

## VANISHING OF EXTENSIONS OF TWISTED VERMA MODULES

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ABSTRACT. We prove a vanishing theorem of extension groups between twisted Verma modules.

### 1. INTRODUCTION

The notion of twisted (or shuffled) Verma modules was introduced by Irving [Irv93] and also developed by Andersen and Lauritzen [AL03]. These are objects of the Bernstein-Gelfand-Gelfand category  $\mathcal{O}$  [BGG76] which correspond to the principal series representations by the Bernstein-Gelfand equivalence of categories [BG80].

In this paper, we prove a vanishing theorem of extension groups between twisted Verma modules. In the case of Verma modules, this theorem was proved by Schmid [Sch81] and Delorme [Del80]. (See also [Car86, Proposition 3.7].)

Let  $\mathfrak{g}$  be a complex semisimple Lie algebra. Fix a Cartan subalgebra  $\mathfrak{h}$  and a Borel subalgebra  $\mathfrak{b}$ . These data define the Bernstein-Gelfand-Gelfand category  $\mathcal{O}$  [BGG76, Definition 1]. Let  $\rho$  be the half-sum of positive roots,  $W$  the Weyl group and  $w_0$  its longest element. We denote  $x < y$  for  $x, y \in W$  if  $x$  is less than  $y$  with respect to the Bruhat order. For  $w \in W$ , let  $\ell(w)$  be the length of  $w$ . For  $x \in W$ , we denote the Verma module with highest weight  $x\rho - \rho$  by  $M(x)$ . Let  $T_w$  be a twisting functor for  $w \in W$  [AL03, 6.1].

To state our main result, we define a subset of the Weyl group  $A_w(x)$  indexed by  $x, w \in W$ . Let  $w = s_1 \cdots s_l$  be a reduced expression and set  $t_i = (s_1 \cdots s_{i-1})s_i(s_1 \cdots s_{i-1})^{-1}$ . Put

$$A_w(x) = \left\{ z \in W \mid \begin{array}{l} \text{for some } 1 \leq i_1 < \cdots < i_r \leq l, z = t_{i_r} \cdots t_{i_1} x \text{ and} \\ t_{i_k} t_{i_{k-1}} \cdots t_{i_1} x < t_{i_{k-1}} \cdots t_{i_1} x \text{ for all } k = 1, \dots, r \end{array} \right\}.$$

Then by [Abe], this set is independent of the choice of a reduced expression. Put

$$\ell(w, x, v, y) = \max\{2\ell(z) - \ell(wx) - \ell(vy) \mid z \in wA_{w^{-1}}(x) \cap vw_0A_{w_0v^{-1}}(w_0y)\}.$$

(If  $wA_{w^{-1}}(x) \cap vw_0A_{w_0v^{-1}}(w_0y) = \emptyset$ , put  $\ell(w, x, v, y) = -\infty$ .)

**Theorem 1.** *For  $w, v, x, y \in W$ ,  $k > \ell(w, x, v, y)$  implies*

$$\text{Ext}_{\mathcal{O}}^k(T_w M(x), T_v M(y)) = 0.$$

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*Remark 2.* If  $w = e$ , then  $wA_{w^{-1}}(x) = x$ . Hence  $\ell(w, x, v, y) = \ell(x) - \ell(vy)$  (or  $-\infty$ ). As a special case, this theorem implies that  $\text{Ext}_{\mathcal{O}}^k(M(x), M(y)) = 0$  for all  $k > \ell(x) - \ell(y)$ . This is a result of Schmid [Sch81] and Delorme [Del80]. Moreover, Carlin [Car86, Theorem 3.8] proved that  $\dim \text{Ext}_{\mathcal{O}}^{\ell(x) - \ell(y)}(M(x), M(y)) = 1$  if  $x \geq y$ .

## 2. PROOF OF THE MAIN THEOREM

In the rest of this paper, we denote  $\text{Ext}_{\mathcal{O}}$  by  $\text{Ext}$ . We use the following lemmas.

- Lemma 3.** (1) *Let  $s$  be a simple reflection and  $w, x \in W$ . Assume that  $ws > w$  and  $sx > x$ . Then we have  $\ell(wsx) > \ell(wx)$ .*  
 (2) *Let  $s$  be a simple reflection and  $t$  a reflection such that  $s \neq t$ . If  $stw < sw$ , then  $tw < w$ .*

*Proof.* (1) Assume that  $\ell(wsx) < \ell(wx)$ . Let  $w = s_1 \cdots s_l$ ,  $x = s_{l+1} \cdots s_r$  be reduced expressions. Put  $t = wsw^{-1}$ . Then we have  $\ell(twx) < \ell(wx)$ . Hence by the strong exchange condition [Hum90, 5.8], for some  $k$  we have  $ts_1 \cdots s_k = s_1 \cdots s_{k-1}$ . If  $k \leq l$ , then  $ws = tw = s_1 \cdots s_{k-1}s_{k+1} \cdots s_l < w$ ; this is a contradiction. If  $k > l$ , then  $sx = w^{-1}twx = s_{l+1} \cdots s_{k-1}s_{k+1} \cdots s_r < x$ ; this is also a contradiction.

(2) Take a positive root  $\beta$  such that  $t$  is a reflection with respect to  $\beta$ . Then  $sts^{-1}$  is a reflection with respect to  $s(\beta)$ . Let  $\check{\beta}$  be the coroot corresponding to  $\beta$ . The assumption  $s \neq t$  implies that  $s(\check{\beta})$  is positive. Since  $sts^{-1}(sw) = stw < sw$ , we have  $\langle s(\check{\beta}), sw\rho \rangle < 0$  by Proposition 5.7 of [Hum90]. Hence  $\langle \check{\beta}, w\rho \rangle < 0$ . Therefore, we have  $tw < w$ .  $\square$

**Lemma 4.** *Let  $s$  be a simple reflection such that  $sw > w$ . Then we have*

$$A_{sw}(x) = \begin{cases} sA_w(sx) & (sx > x), \\ sA_w(x) \cup sA_w(sx) & (sx < x). \end{cases}$$

*Proof.* Fix a reduced expression  $sw = s_1 \cdots s_l$  such that  $s = s_1$ . Then  $w = s_2 \cdots s_l$  is a reduced expression of  $w$ . Put  $t_i = (s_1 \cdots s_{i-1})s_i(s_1 \cdots s_{i-1})^{-1}$  and  $t'_i = s^{-1}t_i s = (s_2 \cdots s_{i-1})s_i(s_2 \cdots s_{i-1})^{-1}$ . Notice that  $t_i \neq t_j$  if  $i \neq j$ .

By the definition,  $z \in A_{sw}(x)$  if and only if there exist  $1 \leq i_1 < \cdots < i_r \leq l$  such that  $z = t_{i_r} \cdots t_{i_1} x$  and  $t_{i_k} t_{i_{k-1}} \cdots t_{i_1} x < t_{i_{k-1}} \cdots t_{i_1} x$  for all  $k = 1, \dots, r$ . Then  $z = st'_{i_r} \cdots t'_{i_1} sx$  and  $st'_{i_k} t'_{i_{k-1}} \cdots t'_{i_1} sx < st'_{i_{k-1}} \cdots t'_{i_1} sx$  for all  $k = 1, \dots, r$ . If  $i_k \neq 1$ , then  $s \neq t'_{i_k}$ . Hence we have  $t'_{i_k} t'_{i_{k-1}} \cdots t'_{i_1} sx < t'_{i_{k-1}} \cdots t'_{i_1} sx$  by Lemma 3 (2). If  $i_1 \neq 1$ , then we have  $sz \in A_w(sx)$ . If  $i_1 = 1$ , then  $x$  must satisfy  $sx = t_{i_1} x < x$  and  $sz \in A_w(x)$ . Therefore we get

$$A_{sw}(x) \subset \begin{cases} sA_w(sx) & (sx > x), \\ sA_w(x) \cup sA_w(sx) & (sx < x). \end{cases}$$

The reverse inclusion follows from the above argument.  $\square$

**Lemma 5.** *If  $vs > v$ , then we have*

$$\ell(w, x, v, y) \begin{cases} = \ell(w, x, vs, sy) & (sy < y), \\ \geq \max\{\ell(w, x, vs, sy), \ell(w, x, vs, y) + 1\} & (sy > y). \end{cases}$$

*Proof.* Fix  $x, w \in W$ . For simplicity, put  $A(v, y) = wA_{w^{-1}}(x) \cap vw_0A_{w_0v^{-1}}(w_0y)$ . First assume that  $sy < y$ . Put  $s' = w_0sw_0$ . Then  $s'$  is also a simple reflection. From

$sy < y$ , we have  $s'w_0y > w_0y$ . By Lemma 4,  $A_{w_0v^{-1}}(w_0y) = s'A_{s'w_0v^{-1}}(s'w_0y) = s'A_{w_0(vs)^{-1}}(w_0sy)$ . Therefore  $A(v, y) = A(vs, sy)$ . Hence the definition of  $\ell$  implies that  $\ell(w, x, vs, sy) = \ell(w, x, v, y)$ . Next, assume that  $sy > y$ . Then using the above argument, we have  $A(v, y) \supset A(vs, sy)$ . Hence we have  $\ell(w, x, v, y) \geq \ell(w, x, vs, sy)$ . By Lemma 4, we also have  $A(v, y) \supset A(vs, y)$  since  $s'w_0v^{-1} < w_0v^{-1}$ . Hence  $\ell(w, x, v, y) \geq \max\{2\ell(z) - \ell(wx) - \ell(vy) \mid z \in A(vs, y)\}$ . By Lemma 3 (1), we have  $\ell(vy) < \ell(vsy)$ . Hence we have  $\ell(w, x, v, y) \geq \ell(w, x, vs, y) + 1$ .  $\square$

**Lemma 6.** Fix  $w, x \in W$ . If  $\text{Ext}^k(T_wM(x), T_{w_0}M(y)) = 0$  for all  $y \in W$  and  $k > \ell(w, x, w_0, y)$ , then  $\text{Ext}^k(T_wM(x), T_vM(y)) = 0$  for all  $v, y \in W$  and  $k > \ell(w, x, v, y)$ .

*Proof.* We prove by downward induction on  $\ell(v)$ . The base case of  $v = w_0$  is just our assumption.

Now assume that  $v \neq w_0$ . Take a simple reflection  $s$  such that  $vs > v$ . Assume first that  $sy < y$ . Then we have  $T_vM(y) \simeq T_{vs}M(sy)$  [AL03, Proposition 6.3]. Hence  $\text{Ext}^k_{\mathcal{O}}(T_wM(x), T_vM(y)) = 0$  for  $k > \ell(w, x, vs, sy) = \ell(w, x, v, y)$ .

Next, assume that  $sy > y$ . In this case, by [AL03, Proposition 6.3], we have the following exact sequence:

$$0 \rightarrow T_{vs}M(y) \rightarrow T_vM(y) \rightarrow T_{vs}M(sy) \rightarrow T_{vs}M(y) \rightarrow 0.$$

Put  $L = \text{Ker}(T_{vs}M(sy) \rightarrow T_{vs}M(y))$ . Then we have an exact sequence  $0 \rightarrow L \rightarrow T_{vs}M(sy) \rightarrow T_{vs}M(y) \rightarrow 0$ . By the corresponding long exact sequence, we have an exact sequence

$$\text{Ext}^{k-1}(T_wM(x), T_{vs}M(y)) \rightarrow \text{Ext}^k(T_wM(x), L) \rightarrow \text{Ext}^k(T_wM(x), T_{vs}M(sy)).$$

If  $k > \ell(w, x, v, y)$ , then by Lemma 5, we have  $k - 1 > \ell(w, x, vs, y)$  and  $k > \ell(w, x, vs, sy)$ . By the inductive hypothesis, we have  $\text{Ext}^{k-1}(T_wM(x), T_{vs}M(y)) = \text{Ext}^k(T_wM(x), T_{vs}M(sy)) = 0$ . Hence we get  $\text{Ext}^k(T_wM(x), L) = 0$ . We also have an exact sequence  $0 \rightarrow T_{vs}M(y) \rightarrow T_vM(y) \rightarrow L \rightarrow 0$ . From the above argument, we have  $\text{Ext}^k(T_wM(x), T_{vs}M(y)) = \text{Ext}^k(T_wM(x), L) = 0$ . Therefore we get  $\text{Ext}^k(T_wM(x), T_vM(y)) = 0$  from the long exact sequence.  $\square$

Let  $D$  be a dualizing functor of  $\mathcal{O}$  (see, for instance, [Hum08, 3.2]).

*Proof of Theorem 1.* It suffices to prove that  $\text{Ext}^k(T_wM(x), T_{w_0}M(y)) = 0$  for all  $k > \ell(w, y, w_0, x)$  by Lemma 6. First we prove this for  $w = e$ . By [AL03, Remark 5.1],  $T_{w_0}M(y) \simeq DM(w_0y)$ . Hence  $\text{Ext}^k(M(x), T_{w_0}M(y)) = 0$  if  $k > 0$  or  $x \neq w_0y$  by [Hum08, Theorem 6.12, Theorem 3.3(c)], so we have proved the theorem in this case. (Note:  $\ell(e, x, w_0, y) = -\infty$  if  $x \neq w_0y$ .)

By Lemma 6,  $\text{Ext}^k(M(x), T_vM(y)) = 0$  for all  $v, x, y \in W$  and  $k > \ell(e, x, v, y)$ . From [AL03, Remark 5.1], we have  $DT_vM(y) = T_{vw_0}M(w_0y)$ . Hence applying  $D$ , we have  $\text{Ext}^k(T_{vw_0}M(w_0y), T_{w_0}M(w_0x)) = 0$  for all  $k > \ell(e, x, v, y)$ . By definition, we have  $\ell(e, x, v, y) = \ell(vw_0, w_0y, w_0, w_0x)$ . Consequently, we have  $\text{Ext}^k(T_vM(y), T_{w_0}M(x)) = 0$  for all  $k > \ell(v, y, w_0, x)$ , from which the theorem follows.  $\square$

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