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VANISHING OF EXTENSIONS OF TWISTED VERMA MODULES

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Abstract. We prove a vanishing theorem of extension groups between twisted Verma modules.

1. Introduction

The notion of twisted (or shuffled) Verma modules was introduced by Irving [Irv93] and also developed by Andersen and Lauritzen [AL03]. These are objects of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [BGG76] which correspond to the principal series representations by the Bernstein-Gelfand equivalence of categories [BG80].

In this paper, we prove a vanishing theorem of extension groups between twisted Verma modules. In the case of Verma modules, this theorem was proved by Schmid [Sch81] and Delorme [Del80]. (See also [Car86, Proposition 3.7].)

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$. These data define the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [BGG76, Definition 1]. Let ρ be the half-sum of positive roots, W the Weyl group and w_0 its longest element. We denote $x < y$ for $x, y \in W$ if x is less than y with respect to the Bruhat order. For $w \in W$, let $\ell(w)$ be the length of w. For $x \in W$, we denote the Verma module with highest weight $x\rho - \rho$ by $M(x)$. Let T_w be a twisting functor for $w \in W$ [AL03, 6.1].

To state our main result, we define a subset of the Weyl group $A_w(x)$ indexed by $x, w \in W$. Let $w = s_1 \cdots s_l$ be a reduced expression and set $t_i = (s_1 \cdots s_{i-1}) s_i (s_1 \cdots s_{i-1})^{-1}$. Put

$$A_w(x) = \left\{ z \in W \;\middle|\; \begin{array}{l} \text{for some } 1 \le i_1 < \cdots < i_r \le l,\ z = t_{i_r} \cdots t_{i_1} x \text{ and} \\ t_{i_k} t_{i_{k-1}} \cdots t_{i_1} x < t_{i_{k-1}} \cdots t_{i_1} x \text{ for all } k = 1, \ldots, r \end{array} \right\}.$$

Then by [Abe], this set is independent of the choice of a reduced expression. Put

$$\ell(w, x, v, y) = \max\{2\ell(z) - \ell(wx) - \ell(vy) \mid z \in wA_{w^{-1}}(x) \cap vw_0 A_{w_0 v^{-1}}(w_0 y)\}.$$

(If $wA_{w^{-1}}(x) \cap vw_0 A_{w_0 v^{-1}}(w_0 y) = \emptyset$, put $\ell(w, x, v, y) = -\infty$.)

Theorem 1. *For $w, v, x, y \in W$, $k > \ell(w, x, v, y)$ implies*

$$\operatorname{Ext}^k_{\mathcal{O}}(T_w M(x), T_v M(y)) = 0.$$

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Remark 2. If $w = e$, then $wA_{w^{-1}}(x) = x$. Hence $\ell(w, x, v, y) = \ell(x) - \ell(vy)$ (or $-\infty$). As a special case, this theorem implies that $\operatorname{Ext}^k_{\mathcal{O}}(M(x), M(y)) = 0$ for all $k > \ell(x) - \ell(y)$. This is a result of Schmid [Sch81] and Delorme [Del80]. Moreover, Carlin [Car86, Theorem 3.8] proved that $\dim \operatorname{Ext}^{\ell(x)-\ell(y)}_{\mathcal{O}}(M(x), M(y)) = 1$ if $x \geq y$.

2. Proof of the main theorem

In the rest of this paper, we denote $\operatorname{Ext}_{\mathcal{O}}$ by Ext. We use the following lemmas.

Lemma 3. (1) *Let s be a simple reflection and $w, x \in W$. Assume that $ws > w$ and $sx > x$. Then we have $\ell(wsx) > \ell(wx)$.*

(2) *Let s be a simple reflection and t a reflection such that $s \neq t$. If $stw < sw$, then $tw < w$.*

Proof. (1) Assume that $\ell(wsx) < \ell(wx)$. Let $w = s_1 \cdots s_l$, $x = s_{l+1} \cdots s_r$ be reduced expressions. Put $t = wsw^{-1}$. Then we have $\ell(twx) < \ell(wx)$. Hence by the strong exchange condition [Hum90, 5.8], for some k we have $ts_1 \cdots s_k = s_1 \cdots s_{k-1}$. If $k \leq l$, then $ws = tw = s_1 \cdots s_{k-1}s_{k+1} \cdots s_l < w$; this is a contradiction. If $k > l$, then $sx = w^{-1}twx = s_{l+1} \cdots s_{k-1}s_{k+1} \cdots s_r < x$; this is also a contradiction.

(2) Take a positive root β such that t is a reflection with respect to β. Then sts^{-1} is a reflection with respect to $s(\beta)$. Let $\check{\beta}$ be the coroot corresponding to β. The assumption $s \neq t$ implies that $s(\check{\beta})$ is positive. Since $sts^{-1}(sw) = stw < sw$, we have $\langle s(\check{\beta}), sw\rho \rangle < 0$ by Proposition 5.7 of [Hum90]. Hence $\langle \check{\beta}, w\rho \rangle < 0$. Therefore, we have $tw < w$. □

Lemma 4. *Let s be a simple reflection such that $sw > w$. Then we have*

$$A_{sw}(x) = \begin{cases} sA_w(sx) & (sx > x), \\ sA_w(x) \cup sA_w(sx) & (sx < x). \end{cases}$$

Proof. Fix a reduced expression $sw = s_1 \cdots s_l$ such that $s = s_1$. Then $w = s_2 \cdots s_l$ is a reduced expression of w. Put $t_i = (s_1 \cdots s_{i-1})s_i(s_1 \cdots s_{i-1})^{-1}$ and $t'_i = s^{-1}t_i s = (s_2 \cdots s_{i-1})s_i(s_2 \cdots s_{i-1})^{-1}$. Notice that $t_i \neq t_j$ if $i \neq j$.

By the definition, $z \in A_{sw}(x)$ if and only if there exist $1 \leq i_1 < \cdots < i_r \leq l$ such that $z = t_{i_r} \cdots t_{i_1} x$ and $t_{i_k} t_{i_{k-1}} \cdots t_{i_1} x < t_{i_{k-1}} \cdots t_{i_1} x$ for all $k = 1, \ldots, r$. Then $z = st'_{i_r} \cdots t'_{i_1} sx$ and $st'_{i_k} t'_{i_{k-1}} \cdots t'_{i_1} sx < st'_{i_{k-1}} \cdots t'_{i_1} sx$ for all $k = 1, \ldots, r$. If $i_k \neq 1$, then $s \neq t'_{i_k}$. Hence we have $t'_{i_k} t'_{i_{k-1}} \cdots t'_{i_1} sx < t'_{i_{k-1}} \cdots t'_{i_1} sx$ by Lemma 3 (2). If $i_1 \neq 1$, then we have $sz \in A_w(sx)$. If $i_1 = 1$, then x must satisfy $sx = t_{i_1} x < x$ and $sz \in A_w(x)$. Therefore we get

$$A_{sw}(x) \subset \begin{cases} sA_w(sx) & (sx > x), \\ sA_w(x) \cup sA_w(sx) & (sx < x). \end{cases}$$

The reverse inclusion follows from the above argument. □

Lemma 5. *If $vs > v$, then we have*

$$\ell(w, x, v, y) \begin{cases} = \ell(w, x, vs, sy) & (sy < y), \\ \geq \max\{\ell(w, x, vs, sy), \ell(w, x, vs, y) + 1\} & (sy > y). \end{cases}$$

Proof. Fix $x, w \in W$. For simplicity, put $A(v, y) = wA_{w^{-1}}(x) \cap vw_0 A_{w_0 v^{-1}}(w_0 y)$. First assume that $sy < y$. Put $s' = w_0 s w_0$. Then s' is also a simple reflection. From

$sy < y$, we have $s'w_0y > w_0y$. By Lemma 4, $A_{w_0v^{-1}}(w_0y) = s'A_{s'w_0v^{-1}}(s'w_0y) = s'A_{w_0(vs)^{-1}}(w_0sy)$. Therefore $A(v,y) = A(vs, sy)$. Hence the definition of ℓ implies that $\ell(w,x,vs,sy) = \ell(w,x,v,y)$. Next, assume that $sy > y$. Then using the above argument, we have $A(v,y) \supset A(vs,sy)$. Hence we have $\ell(w,x,v,y) \geq \ell(w,x,vs,sy)$. By Lemma 4, we also have $A(v,y) \supset A(vs,y)$ since $s'w_0v^{-1} < w_0v^{-1}$. Hence $\ell(w,x,v,y) \geq \max\{2\ell(z) - \ell(wx) - \ell(vy) \mid z \in A(vs,y)\}$. By Lemma 3 (1), we have $\ell(vy) < \ell(vsy)$. Hence we have $\ell(w,x,v,y) \geq \ell(w,x,vs,y) + 1$. $\square$

Lemma 6. *Fix $w, x \in W$. If $\operatorname{Ext}^k(T_wM(x), T_{w_0}M(y)) = 0$ for all $y \in W$ and $k > \ell(w,x,w_0,y)$, then $\operatorname{Ext}^k(T_wM(x), T_vM(y)) = 0$ for all $v, y \in W$ and $k > \ell(w,x,v,y)$.*

Proof. We prove by downward induction on $\ell(v)$. The base case of $v = w_0$ is just our assumption.

Now assume that $v \neq w_0$. Take a simple reflection s such that $vs > v$. Assume first that $sy < y$. Then we have $T_vM(y) \simeq T_{vs}M(sy)$ [AL03, Proposition 6.3]. Hence $\operatorname{Ext}^k_{\mathcal{O}}(T_wM(x), T_vM(y)) = 0$ for $k > \ell(w,x,vs,sy) = \ell(w,x,v,y)$.

Next, assume that $sy > y$. In this case, by [AL03, Proposition 6.3], we have the following exact sequence:

$$0 \to T_{vs}M(y) \to T_vM(y) \to T_{vs}M(sy) \to T_{vs}M(y) \to 0.$$

Put $L = \operatorname{Ker}(T_{vs}M(sy) \to T_{vs}M(y))$. Then we have an exact sequence $0 \to L \to T_{vs}M(sy) \to T_{vs}M(y) \to 0$. By the corresponding long exact sequence, we have an exact sequence

$$\operatorname{Ext}^{k-1}(T_wM(x), T_{vs}M(y)) \to \operatorname{Ext}^k(T_wM(x), L) \to \operatorname{Ext}^k(T_wM(x), T_{vs}M(sy)).$$

If $k > \ell(w,x,v,y)$, then by Lemma 5, we have $k-1 > \ell(w,x,vs,y)$ and $k > \ell(w,x,vs,sy)$. By the inductive hypothesis, we have $\operatorname{Ext}^{k-1}(T_wM(x), T_{vs}M(y)) = \operatorname{Ext}^k(T_wM(x), T_{vs}M(sy)) = 0$. Hence we get $\operatorname{Ext}^k(T_wM(x), L) = 0$. We also have an exact sequence $0 \to T_{vs}M(y) \to T_vM(y) \to L \to 0$. From the above argument, we have $\operatorname{Ext}^k(T_wM(x), T_{vs}M(y)) = \operatorname{Ext}^k(T_wM(x), L) = 0$. Therefore we get $\operatorname{Ext}^k(T_wM(x), T_vM(y)) = 0$ from the long exact sequence. $\square$

Let D be a dualizing functor of $\mathcal{O}$ (see, for instance, [Hum08, 3.2]).

Proof of Theorem 1. It suffices to prove that $\operatorname{Ext}^k(T_wM(x), T_{w_0}M(y)) = 0$ for all $k > \ell(w,y,w_0,x)$ by Lemma 6. First we prove this for $w = e$. By [AL03, Remark 5.1], $T_{w_0}M(y) \simeq DM(w_0y)$. Hence $\operatorname{Ext}^k(M(x), T_{w_0}M(y)) = 0$ if $k > 0$ or $x \neq w_0y$ by [Hum08, Theorem 6.12, Theorem 3.3(c)], so we have proved the theorem in this case. (Note: $\ell(e,x,w_0,y) = -\infty$ if $x \neq w_0y$.)

By Lemma 6, $\operatorname{Ext}^k(M(x), T_vM(y)) = 0$ for all $v, x, y \in W$ and $k > \ell(e,x,v,y)$. From [AL03, Remark 5.1], we have $DT_vM(y) = T_{vw_0}M(w_0y)$. Hence applying D, we have $\operatorname{Ext}^k(T_{vw_0}M(w_0y), T_{w_0}M(w_0x)) = 0$ for all $k > \ell(e,x,v,y)$. By definition, we have $\ell(e,x,v,y) = \ell(vw_0, w_0y, w_0, w_0x)$. Consequently, we have $\operatorname{Ext}^k(T_vM(y), T_{w_0}M(x)) = 0$ for all $k > \ell(v,y,w_0,x)$, from which the theorem follows. $\square$

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