kappa} S_\rho$ ;  $Q := \sum_{n < \omega} Q_n$ ;
- (4)  $P_{\alpha, \beta} := P_\alpha \cap P_\beta = \langle p_n^{-1} : n \in A_\alpha \cap A_\beta \rangle$ ;  $\delta_{\alpha, \beta} := \text{type}(P_{\alpha, \beta}) = \delta_\alpha \wedge \delta_\beta$ ;
- (5)  $\widehat{Q}_\alpha := \sum_{n \in A_\alpha} Q_n$ ;  $\tau_\alpha := \text{type}(\widehat{Q}_\alpha)$ ;
- (6)  $Q_{\alpha, \beta} := \sum_{n \in A_\alpha \cap A_\beta} Q_n$ ;  $\tau_{\alpha, \beta} := \text{type}(Q_{\alpha, \beta})$ ;
- (7)  $\sigma_\alpha := \text{type}(S_\alpha)$ ;  $S_{\alpha, \beta} = S_\alpha \cap S_\beta$ ;  $\sigma_{\alpha, \beta} = \text{type}(S_{\alpha, \beta}) = \sigma_\alpha \wedge \sigma_\beta$ .

Recall from [10, after Definition 2.2] the following conditions:

- (2.1)  $P \cap Q = \mathbb{Z}$  and  $\forall n < \omega : Q_n \cap \sum_{i: i < \omega, i \neq n} Q_i = \mathbb{Z}$ ;
- (2.2)  $S \cap Q = \mathbb{Z}$  and  $S \cap P = \mathbb{Z}$ ;
- (2.3)  $R \cap P = \mathbb{Z}$  and  $R \cap Q = \mathbb{Z}$ .

All these conditions are satisfied if  $R = S = Q = \mathbb{Z}$ , but it is not difficult to exhibit many other examples of rational groups satisfying these conditions. In fact, in our application in Section 3, all groups will be of this kind, i.e. will satisfy (2.1)-(2.3). Moreover, they will also satisfy  $S_\alpha = \mathbb{Z}$  for all  $\alpha < \kappa$ ; hence we drop this part from the data and call the class of such Hawaiian groups  $\mathcal{B}_\kappa(\mathcal{R})$ . If all participating groups  $R_n, S_\alpha$  and  $Q_n$  are chosen to be the group of integers, then the class of Hawaiian groups with these specific data was considered in [12]. Our strategy is to change the groups  $R_n$  and  $Q_n$  for  $n < \omega$  and to prove that the results from [12] still hold for this larger class of groups and then use the results from [10] to say something about the endomorphism rings of these groups. We therefore state the more general results from [10] that we will need in a form adjusted to our particular setting.

In order to say something about the endomorphism rings of Hawaiian groups, Mader and the author made stronger assumptions on the defining sequences  $\mathcal{A}$  (see [10, Definition 3.1]).

**Definition 2.3.** Let  $\aleph$  be an infinite cardinal. A set of types  $\{\delta_\alpha \mid \alpha < \aleph\}$  is called a *strong anti-chain of size  $\aleph$*  if it satisfies the following condition.

If  $\alpha < \aleph$  and  $E \subseteq \aleph$  is a finite subset of  $\aleph$  such that  $\alpha \notin E$ , then  $\delta_\alpha$  and  $\bigwedge_{\rho \in E} \delta_\rho$  are incomparable.

Some easy combinatorics show that there exist strong anti-chains of any cardinality less than or equal to the continuum (see [10, Theorem 3.4]).

We now define strong Hawaiian groups as in [10, Definition 3.5]. Recall that in the context of Hawaiian groups we have  $P_\alpha := \langle p_n^{-1} : n \in A_\alpha \rangle$  and  $\delta_\alpha := \text{type}(P_\alpha)$ .

**Definition 2.4.** A Hawaiian group  $B = B_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R})$  is called a *strong Hawaiian group* if  $\{\delta_\alpha \mid \alpha < \kappa\}$  is a strong anti-chain of size  $\kappa$ . We denote the strong Hawaiian groups with the specified data by  $B_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}^*, \mathcal{P})$ .

We now pass to our specialized setting, namely, strong Hawaiian groups satisfying (2.1) - (2.3). One of the main results from [10, Theorem 2.11 and Theorem 3.6] is the following.

**Theorem 2.5.** *Let  $B = B_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}^*, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R})$  be a strong Hawaiian group satisfying (2.1) - (2.3) and assume  $\text{cf}(\kappa) > \aleph_0$ . Then  $B$  is finitely Butler but not a  $B_2$ -group.*

We will now show that the groups from  $\mathcal{B}_\kappa(\mathcal{R})$  are even pure subgroups of completely decomposable groups.

**Theorem 2.6.** *Every Hawaiian group  $B$  in  $\mathcal{B}_\kappa(\mathcal{R})$  satisfying (2.1) - (2.3) is a pure subgroup of some completely decomposable group.*

*Proof.* Let  $B \in \mathcal{B}_\kappa(\mathcal{R})$  be a Hawaiian group satisfying (2.1) - (2.3). Then  $B$  is of the form

$$B = \left\langle \bigoplus_{n < \omega} R_n x_n \oplus \bigoplus_{\alpha < \kappa} \mathbb{Z} y_\alpha, p_n^{-1} Q_n (y_\alpha - x_n) : \alpha < \kappa, n \in A_\alpha \right\rangle$$

and satisfies (2.1) to (2.3). It will be important in the sequel that this implies that the  $Q_n$ ,  $R_n$  and  $A_\alpha$  involve disjoint sets of primes. We now define a completely decomposable group  $D$  via

$$D = \bigoplus_{\alpha < \kappa} (P_\alpha + \widehat{Q}_\alpha) b_\alpha \oplus \bigoplus_{n < \omega} (Q_n + R_n) e_n \oplus Ra.$$

We claim that  $B$  can be embedded purely into  $D$  and define  $\varphi : B \rightarrow D$  via

- $\varphi(x_n) = a + p_n e_n$  if  $n < \omega$ ,
- $\varphi(y_\alpha) = a + b_\alpha$  if  $\alpha < \kappa$ ,
- $\varphi(p_n^{-1}(y_\alpha - x_n)) = p_n^{-1} b_\alpha - e_n$  if  $n \in A_\alpha$  and  $\alpha < \kappa$ .

By the definition of  $D$  and the choice of  $A_\alpha$  ( $\alpha < \kappa$ ) this is a well-defined homomorphism from  $B$  into  $D$ . We have to prove that  $\varphi$  is a pure embedding. Clearly,  $\varphi$  is injective on  $\bigoplus_{n < \omega} R_n x_n \oplus \bigoplus_{\alpha < \kappa} \mathbb{Z} y_\alpha$  and this subgroup is essential in  $B$ ; hence  $\varphi$  is injective. Therefore it is enough to show that  $\varphi$  maps  $B$  purely into  $D$ . Thus, let  $p \in \Pi$  be a prime and assume that  $x \in B$  such that  $\varphi(x) \in pD$ . Then  $x$  is of the form

$$(2.4) \quad x = \sum_{n < \omega} z'_n x_n + \sum_{\alpha < \kappa} z'_\alpha y_\alpha + \sum_{(n, \alpha) : \alpha < \kappa, n \in A_\alpha} \frac{t_n^\alpha}{p^n} (y_\alpha - x_n)$$

for some  $z'_n \in R_n$ ,  $z'_\alpha \in \mathbb{Z}$  and  $t_n^\alpha \in Q_n$ . Note that almost all coefficients in equation (2.4) are zero. We have to prove that  $x \in pB$ . Since  $\varphi(x) \in pD$  there are

$z_\alpha \in (P_\alpha + \sum_{n:n \in A_\alpha} Q_n)$  and  $z_n \in (Q_n + R_n)$ ,  $z_a \in \sum_{n < \omega} R_n$  such that

$$\begin{aligned}
 & p \left( \sum_{\alpha < \kappa} z_\alpha b_\alpha + \sum_{n < \omega} z_n e_n + z_a a \right) = \varphi(x) \\
 (2.5) \quad & = \varphi \left( \sum_{n < \omega} z'_n x_n + \sum_{\alpha < \kappa} z'_\alpha y_\alpha + \sum_{(n,\alpha): \alpha < \kappa, n \in A_\alpha} \frac{t_n^\alpha}{p_n} (y_\alpha - x_n) \right) \\
 & = \sum_{n < \omega} z'_n (a + p_n e_n) + \sum_{\alpha < \kappa} z'_\alpha (a + b_\alpha) + \sum_{(n,\alpha): \alpha < \kappa, n \in A_\alpha} \left( \frac{t_n^\alpha}{p_n} b_\alpha - t_n^\alpha e_n \right).
 \end{aligned}$$

Equating coefficients yields

- (i)  $p z_\alpha = z'_\alpha + \sum_{n:n \in A_\alpha} \frac{t_n^\alpha}{p_n}$  for  $\alpha < \kappa$ ,
- (ii)  $p z_n = p_n z'_n - \sum_{\alpha:n \in A_\alpha} t_n^\alpha$  for  $n < \omega$ ,
- (iii)  $p z_a = \sum_{n < \omega} z'_n + \sum_{\alpha < \kappa} z'_\alpha$ .

We now distinguish two cases, namely, if  $p = p_m \in \mathcal{P}$  for some  $m < \omega$  or  $p \neq p_m$  for all  $m < \omega$ . Since both cases are similar, we will assume that  $p = p_m$  for some  $m \in \omega$  and leave the second case to the reader. We choose  $0 \neq s \in \mathbb{Z}$  minimal such that  $p_m s z_\alpha \in \mathbb{Z}$  for all  $\alpha < \kappa$ . By the choice of  $R_n$  and  $Q_n$  (see (2.1) and (2.3)) we conclude that  $\frac{1}{p} \notin Q_n$  and  $\frac{1}{p} \notin R_n$  for all  $n < \omega$ . Moreover, since  $h_{p_n}^{\mathcal{P}}(1)$  is either 1 or 0 for all  $\alpha < \kappa$  and  $n < \omega$  we obtain  $\gcd(s, p_m) = 1$ . Let  $k, l \in \mathbb{Z}$  such that  $ks + lp_m = 1$ . Then

$$\begin{aligned}
 x &= \sum_{n < \omega} \left( z'_n - \sum_{\alpha:n \in A_\alpha} \frac{kst_n^\alpha}{p_n} \right) x_n + \sum_{\alpha < \kappa} \left( z'_\alpha + \sum_{n:n \in A_\alpha} \frac{kst_n^\alpha}{p_n} \right) y_\alpha \\
 &+ \sum_{(n,\alpha): \alpha < \kappa, n \in A_\alpha} lp_m \frac{t_n^\alpha}{p_n} (y_\alpha - x_n).
 \end{aligned}$$

Now, the last term  $\sum_{(n,\alpha): n \in A_\alpha} lp_m \frac{t_n^\alpha}{p_n} (y_\alpha - x_n) \in p_m B$ , and  $\sum_{n:n \in A_\alpha} \frac{kst_n^\alpha}{p_n} \in \mathbb{Z}$  as well as  $\sum_{\alpha:n \in A_\alpha} \frac{kst_n^\alpha}{p_n} \in \mathbb{Z}$  by Lemma 2.5 from [10]. Hence we may assume without loss of generality that  $x$  is of the form

$$x = \sum_{n < \omega} z'_n x_n + \sum_{\alpha < \kappa} z'_\alpha y_\alpha$$

for some  $z'_n \in R_n$  and  $z'_\alpha \in \mathbb{Z}$ . Equating coefficients we obtain as before

- (iv)  $p_m z_\alpha = z'_\alpha$  for  $\alpha < \kappa$ ,
- (v)  $p_m z_n = p_n z'_n$  for  $n < \omega$ ,
- (vi)  $p_m z_a = \sum_{n < \omega} z'_n + \sum_{\alpha < \kappa} z'_\alpha$ .

Now write  $x$  as

$$\begin{aligned} x &= \sum_{n < \omega, n \neq m} z'_n x_n + z'_m x_m + \sum_{\alpha: m \notin A_\alpha} z'_\alpha y_\alpha + \sum_{\alpha: m \in A_\alpha} z'_\alpha y_\alpha \\ &= \sum_{n < \omega, m \neq n} z'_n x_n + (z'_m + \sum_{\alpha: m \in A_\alpha} z'_\alpha) x_m + \sum_{\alpha: m \notin A_\alpha} z'_\alpha y_\alpha \\ &\quad + p_m \sum_{\alpha: m \in A_\alpha} z'_\alpha p_m^{-1} (y_\alpha - x_m). \end{aligned}$$

Again, the last term  $p_m \sum_{\alpha: m \in A_\alpha} z'_\alpha p_m^{-1} (y_\alpha - x_m) \in p_m B$ ; hence we may assume without loss of generality that it is zero, so

$$x = \sum_{n < \omega, n \neq m} z'_n x_n + z'_m x_m + \sum_{\alpha: m \notin A_\alpha} z'_\alpha y_\alpha.$$

Applying  $\varphi$  and equating coefficients now yields

- (vii)  $p_m z_\alpha = z'_\alpha$  for all  $\alpha$  such that  $m \notin A_\alpha$ ,
- (viii)  $p_m z_n = p_n z'_n$  for  $n \neq m$ ,
- (ix)  $p_m z_a = (\sum_{\alpha: m \notin A_\alpha} z'_\alpha + \sum_{n < \omega} z'_n)$ .

Hence  $z'_\alpha \in p_m \mathbb{Z}$  by (vii) and  $z'_n \in p_m \mathbb{Z}$  by (viii) and (ix), so  $x \in p_m B$ . This finishes the proof. □

Finally, we recall some results from [10] on the endomorphism rings of strong Hawaiian groups. We need to explain some more notation used for classes of Hawaiian groups in [10], namely,

- $B_\kappa(\mathcal{R}^*, \mathcal{Q}, \mathcal{A}^*, \mathcal{P})$  is the group  $B_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}^*, \mathcal{P})$ , where  $\mathcal{R}$  is such that the types  $\text{type}(R_n)$  are pairwise incomparable;
- $B_\kappa(\mathcal{A}^*, \mathcal{P})$  (respectively  $B_\kappa(\mathcal{A}, \mathcal{P})$ ) is the group  $B_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}^*, \mathcal{P})$  (respectively  $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}, \mathcal{P})$ ), where  $\mathcal{R}$  and  $\mathcal{Q}$  are all sequences of the integers  $\mathbb{Z}$ .

The following lemma shows that the Hawaiian groups may have many free summands but may also be indecomposable.

**Lemma 2.7** ([10, Corollary 4.7 and Corollary 4.8]). *Let  $B = B_\kappa(\mathcal{A}, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R})$ . Then  $|\text{Hom}_{\mathbb{Z}}(B, \mathbb{Z})| \geq 2^{\aleph_0}$ . Moreover, if  $B = B_\kappa(\mathcal{A}^*, \mathcal{P})$  and  $A_\alpha$  is infinite for all  $\alpha < \kappa$ , then*

$$\text{Hom}_{\mathbb{Z}}(B, \mathbb{Z}) \cong \prod_{n < \omega} \mathbb{Z} \quad \text{and} \quad |\text{Hom}_{\mathbb{Z}}(B, \mathbb{Z})| = 2^{\aleph_0}.$$

Recall that the nucleus  $\text{nuc}(R)$  of a rational group is the largest subring of  $R$  or equivalently  $\text{End}(R)$ .

**Lemma 2.8** ([10, Corollary 5.6]). *Let  $B = B_\kappa(\mathcal{R}^*, \mathcal{Q}, \mathcal{A}^*, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R})$  satisfy (2.1)–(2.3) and the additional property that  $q_0 R = R$  for some prime number  $q_0 \notin \mathcal{P}$  and  $\{\text{type}(R_n) : n < \omega\}$  is an anti-chain. Then  $f \in \text{End}_{\mathbb{Z}}(B)$  if and only if  $f$  is induced by a linear transformation  $y_\alpha \mapsto r y_\alpha$ ,  $x_n \mapsto r x_n - \epsilon_n p_n x_n$ , where  $r \in \mathbb{Z}$  and  $\epsilon_n \in \text{nuc}(R_n)$ .*

Finally, we have

**Lemma 2.9** ([10, Corollary 5.7]). *Let  $B = B_\kappa(\mathcal{R}^*, \mathcal{Q}, \mathcal{A}^*, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R})$  satisfy (2.1)–(2.3) and the additional property that  $q_0R = R$  for some prime number  $q_0 \notin \mathcal{P}$  and  $\{\text{type}(R_n) : n < \omega\}$  is an anti-chain. Assume that  $Q_n \not\subseteq R_n$  for all  $n < \omega$ . Then  $\text{End}_{\mathbb{Z}}(B) = \mathbb{Z}$ .*

3.  $B_1$ -GROUPS

In this section we shall use [12] together with the results from the previous section to show that the existence of  $B_1$ -groups that are not  $B_2$ -groups but have a certain prescribed endomorphism ring is consistent with ZFC. We first need to define a particular ring. Let  $\Pi_p = \{p_n : n < \omega\}$  be an infinite set of primes.

**Definition 3.1.** Let  $L = (\mathbb{Z} \times \prod_{n < \omega} p_n \mathbb{Z}, +, *)$  be the ring defined as follows for the operations  $+$  and  $*$  for  $(r, p_n r_n : n < \omega), (s, p_n s_n : n < \omega) \in L$ :

- (1)  $(r, p_n r_n : n < \omega) + (s, p_n s_n : n < \omega) = (r + s, p_n(r_n + s_n) : n < \omega)$ ;
- (2)  $(r, p_n r_n : n < \omega) * (s, p_n s_n : n < \omega) = (rs, p_n(rs_n + sr_n - p_n r_n s_n) : n < \omega)$ .

The following lemma is easily verified.

**Lemma 3.2.** *Let  $L = (\mathbb{Z} \times \prod_{n < \omega} p_n \mathbb{Z}, +, *)$  be defined as in Definition 3.1. Then  $L$  is a unital associative ring such that*

- (1)  $1_L = (1, 0 : n < \omega)$ .
- (2)  $0_L = (0, 0 : n < \omega)$ .
- (3)  $L$  is commutative.
- (4)  $L$  does not contain idempotents  $\neq 0_r, 1_L$ .
- (5)  $L$  contains (many) zero divisors; for instance, if  $m \in \mathbb{Z}$ , then  $(p_m, 1 : n < \omega)$  is a zero divisor.
- (6) the group of units  $U(L)$  satisfies  $U(L) = \{1_L, -1_L\}$ .
- (7) the nil radical  $\text{nil}(L)$  of  $L$  satisfies  $\text{nil}(L) = \{0_L\}$ .
- (8)  $\bar{s} = (s, p_n s_n : n < \omega) \in L$  is von Neumann regular if and only if  $\bar{s}^3 = \bar{s}$ .
- (9)  $\prod_{n < \omega} p_n \mathbb{Z}$  is a two-sided ideal of  $L$  and the multiplication is given by  $(p_n r_n : n < \omega) * (p_n s_n : n < \omega) = (-r_n s_n p_n^2 : n < \omega)$ .

*Proof.* Claims (1), (2), (3), (5) and (9) are easy to check. In order to prove (4) through (8) assume that  $\bar{r} = (r, p_n r_n : n < \omega) \in L$  and  $\bar{s} = (s, p_n s_n : n < \omega) \in L$ . Then

$$(3.1) \quad \bar{r} * \bar{s} = (rs, (rs_n + sr_n - r_n s_n p_n)p_n : n < \omega).$$

To check (4) assume that  $\bar{r}^2 = \bar{r}$ . By equation (3.1) it follows that

$$(3.2) \quad r^2 = r \quad \text{and} \quad (2rr_n - r_n^2 p_n)p_n = r_n p_n$$

for all  $n < \omega$ . Thus  $r = 1$  or  $r = 0$  and  $r_n = (2rr_n - r_n^2 p_n)$  for all  $n < \omega$ . If  $r = 0$ , then  $r_n = 0$  follows and we obtain  $\bar{r} = 0_L$ . If  $r = 1$ , then  $r_n = r_n^2 p_n$  follows and again we get  $r_n = 0$ . Thus  $\bar{r} = 1_L$  in this case.

Now assume that  $\bar{r} * \bar{s} = 1_L$ . Hence

$$(3.3) \quad rs = 1 \quad \text{and} \quad (rs_n + sr_n - r_n s_n p_n)p_n = 0$$

for all  $n < \omega$  by equation (3.1). Thus  $r = s \in \{1, -1\}$  and  $|s_n + r_n| = |r_n s_n p_n|$  follows, which immediately implies  $r_n = s_n = 0$  and hence  $\bar{s}, \bar{r} \in \{1_L, -1_L\}$ , which shows (6).

Similar calculations prove (7) and (8). □

As in Lemma 2.8 we let  $\mathcal{R}_1 = (R_n^1 : n < \omega)$  and  $\mathcal{Q}_1 = (\mathbb{Z} : n < \omega)$  be such that the types  $\{\text{type}(R_n^1) : n < \omega\}$  form an anti-chain and assume that  $\text{nuc}(R_n^1) = \mathbb{Z}$  for all  $n < \omega$ .

**Proposition 3.3.** *Let  $\kappa \leq 2^{\aleph_0}$  be an infinite cardinal and  $B = B_\kappa(\mathcal{R}_1^*, \mathcal{Q}_1, \mathcal{A}^*, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R}_1)$  a strong Hawaiian group satisfying (2.1) - (2.3). Moreover, let  $L = (\mathbb{Z} \times \prod_{n < \omega} p_n \mathbb{Z}, +, *)$  be the ring from Definition 3.1. Then  $\text{End}_{\mathbb{Z}}(B) \cong L$  as rings.*

*Proof.* Let  $B$  and  $L$  be as stated in Proposition 3.3. Then any endomorphism  $f \in \text{End}_{\mathbb{Z}}(B)$  satisfies  $f(y_\alpha) = r^f y_\alpha$  and  $f(x_n) = r^f x_n - p_n r_n^f x_n$  for some  $r^f, r_n^f \in \mathbb{Z}$  and all  $\alpha < \kappa, n < \omega$  by Lemma 2.8. We define a map  $\Phi : \text{End}_{\mathbb{Z}}(B) \rightarrow L$  by sending  $f$  to the sequence  $(r^f, r_n^f : n < \omega)$ . It is now straightforward to check that  $\Phi$  is a ring isomorphism by the choice of the operations  $+$  and  $*$  in  $L$ .  $\square$

Our main theorem is

**Theorem 3.4.** *Let  $\kappa \geq \aleph_4$  be a regular uncountable cardinal. Then the following is consistent with ZFC and  $2^{\aleph_0} = \kappa$ .*

- (1) *There is a  $B_1$ -group  $B$  of cardinality  $\kappa$  that is a pure subgroup of some completely decomposable group but is not a  $B_2$ -group and satisfies  $|\text{Hom}_{\mathbb{Z}}(B, \mathbb{Z})| = \kappa$ . In fact,  $\text{Hom}_{\mathbb{Z}}(B, \mathbb{Z}) \cong \prod_{n < \omega} \mathbb{Z}$ .*
- (2) *There is a  $B_1$ -group  $B$  of cardinality  $\kappa$  that is a pure subgroup of some completely decomposable group but is not a  $B_2$ -group and  $\text{End}_{\mathbb{Z}}(B) \cong (\mathbb{Z} \times \prod_{n < \omega} p_n \mathbb{Z}, +, *)$  as rings. In particular,  $B$  is indecomposable and  $\text{Aut}_{\mathbb{Z}}(B) = \{id, -id\}$ .*
- (3) *There is a  $B_1$ -group  $B$  of cardinality  $\kappa$  that is a pure subgroup of some completely decomposable group but is not a  $B_2$ -group and satisfies  $\text{End}_{\mathbb{Z}}(B) = \mathbb{Z}$ .*

*Proof.* The proof follows [12, Theorem 4.3] with some algebraic modifications. We therefore recall it briefly and point out the changes as well as prove the new algebraic facts. Let  $M$  be a countable transitive model of ZFC in which the generalized continuum hypothesis holds, i.e.  $2^\lambda = \lambda^+$  for all infinite cardinals  $\lambda$ . Moreover, let  $\kappa \geq \aleph_4$  be regular and let  $\mathfrak{C}$  be the forcing of adding  $\kappa$  Cohen reals, i.e.  $\mathfrak{C} = \{f : \kappa \rightarrow 2 \mid \text{Dom}(f) \subseteq \kappa \text{ is finite}\}$ . Then  $\mathfrak{C}$  preserves cardinals and cofinalities and  $2^{\aleph_0} = \kappa$  in any extension model  $M[H]$  of  $M$ , where  $H$  is a  $\mathfrak{C}$ -generic filter. Let  $\tilde{\eta}_\alpha$  denote the  $\mathfrak{C}$ -names for the Cohen reals  $\eta_\alpha(H) = \tilde{\eta}_\alpha : \omega \rightarrow 2$  for  $\alpha < \kappa$ . Moreover, let  $\tilde{A}_\alpha = \{n < \omega : \tilde{\eta}_\alpha(n) = 1\}$ . Let  $\tilde{A} = (\tilde{A}_\alpha : \alpha < \kappa)$ . We now consider strong Hawaiian groups  $B = \bar{B}_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}^*, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R})$  for some data  $\mathcal{R}, \mathcal{Q}$  satisfying (2.1) - (2.3). In fact we shall choose  $\mathcal{R}, \mathcal{Q}$  differently for (1), (2) and (3). Let  $\Pi = \Pi_1 \cup \Pi_2 \cup \Pi_3$  be the union of three disjoint infinite sets. Moreover, enumerate  $\Pi_1 = \{p_n : n < \omega\}$  increasingly and put  $\mathcal{P} = \Pi_1 = (p_n : n < \omega)$  for notational reasons. Finally, let  $\Pi_2 = \bigcup_{n < \omega} \Pi_2^n$  as well as  $\Pi_3 = \bigcup_{n < \omega} \Pi_3^n$  be the disjoint union of infinite subsets  $\Pi_2^n$  and  $\Pi_3^n$  respectively such that  $p > k$  for all  $p \in \Pi_3^k$  and  $k \in \omega$ .

- For (1) let  $\mathcal{R} = \mathcal{Q} = (\mathbb{Z} : n < \omega)$ .
- For (2) put  $R_n = \langle p^{-1} : p \in \Pi_2^n \rangle$  and let  $\mathcal{R} = (R_n : n < \omega)$  and  $\mathcal{Q} = (\mathbb{Z} : n < \omega)$ .
- For (3) let  $\mathcal{R} = (R_n : n < \omega)$  as in (2) and  $\mathcal{Q} = (Q_n : n < \omega)$ , where  $Q_n = \langle p^{-1} : p \in \Pi_3^n \rangle$ .



It follows that any  $B = \bar{B}_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}^*, \mathcal{P}) \in \mathcal{B}_\kappa(\mathcal{R})$  satisfying (2.1) - (2.3) is a strong Hawaiian group and Theorems 2.6 and 2.5 prove that  $B$  is not a  $B_2$ -group but a pure subgroup of some completely decomposable group. Moreover, Lemma 2.7, Proposition 3.3 and Lemma 2.9 show that strong Hawaiian groups in  $\mathcal{B}_\kappa(\mathcal{R})$  have the desired endomorphism ring noting that  $\text{nuc}(R_n) = \mathbb{Z}$  for all  $n < \omega$ . We need to show that in some extension model  $M[H]$  of  $M$  there is a strong Hawaiian group in  $\mathcal{B}_\kappa(\mathcal{R})$  which is a  $B_1$ -group. The proof for (1) is exactly as in [12], and the proof of (2) is easier with less modifications than the one for (3). Hence we shall only prove (3) using  $\bar{B}_\kappa(\mathcal{R}, \mathcal{Q}, \mathcal{A}^*, \mathcal{P})$  as in (3) and satisfying (2.1) - (2.3). Thus the groups under consideration are of the form

$$(3.4) \quad \tilde{B} = B(\tilde{\mathcal{A}}) = \left\langle \bigoplus_{n < \omega} R_n x_n \oplus \bigoplus_{\alpha < \kappa} \mathbb{Z} y_\alpha, p_n^{-1} Q_n(y_\alpha - x_n) : n \in \tilde{A}_\alpha, \alpha < \kappa \right\rangle$$

with  $R_n = \langle p^{-1} : p \in \Pi_2^n \rangle$  and  $Q_n = \langle p^{-1} : p \in \Pi_3^n \rangle$  for  $n < \omega$ . The main difference to [12] is that the generators  $x_n$  and  $(y_\alpha - x_n)$  have types larger than  $\mathbb{Z}$  in our situation.

By way of contradiction assume that  $\mathfrak{C}$  forces that  $\tilde{B}$  is not a  $B_1$ -group; hence  $\text{Bext}_{\mathbb{Z}}^1(\tilde{B}, \tilde{T}) \neq 0$  for some torsion group  $\tilde{T}$ . Let

$$(3.5) \quad (\text{E}) \quad 0 \rightarrow \tilde{T} \xrightarrow{id} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{B} \rightarrow 0$$

be forced to be a non-splitting balanced exact sequence with  $\tilde{T}$ . Thus there exists  $r^* \in \mathfrak{C}$  such that

$$(3.6) \quad r^* \Vdash "0 \rightarrow \tilde{T} \xrightarrow{id} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{B} \rightarrow 0 \text{ is balanced exact.}"$$

We now choose preimages  $\tilde{g}_\alpha \in \tilde{G}$  of  $y_\alpha$  under  $\tilde{\varphi}$  for all  $\alpha < \kappa$ . Similarly let  $\tilde{x}_n \in \tilde{G}$  be a preimage for  $x_n$  under  $\tilde{\varphi}$  for  $n < \omega$  such that  $p|\tilde{x}_n$  in  $\tilde{G}$  for all  $p \in \Pi_2^n$ . Note that such a choice is possible since the sequence (E) is balanced. It is our aim to show that the balanced exact sequence (3.5) is forced to split. Hence it is enough to prove that the homomorphism  $\tilde{\varphi}$  is right-invertible; i.e. we have to find  $\tilde{\psi} : \tilde{B} \rightarrow \tilde{G}$  such that  $\tilde{\varphi}\tilde{\psi} = id_{\tilde{B}}$ . Therefore it is necessary to find preimages of the generators of  $\tilde{B}$  in  $\tilde{G}$  such that equations satisfied in  $\tilde{B}$  also hold in  $\tilde{G}$ . For simplicity let us assume for the moment that we work in a fixed extension model  $M[H]$ . We need the following definition.

**Definition 3.5.** Let  $\alpha < \kappa$  and  $t \in T$  be arbitrary. Then the set  $R_{\alpha,t}$  is defined as

$$R_{\alpha,t} = \{n \in A_\alpha : g_\alpha - t - \bar{x}_n \text{ is not divisible by some } p \in \{p_n\} \cup \Pi_3^n\}.$$

A purely group theoretic argument was used in [12, Theorem 4.3 and Lemma 4.5] to show that, if for every  $\alpha < \kappa$  there is a  $t_\alpha \in T$  such that  $R_{\alpha,t_\alpha}$  is finite, then  $\varphi$  is invertible. We have to adjust the proof to our situation.

*Fact 3.6.* Let  $\alpha < \kappa$  and let  $t \in T$  such that  $R_{\alpha,t}$  is finite. Then there exists  $t_\alpha \in T$  such that  $R_{\alpha,t_\alpha} = \emptyset$ .

*Proof.* Since  $R_{\alpha,t}$  is finite we may assume without loss of generality that  $R_{\alpha,t}$  has minimal cardinality. Assume that  $R_{\alpha,t}$  is not empty and fix  $n \in R_{\alpha,t}$ . By the primary decomposition theorem we decompose  $T$  as

$$T = T_{p_n} \oplus T' \oplus T_{\Pi_3^n},$$

where  $T_{p_n}$  denotes the  $p_n$ -primary component of  $T$  and  $T_{\Pi_3^n} = \bigoplus_{v \in \Pi_3^n} T_v$ . Since  $n \in A_\alpha$  it follows that  $p$  divides  $(y_\alpha - x_n)$  for all  $p \in \{p_n\} \cup \Pi_3^n$ . Hence there exists  $z \in G$  such that

$$(3.7) \quad \varphi(z) = (y_\alpha - x_n) \text{ and } p|z \text{ for all } p \in \{p_n\} \cup \Pi_3^n$$

since the sequence (E) is balanced. Thus

$$(3.8) \quad (g_\alpha - t - \bar{x}_n) - z \in T = T_{p_n} \oplus T' \oplus T_{\Pi_3^n}$$

and therefore there exist  $t_0 \in T_{p_n}$ ,  $t_1 \in T'$  and  $t_2 \in T_{\Pi_3^n}$  such that

$$(3.9) \quad (g_\alpha - t - \bar{x}_n) - z = t_0 + t_1 + t_2.$$

Let  $t' = t + t_0 + t_2$ . We claim that  $R_{\alpha,t'}$  has smaller cardinality than  $R_{\alpha,t}$ , a contradiction. By the choice of  $t'$  we have

$$(3.10) \quad (g_\alpha - t' - \bar{x}_n) = g_\alpha - t - t_0 - t_2 - \bar{x}_n = (z + t_1),$$

which is divisible by all  $p \in \{p_n\} \cup \Pi_3^n$  since  $z$  and  $t_1$  are. Hence  $n \notin R_{\alpha,t'}$ . But on the other side, if  $m \notin R_{\alpha,t}$ , then all  $p \in \{p_m\} \cup \Pi_3^m$  divide  $(g_\alpha - t - \bar{x}_m)$  and thus every  $p \in \{p_m\} \cup \Pi_3^m$  divides  $(g_\alpha - (t' - t_0 - t_2) - \bar{x}_m)$ . Since  $p_n \neq p_m$  and  $\Pi_3^n \cap \Pi_3^m = \emptyset$  it follows that  $p$  divides  $t_0 + t_2$  and therefore  $p$  divides  $(g_\alpha - t' - \bar{x}_m)$  for every  $p \in \{p_m\} \cup \Pi_3^m$ . Hence  $m \notin R_{\alpha,t'}$  showing that  $R_{\alpha,t'}$  is strictly smaller than  $R_{\alpha,t}$ . This finishes the proof.  $\square$

*Fact 3.7.* Assume that for every  $\alpha < \kappa$  there exists  $t_\alpha \in T$  such that  $R_{\alpha,t_\alpha}$  is finite. Then  $\varphi$  is invertible and hence the sequence (E) splits.

*Proof.* By Fact 3.6 we may assume without loss of generality that for every  $\alpha < \kappa$  the set  $R_{\alpha,t_\alpha}$  is empty. Thus for each  $n \in A_\alpha$  we can find  $z_{\alpha,n,p} \in G$  such that

$$pz_{\alpha,n,p} = g_\alpha - \bar{x}_n - t_\alpha$$

for every  $p \in \{p_n\} \cup \Pi_3^n$ . We now define a homomorphism  $\psi : B \rightarrow G$  as follows:

- (1)  $\psi(x_n) = \bar{x}_n$  ( $n < \omega$ );
- (2)  $\psi(y_\alpha) = g_\alpha - t_\alpha$  ( $\alpha < \kappa$ );
- (3)  $\psi(p^{-1}(y_\alpha - x_n)) = z_{\alpha,n,p}$  ( $\alpha < \kappa$ ,  $n \in A_\alpha$ ,  $p \in \{p_n\} \cup \Pi_3^n$ ).

We leave it to the reader to check that (1), (2) and (3) induce a well-defined homomorphism  $\psi : B \rightarrow G$  satisfying  $\varphi\psi = id_B$ .  $\square$

*Continuation of the proof of Theorem 3.4.* By Fact 3.7 it remains to find for every  $\alpha < \kappa$  an element  $t_\alpha \in T$  such that the set  $R_{\alpha,t_\alpha}$  is finite. Here the forcing was used in [12]. Since the sequence (E) is forced to be balanced exact there exist homomorphisms

$$(3.11) \quad \tilde{\psi}_{\alpha,\beta} : \tilde{Y}_{\alpha,\beta} \rightarrow \tilde{G} \text{ such that } \tilde{\varphi}\tilde{\psi}_{\alpha,\beta} = id_{\tilde{Y}_{\alpha,\beta}},$$

where  $\tilde{Y}_{\alpha,\beta} = \langle y_\beta - y_\alpha \rangle_*$ . Let  $\tilde{h}_{\alpha,\beta} = \tilde{\psi}_{\alpha,\beta}(y_\beta - y_\alpha) \in \tilde{G}$ ; hence

$$(3.12) \quad \tilde{t}_{\alpha,\beta} = \tilde{h}_{\alpha,\beta} - (\tilde{g}_\beta - \tilde{g}_\alpha) \in \tilde{T}.$$

Since  $\tilde{T}$  is a torsion group we may let  $\tilde{m}_{\alpha,\beta} < \omega$  such that

$$(3.13) \quad \tilde{m}_{\alpha,\beta} = \text{ord}(\tilde{t}_{\alpha,\beta}).$$

We can now easily show

*Fact 3.8.*  $r^* \Vdash$  “If  $n > \tilde{m}_{\alpha,\beta}$ , then  $p$  divides  $(\tilde{g}_\beta - \tilde{g}_\alpha)$  for every  $n \in \tilde{A}_\alpha \cap \tilde{A}_\beta$  and  $p \in \{p_n\} \cup \Pi_3^n$ ”.

*Proof.* Fix an extension model  $M[H]$  of  $M$ . Let  $n > m_{\alpha,\beta}$  and  $n \in A_\beta \cap A_\alpha$ . Then  $p > m_{\alpha,\beta}$  follows for all  $p \in \{p_n\} \cup \Pi_3^n$  since the primes  $p_m$  are increasing and  $p > n$  for all  $p \in \Pi_1^n$  by the choice of  $\Pi_3^n$ . Therefore  $\gcd(p, m_{\alpha,\beta}) = 1$  and thus  $p$  divides  $(h_{\alpha,\beta} - (g_\beta - g_\alpha))$  for all  $p \in \{p_n\} \cup \Pi_3^n$ . Moreover,  $h_{\alpha,\beta} = \psi_{\alpha,\beta}(y_\beta - y_\alpha)$  is divisible by  $p$  for all  $p \in \{p_n\} \cup \Pi_3^n$  since  $n \in A_\alpha \cap A_\beta$ . Hence  $p$  divides  $(g_\beta - g_\alpha)$  for all  $p \in \{p_n\} \cup \Pi_3^n$ .  $\square$

Now let  $r^* \leq r_{\alpha,\beta} \in \mathfrak{C}$  be such that  $r_{\alpha,\beta}$  forces the value  $m_{\alpha,\beta}$  to  $\tilde{m}_{\alpha,\beta}$ , i.e.

$$(3.14) \quad r_{\alpha,\beta} \Vdash \text{“}\tilde{m}_{\alpha,\beta} = m_{\alpha,\beta}\text{”}.$$

In [12, Fact 4.7 - Fact 4.11] it was then shown by using a theorem due to Erdős and Rado that there is a subset  $\Gamma \subseteq \kappa$  such that  $\Gamma = \{\alpha_\epsilon : \epsilon < \aleph_1\}$  and, among other properties, whenever  $\alpha_\epsilon, \alpha_\rho \in \Gamma$ , then  $m_{\alpha_\epsilon, \alpha_\rho} = m^*$  for some fixed integer  $m^*$ . This was then used to prove that there exists a forcing condition  $s^* \geq r^*$  and a set  $\tilde{Y} \subseteq \Gamma$  such that

*Fact 3.9.*  $s^* \Vdash \text{“}|\tilde{Y}| = \aleph_1\text{”}.$

Moreover,  $s^*$  is strong enough to force the following.

*Fact 3.10.*  $s^* \Vdash \text{“If } \alpha_\epsilon, \alpha_\rho \in \tilde{Y} \text{ and } n \in \tilde{A}_{\alpha_\epsilon} \cap \tilde{A}_{\alpha_\rho} \setminus [0, m^*], \text{ then } p \text{ divides } \tilde{g}_{\alpha_\epsilon} - \tilde{g}_{\alpha_\rho} \text{ for all } p \in \{p_n\} \cup \Pi_3^n\text{”}.$

The proofs of the above two facts are purely set theoretical and involve only the functions  $r_{\alpha,\beta}$ . The only algebraic part appears in the proof of Fact 3.10. Given  $\alpha_\epsilon, \alpha_\rho \in \tilde{Y}$  and  $n \in \tilde{A}_{\alpha_\epsilon} \cap \tilde{A}_{\alpha_\rho} \setminus [0, m^*]$  the main point in [12, Fact 4.12] is to construct a forcing condition  $s^+ \geq s^*$  and an ordinal  $\alpha_\gamma \in \tilde{Y}$  such that  $n \in \tilde{A}_{\alpha_\gamma}$  and  $s^+ \geq r_{\alpha_\epsilon, \alpha_\gamma}, r_{\alpha_\gamma, \alpha_\rho}$ . Thus  $s^+$  forces that  $m_{\alpha_\epsilon, \alpha_\gamma} = m^* = m_{\alpha_\gamma, \alpha_\rho}$  and therefore  $p$  divides  $\tilde{g}_{\alpha_\epsilon} - \tilde{g}_{\alpha_\gamma}$  and  $p$  divides  $\tilde{g}_{\alpha_\gamma} - \tilde{g}_{\alpha_\rho}$  for all  $p \in \{p_n\} \cup \Pi_3^n$  by Fact 3.8. Hence  $p$  divides  $\tilde{g}_{\alpha_\epsilon} - \tilde{g}_{\alpha_\rho}$  for all  $p \in \{p_n\} \cup \Pi_3^n$ .

Finally, using the same arguments we can adjust [12, Fact 4.13] to prove that

*Fact 3.11.*  $s^* \Vdash \text{“For every } \beta < \kappa \text{ there exists } m_\beta < \omega \text{ such that for all } m_\beta < n \in \tilde{A}_\beta \text{ and } \alpha_\epsilon \in \tilde{Y} \text{ with } n \in \tilde{A}_{\alpha_\epsilon} \text{ we have that } p \text{ divides } \tilde{g}_\beta - \tilde{g}_{\alpha_\epsilon} \text{ for all } p \in \{p_n\} \cup \Pi_3^n\text{”}.$

We need to show that Fact 3.11 implies Theorem 3.4; in fact, using Fact 3.11 we prove that the set  $R_{\beta,0}$  is contained in  $[0, m_\beta) \cap \mathbb{Z}$  for all  $\beta < \kappa$  and hence is finite whenever  $s^* \in H$ . Choose an extension model  $M[H]$  of  $M$  for some generic filter  $H$  containing  $s^*$ . As in [12, after Fact 4.13] we claim that we can choose  $\hat{x}_n \in G$  such that

- (1)  $\varphi(\hat{x}_n) = x_n$  and  $p$  divides  $\hat{x}_n$  for all  $p \in \Pi_2^n$ ;
- (2) if  $n > m^*$  and  $\alpha \in Y$  such that  $n \in A_\alpha$ , then  $p$  divides  $g_\alpha - \hat{x}_n$  for all  $p \in \{p_n\} \cup \Pi_3^n$ .

Let  $n > m^*$  and choose  $\alpha \in Y$  such that  $n \in A_\alpha$ . Let  $k_n \in G$  such that  $\varphi(k_n) = (y_\alpha - x_n)$  and  $p$  divides  $k_n$  for all  $p \in \{p_n\} \cup \Pi_3^n$ . Note that  $k_n$  exists since the sequence (E) is balanced. Put  $\hat{\hat{x}}_n = -k_n + g_\alpha$ . Then  $\varphi(\hat{\hat{x}}_n) = x_n$  and therefore  $\hat{\hat{x}}_n - \hat{x}_n \in T$ . Thus  $\bar{x}_n = \hat{\hat{x}}_n + t$  for some  $t \in T$ . Write  $t = t_1 + t_2$  with  $t_1 \in T'$  and  $t_2 \in T_{\Pi_3^n} \oplus T_{p_n}$ , where  $T = T_{\Pi_3^n} \oplus T_{p_n} \oplus T'$ . Then  $\bar{x}_n - t_2 = \hat{\hat{x}}_n + t_1$  is divisible by all  $p \in \Pi_2^n$  since  $\Pi_3^n \cap \Pi_2^n = \emptyset$ . Let  $\hat{x}_n = \hat{\hat{x}}_n + t_1$ . Then (1) is satisfied. Moreover,  $g_\alpha - \hat{x}_n = g_\alpha - \hat{\hat{x}}_n - t_1 = g_\alpha + k_n - g_\alpha - t_1 = -(t_1 - k_n)$  is divisible by all  $p \in \{p_n\} \cup \Pi_3^n$  since  $t_1 \in T'$ . This finishes the proof.

Now Fact 3.11 ensures that  $R_{\beta,0}$  is contained in  $[0, m_\beta)$ , for if  $m_\beta < n \in A_\beta$ , choose  $\alpha_\epsilon \in Y$  such that  $n \in A_{\alpha_\epsilon}$  (the one which was used when choosing the  $\hat{x}_n$ 's). Then we have by the choice of  $\hat{x}_n$  that  $p$  divides  $g_{\alpha_\epsilon} - \hat{x}_n$  for all  $p \in \{p_n\} \cup \Pi_3^n$ , and by Fact 3.11 we have that  $p$  divides  $g_\beta - g_{\alpha_\epsilon}$  and hence  $p$  divides  $g_\beta - \hat{x}_n$  for all  $p \in \{p_n\} \cup \Pi_3^n$ . Thus  $n \notin R_{[\beta,0)}$  and  $R_{[\beta,0)} \subseteq [0, m_\beta)$  follows. Therefore the proof of Fact 3.11 finishes the proof of Theorem 3.4.  $\square$

#### ACKNOWLEDGMENTS

The author would like to thank the Department of Mathematics at the University of Hawaii for their kind hospitality. In particular he is grateful to Professor Adolf Mader for his friendship and helpful discussions.

#### REFERENCES

1. D. M. Arnold, *Abelian Groups and Representations of Finite Partially Ordered Sets*, CMS Books in Mathematics, Springer-Verlag, New York (2000). MR1764257 (2001g:16030)
2. L. Bican and L. Salce, *Butler groups of infinite rank*, Abelian Group Theory, Lecture Notes in Math. 1006, Springer-Verlag, Berlin (1983), 171–189. MR722617 (86c:20050)
3. M.C.R. Butler, *A class of torsion-free abelian groups of finite rank*, Proc. London Math. Soc. 15 (1965), 680–698. MR0218446 (36:1532)
4. P.C. Eklof and A. Mekler, *Almost Free Modules. Set-Theoretic Methods (revised edition)*, North-Holland Math. Library, North-Holland, Amsterdam (1990). MR1055083 (92e:20001)
5. L. Fuchs, *Infinite Abelian Groups, Vols. I and II*, Academic Press (1970 and 1973). MR0255673 (41:333), MR0349869 (50:2362)
6. L. Fuchs and M. Magidor, *Butler groups of arbitrary cardinality*, Israel J. Math. 84 (1993), 239–263. MR1244670 (95a:20058)
7. T. Jech, *Set Theory*, Academic Press, New York (1973). MR506523 (80a:03062)
8. K. Kunen, *Set Theory. An Introduction to Independence Proofs*, Studies in Logic and the Foundations of Mathematics 102, North-Holland, Amsterdam-New York (1980). MR597342 (82f:03001)
9. A. Mader, *Almost completely decomposable groups*, volume 13 of *Algebra, Logic and Applications*, Gordon and Breach Science Publishers, Amsterdam (2000). MR1751515 (2001c:20114)
10. A. Mader and L. Strümgmann, *A class of Butler groups and their endomorphism rings*, Hokkaido Math. J. XXXVII (2008), 399–425. MR2415908 (2009d:20134)
11. S. Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin (1998). MR1623206 (98m:03002)
12. S. Shelah and L. Strümgmann, *It is consistent with ZFC that  $B_1$ -groups are not  $B_2$* , Forum Math. 15 (2003), 507–524. MR1978332 (2004j:20108)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DUISBURG-ESSEN, CAMPUS ESSEN, 45117 ESSEN, GERMANY

*E-mail address:* lutz.struengmann@uni-due.de