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COMPACT COMPOSITION OPERATORS ON BMOA AND THE BLOCH SPACE

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ABSTRACT. We give a new and simple compactness criterion for composition operators C_φ on BMOA and the Bloch space in terms of the norms of φ^n in the respective spaces.

1. Introduction

The study of composition operators on various Banach spaces of analytic functions is currently a very active field of complex and functional analysis. There are still many unsolved problems, some old and some new, that are of interest to numerous mathematicians. In this article, we obtain a new result about an old problem, namely, the compactness of composition operators on BMOA (the space of analytic functions with bounded mean oscillation) and the Bloch space on the unit disk.

Throughout the paper, we use φ to denote an analytic self-map of the unit disk $\Delta = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. The composition operator C_φ is defined by $C_\varphi(f) = f \circ \varphi$, $f \in H(\Delta)$, where $H(\Delta)$ is the space of all analytic functions in Δ.

The compactness of composition operators on BMOA has been studied in several papers. Bourdon, Cima, and Matheson first obtained a compactness condition for C_φ on BMOA that involves the symbol function φ and the unit ball of BMOA; see Theorem 3.1 in [2]. W. Smith then found a characterization of compact C_φ on BMOA that only involves the symbol function φ; see Theorem 1.1 in [9]. Smith's condition consists of two parts: one in terms of the Nevanlinna counting function of φ and the other in terms of a certain mapping property of the boundary function $\varphi(e^{it})$. From Smith's work in [9], the first author derived a set of relatively simple compactness conditions in [11] for the action of C_φ on BMOA. For the Bloch space, Madigan and Matheson [6] and Tjani [10] gave various compactness conditions for C_φ.

The compactness criterion for C_φ on BMOA that was obtained in [11] also consists of two parts: one involves the powers of the symbol function φ and the other concerns the action of C_φ on Möbius maps. In this paper we show that the part of Wulan's condition in [11] that involves Möbius maps is abundant and thus can be

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dropped. We also show that the correponding result holds for the Bloch space as well. Thus our main result can be stated as follows.

Main Theorem. *Let X denote BMOA or the Bloch space on the unit disk. Then a composition operator $C_\varphi : X \to X$ is compact if and only if $\|\varphi^n\|_X \to 0$ as $n \to \infty$.*

2. Compactness on BMOA

Recall that an analytic function f in Δ belongs to the Hardy space H^p, $0 < p < \infty$, if

$$\|f\|_{H^p}^p = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

For $p = \infty$, $f \in H^\infty$ if

$$\|f\|_{H^\infty} = \sup_{z\in\Delta} |f(z)| < \infty.$$

For any point $a \in \Delta$ we write

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad z \in \Delta.$$

The space BMOA can then be defined as those functions $f \in H^2$ with the property that

$$\|f\|_* = \sup_{a\in\Delta} \|f \circ \sigma_a - f(a)\|_{H^2} < \infty.$$

See [1]. It is well known that BMOA is a Banach space with the norm

$$\|f\|_{\mathrm{BMOA}} = |f(0)| + \|f\|_*.$$

Note that (see [5] for example)

$$\|f\|_* \approx \sup_{a\in\Delta} \int_\Delta |f'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z),$$

where dA is an area measure on Δ normalized so that $A(\Delta) = 1$.

For an arc $I \subset \partial\Delta$ let $R(I)$ denote the Carleson square

$$R(I) = \{re^{i\theta} \in \Delta : 1 - |I| < r < 1, e^{i\theta} \in I\}.$$

Recall that a well-known result of L. Carleson [4] states that for a positive measure μ on Δ the following conditions are equivalent:

(a) There is a positive constant C such that

$$\int_\Delta |f(z)|^2 \, d\mu(z) \le C\|f\|_{H^2}^2$$

for all $f \in H^2$.

(b) There exists a positive constant C such that $\mu(R(I)) \le C|I|$ for all subarcs I of $\partial\Delta$.

When the above conditions hold, the measure μ is called a Carleson measure. It is also well known (see [5]) that an analytic function f in Δ is in BMOA if and only if $|f'(z)|^2(1 - |z|^2)\, dA(z)$ is a Carleson measure.

In the remainder of this paper we fix an analytic self-map φ of the unit disk and let $\Phi_r(z)$ denote the characteristic function of the set

$$\Omega_r = \{z \in \Delta : |\varphi(z)| > r\}, \qquad 0 < r < 1.$$

We can then state the compactness criterion for C_φ on BMOA given by Bourdon, Cima, and Matheson in [2] as follows: the operator C_φ is compact on BMOA if and only if for every $\varepsilon > 0$ there exists some $r \in (0,1)$ such that

$$\int_{R(I)} \Phi_r(z)(1-|z|^2)|f'(\varphi(z))|^2|\varphi'(z)|^2\,dA(z) \le \varepsilon|I|$$

for every arc $I \subset \partial\Delta$ and every unit vector f in BMOA.

To state Smith's criterion for compact composition operators on BMOA, we write

$$N(\varphi, w) = \sum_{\varphi(z)=w} \log(1/|z|), \qquad w \in \Delta \setminus \{\varphi(0)\},$$

for the classical Nevanlinna counting function of φ. Then the main result in [9] can be stated as follows: C_φ is compact on BMOA if and only if

$$\lim_{|\varphi(a)|\to 1} \sup_{0<|w|<1} |w|^2 N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w) = 0$$

and

$$\lim_{t\to 1} \sup_{\{a:|\varphi(a)|\le R\}} m(\sigma_a(E(\varphi,t))) = 0$$

for all $0 < R < 1$, where

$$E(\varphi,t) = \{e^{i\theta} : |\varphi(e^{i\theta})| > t\}, \quad 0 < t < 1,$$

and $m(A)$ denotes the one-dimensional Lebesgue measure of a set $A \subset \partial\Delta$.

Based on results of Bourdon, Cima, and Matheson in [2] and Smith's work in [9], a more straightforward description for the compactness of C_φ on BMOA was obtained by Wulan in [11]. More specifically, Wulan's theorem states that C_φ is compact on BMOA if and only if

$$\lim_{|a|\to 1} \|\sigma_a \circ \varphi\|_* = 0 \tag{1}$$

and

$$\lim_{n\to\infty} \|\varphi^n\|_* = 0. \tag{2}$$

We now show that the first condition (1) in Wulan's theorem above can be dropped.

Theorem 1. *The composition operator C_φ is compact on BMOA if and only if condition* (2) *holds.*

Proof. First assume that C_φ is compact on BMOA. Then C_φ maps every bounded set in BMOA to a set whose closure in BMOA is compact in the norm topology. Let $\{f_n\}$ be a bounded sequence in BMOA that converges to 0 pointwise. Since the closure of $\{C_\varphi(f_n)\}$ is compact in BMOA, it contains a convergent subsequence, say, $\|C_\varphi(f_{n_k}) - f\| \to 0$ as $k \to \infty$. Since norm convergence in BMOA implies pointwise convergence, we have $f_{n_k}(\varphi(z)) \to f(z)$ as $k \to \infty$. But $\{f_{n_k}\}$ converges to 0 pointwise, so $f(z) = 0$ and $\|C_\varphi(f_{n_k})\| \to 0$ as $k \to \infty$. By working with subsequences of $\{f_n\}$ in the first place, we conclude that $\|C_\varphi(f_n)\| \to 0$ whenever $\{f_n\}$ is a bounded sequence in BMOA that converges to 0 pointwise.

Since the sequence $\{z^n\}$ is bounded in BMOA and it converges to 0 pointwise, the compactness of C_φ on BMOA implies that condition (2) holds.

Next we assume that condition (2) holds. To show that C_φ is compact on BMOA, it suffices to verify condition (1).

It is easy to show that the Maclaurin expansion of the Möbius map σ_a is given by

$$\sigma_a(z) = a - (1 - |a|^2) \sum_{n=0}^{\infty} \overline{a}^n z^{n+1}.$$

It follows from the triangle inequality that

$$\|\sigma_a \circ \varphi\|_* \leq (1 - |a|^2) \sum_{n=0}^{\infty} |a|^n \|\varphi^{n+1}\|_*. \tag{3}$$

If (2) holds, then for any $\varepsilon > 0$ we can find a positive integer N such that $\|\varphi^n\|_* < \varepsilon$ for all $n > N$. Combining this with (3), we obtain

$$\|\sigma_a \circ \varphi\|_* \leq (1 - |a|^2) \sum_{n=0}^{N} |a|^n \|\varphi^{n+1}\|_* + 2\varepsilon. \tag{4}$$

Since $\{\|\varphi^n\|_*\}$ is a bounded sequence, letting $|a| \to 1$ in (4) leads to

$$\limsup_{|a|\to 1} \|\sigma_a \circ \varphi\|_* \leq 2\varepsilon.$$

Since ε is arbitrary, we conclude that the condition in (2) implies

$$\lim_{|a|\to 1} \|\sigma_a \circ \varphi\|_* = 0.$$

This completes the proof of the theorem. □

Although not as elegant as the criterion we have just presented, the compactness characterizations found in [2] and [9] actually lead to some useful geometric conditions sufficient for the compactness of C_φ on BMOA. See Section 5 of [2] and Section 4 of [9]. Moreover, the authors of [2] construct a proof, based on their characterization, that compactness of C_φ on BMOA implies its compactness on the Hardy space H^2; see Theorem 4.1 of [2]. The same arguments are also used in Corollary 2.3 of [11]. The referee of the present paper pointed out that appropriate attribution should have been given in [11] concerning the proof mentioned above. We agree and are grateful to the referee for a very careful reading of our paper and for suggesting several other changes that improved the presentation of the paper.

3. Compactness on the Bloch space

Recall that the Bloch space $\mathcal{B}$ consists of those analytic functions f on Δ for which

$$\|f\|_{\mathcal{B}} = \sup_{z\in\Delta} (1 - |z|^2)|f'(z)| < \infty. \tag{5}$$

We now prove that an analogue of Theorem 1 holds for the Bloch space as well.

Theorem 2. *Let φ be an analytic self-map of Δ. Then C_φ is compact on the Bloch space $\mathcal{B}$ if and only if*

$$\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} = 0. \tag{6}$$

Proof. Since $\{z^n\}$ is bounded in the Bloch space $\mathcal{B}$ and converges to 0 uniformly on compact subsets of Δ, the same argument from the first paragraph of the proof of Theorem 1 shows that the compactness of C_φ on $\mathcal{B}$ implies that (6) holds.

To show that condition (6) implies the compactness of C_φ, it suffices for us to prove that $\|C_\varphi f_n\|_{\mathcal{B}} \to 0$ as $n \to \infty$ whenever $\{f_n\}$ is a bounded sequence in $\mathcal{B}$ that converges to 0 uniformly on compact subsets of Δ.

So for the rest of this proof we assume that (6) holds and let $\{f_n\}$ be a bounded sequence in $\mathcal{B}$ that converges to 0 uniformly on compact subsets of Δ. Given $\varepsilon > 0$, there exists a positive integer $N > 2$ such that $\|\varphi^n\|_{\mathcal{B}} < \varepsilon$ for $n \geq N$. It follows easily from Cauchy's formula that $\{f_n'\}$ also converges to 0 uniformly on compact subsets of Δ.

For any positive integer $k \geq 2$ we consider the set

$$\Delta_k = \left\{ z \in \Delta : \frac{k-2}{k-1} \leq |\varphi(z)| \leq \frac{k-1}{k} \right\}. \tag{7}$$

It is clear that

$$\|C_\varphi(f_n)\|_{\mathcal{B}} = \sup_{z\in\Delta} |f_n'(\varphi(z))||\varphi'(z)|(1-|z|^2) \leq I_1(n) + I_2(n), \tag{8}$$

where

$$I_1(n) = \sup\{|f_n'(\varphi(z))||\varphi'(z)|(1-|z|^2) : |\varphi(z)| \leq (N-1)/N\},$$

and

$$I_2(n) = \sup_{k>N} \sup_{z\in\Delta_k} |f_n'(\varphi(z))||\varphi'(z)|(1-|z|^2).$$

Since $|\varphi'(z)|(1-|z|^2) \leq 1$ for all $z \in \Delta$ (Schwarz lemma) and $f_n' \to 0$ uniformly on compact subsets of Δ, we have $I_1(n) \to 0$ as $n \to \infty$.

To estimate $I_2(n)$, we observe that for each $k > 2$ the function

$$h(x) = kx^{k-1}(1-x), \qquad \frac{k-2}{k-1} \leq x \leq \frac{k-1}{k},$$

is increasing. So for any x between $(k-2)/(k-1)$ and $(k-1)/k$ we have

$$\frac{1}{e} \leq h\left(\frac{k-2}{k-1}\right) \leq h(x) \leq h\left(\frac{k-1}{k}\right) \leq \frac{1}{2}.$$

From this we deduce that for each $k > 2$,

$$\inf_{z\in\Delta_k} k|\varphi(z)|^{k-1}(1-|\varphi(z)|) \geq \frac{1}{e}. \tag{9}$$

It follows that

$$\begin{aligned}
I_2(n) &= \sup_{k>N} \sup_{z\in\Delta_k} |f_n'(\varphi(z))||\varphi'(z)|(1-|z|^2) \\
&= \sup_{k>N} \sup_{z\in\Delta_k} \frac{|f_n'(\varphi(z))|(1-|\varphi(z)|^2)k|\varphi(z)|^{k-1}|\varphi'(z)|(1-|z|^2)}{k|\varphi(z)|^{k-1}(1-|\varphi(z)|^2)} \\
&\leq e\|f_n\|_{\mathcal{B}} \sup_{k>N} \sup_{z\in\Delta} \left\{k|\varphi(z)|^{k-1}|\varphi'(z)|(1-|z|^2)\right\} \\
&= e\|f_n\|_{\mathcal{B}} \sup_{k>N} \|\varphi^k\|_{\mathcal{B}} \\
&\leq e\varepsilon\|f_n\|_{\mathcal{B}}.
\end{aligned}$$

Since the sequence $\{\|f_n\|_{\mathcal{B}}\}$ is bounded, we take the upper limit in (8) to obtain

$$\limsup_{n\to\infty} \|C_\varphi f_n\|_{\mathcal{B}} \le M\varepsilon,$$

where M is a positive constant independent of ε. Since ε is arbitrary, we conclude that

$$\lim_{n\to\infty} \|C_\varphi(f_n)\|_{\mathcal{B}} = 0.$$

This proves the compactness of C_φ on $\mathcal{B}$ and completes the proof of the theorem. $\square$

Theorem 2, together with a result of Madigan and Matheson [6], shows that the condition

$$\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} = 0$$

is equivalent to

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} = 0,$$

which, according to [10], is also equivalent to

$$\lim_{|a|\to 1} \|C_\varphi \sigma_a\|_{\mathcal{B}} = 0.$$

This gives Theorem 3 below. But we think a direct proof without appealing to the notion of composition operators should be of some independent interest.

Theorem 3. *Let φ be an analytic self-map of Δ. Then the following conditions are equivalent:*

(a) $\|\varphi^n\|_{\mathcal{B}} \to 0$ *as* $n\to\infty$.
(b) $\|\sigma_a(\varphi)\|_{\mathcal{B}} \to 0$ *as* $|a| \to 1^-$.
(c) $(1-|z|^2)|\varphi'(z)|/(1-|\varphi(z)|^2) \to 0$ *as* $|\varphi(z)| \to 1^-$.

Proof. To show that (a) implies (b), we employ the same argument used in the proof of Theorem 1. The interested reader should have no trouble filling in the details.

If condition (b) holds, then for any given $\varepsilon > 0$ there is an $r \in (0,1)$ such that $\|\sigma_a(\varphi)\|_{\mathcal{B}} < \varepsilon$ whenever $r < |a| < 1$. In particular, if $z \in \Delta$ satisfies $|\varphi(z)| > r$, then $\|\sigma_{\varphi(z)}(\varphi)\|_{\mathcal{B}} < \varepsilon$. Thus

$$\sup_{w\in\Delta} \frac{1-|\varphi(z)|^2}{|1-\overline{\varphi(z)}\varphi(w)|^2} |\varphi'(w)|(1-|w|^2) < \varepsilon.$$

Taking $w = z$ in the above supremum, we obtain

$$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2}(1-|z|^2) < \varepsilon.$$

This shows that (b) implies (c).

It remains to show that (c) implies (a). So we assume

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2} = 0.$$

Given $\varepsilon > 0$ there exists a $\delta \in (0,1)$ such that

$$\frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2} < \varepsilon \tag{10}$$

whenever $\delta < |\varphi(z)| < 1$. By elementary calculations, the function

$$h(x) = nx^{n-1}(1-x), \qquad 0 \leq x \leq 1,$$

attains its maximum value, which is ≤ 1, at the point $(n-1)/n$. Therefore,

$$n|\varphi(z)|^{n-1}(1-|\varphi(z)|) \leq 1$$

for all $z \in \Delta$. It follows that

$$n|\varphi(z)|^{n-1}|\varphi'(z)|(1-|z|^2) \leq \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}.$$

This along with (10) shows that

$$\sup_{\delta<|\varphi(z)|<1} n|\varphi(z)|^{n-1}|\varphi'(z)|(1-|z|^2) < \varepsilon. \tag{11}$$

On the other hand,

$$\sup_{|\varphi(z)|\leq\delta} n|\varphi(z)|^{n-1}|\varphi'(z)|(1-|z|^2) \leq n\delta^{n-1}\|\varphi\|_{\mathcal{B}}. \tag{12}$$

Combining (11) and (12), we see that

$$\limsup_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} \leq \varepsilon.$$

Since ε is arbitrary, we conclude that $\|\varphi^n\|_{\mathcal{B}} \to 0$ as $n \to \infty$. This proves that (c) implies (a). □

4. Further remarks

We conclude the article with several remarks and questions.

First, the second part of the proof of Theorem 1 works for much more general spaces than BMOA and the Bloch space. Thus for a large class of spaces $(X, \| \ \|)$, where $\| \ \|$ is a semi-norm on X such that $\|c\| = 0$ for any constant function c, the condition $\|\sigma_a(\varphi)\| \to 0$ $(|a| \to 1^-)$ is a consequence of the condition $\|\varphi^n\| \to 0$ $(n \to \infty)$.

Second, for the Bloch space, the condition $\|\varphi^n\| \to 0$ $(n \to \infty)$ is equivalent to the condition $\|\sigma_a(\varphi)\| \to 0$ $(|a| \to 1^-)$. We have been unable to determine if this holds for BMOA as well.

Finally, there is a class of Möbius invariant spaces that are closely related to both BMOA and the Bloch space, namely, the so-called Q_p spaces. See [12]. A characterization for compact composition operators on these spaces is still lacking, despite the effort by several mathematicians in recent years. It is our hope that this article will further motivate someone to solve the problem for Q_p spaces.

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