

BETTI NUMBERS AND INJECTIVITY RADII

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Dedicated to José María Montesinos on the occasion of his 65th birthday

ABSTRACT. We give lower bounds on the maximal injectivity radius for a closed hyperbolic 3-manifold with first Betti number 2 under some additional topological hypotheses.

The theme of this paper is the connection between topological properties of a closed orientable hyperbolic 3-manifold M and the maximal injectivity radius of M . In [4] we showed that if the first Betti number of M is at least 3, then the maximal injectivity radius of M is at least $\log 3$. By contrast, the best known lower bound for the maximal injectivity radius of M with no topological restriction on M is the lower bound of $\operatorname{arcsinh}(\frac{1}{4}) = 0.24746\dots$ due to Przeworski [7]. One of the results of this paper, Corollary 4, gives a lower bound of 0.32798 for the case where the first Betti number of M is 2 and M does not contain a “fibroid” (see below). Our main result, Theorem 3, is somewhat stronger than this.

The proofs of our results combine a result due to Andrew Przeworski [7] with results from [5] and [6].

The results of [5] and [6] were motivated by applications to the study of hyperbolic volume, and these applications were superseded by the results of [2]. The applications presented in the present paper do not seem to be accessible by other methods.

As in [5], we define a *book of I -bundles* to be a compact, connected, orientable topological 3-manifold (with boundary) W which has the form $W = \mathcal{P} \cup \mathcal{B}$, where

- \mathcal{P} is an I -bundle over a non-empty compact 2-manifold-with-boundary,
- each component of \mathcal{B} is homeomorphic to $D^2 \times S^1$,
- the set $\mathcal{A} = \mathcal{P} \cap \mathcal{B}$ is the vertical boundary of the I -bundle \mathcal{P} , and
- each component of \mathcal{A} is an annulus in $\partial\mathcal{B}$ which is homotopically non-trivial in \mathcal{B} .

(Note that this terminology differs slightly from that of [1], where it is the triple $(W, \mathcal{B}, \mathcal{P})$ that is called a book of I -bundles.)

As in [5], we define a *fibroid* in a closed, connected, orientable 3-manifold M to be a connected incompressible surface F with the property that each component of the compact manifold obtained by cutting M along F is a book of I -bundles.

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Note that in defining a fibroid to be connected, we are following the convention of [5] rather than that of [6].

We define a function $R(x)$ for $0 < x < \log 3$ by

$$(1) \quad R(x) = \frac{1}{2} \operatorname{arccosh} \left(\frac{e^{2x} + 2e^x + 5}{(\cosh \frac{x}{2})(e^x - 1)(e^x + 3)} \right).$$

As in [3, Section 10], we define a function $V(x)$ for $0 < x < \log 3$ by

$$(2) \quad V(x) = \pi x \sinh^2 R(x) = \frac{\pi x}{e^x - 1} \left(\frac{e^{2x} + 2e^x + 5}{2(e^x + 3) \cosh(x/2)} \right) - \frac{\pi x}{2}.$$

Thus, in a closed hyperbolic 3-manifold, if a geodesic of length ℓ is the core of an embedded tube of radius $R(\ell)$, then this tube has volume $V(\ell)$.

The following result is implicit in [6], but we will supply a proof for the sake of comprehensibility.

Proposition 1. *Let M be a closed hyperbolic 3-manifold. Suppose that there is an infinite subset \mathcal{N} of $H_2(M; \mathbb{Z})$ such that every element of \mathcal{N} is represented by some connected, incompressible surface which is not a fibroid. Let λ be a positive number less than $\log 3$. Then either the maximal injectivity radius of M is at least $\lambda/2$ or M contains a closed geodesic C of length at most λ such that the maximal tube about C has radius at least $R(\lambda)$ and volume at least $V(\lambda)$, where $R(\lambda)$ and $V(\lambda)$ are defined by (1) and (2).*

Proof. The hypothesis implies in particular that $H_2(M; \mathbb{Z})$ has infinitely many primitive elements, and so the first Betti number $\beta_1(M)$ is at least 2. If $\beta_1(M) \geq 3$, then according to [4, Corollary 10.4], the maximal injectivity radius is at least $\frac{1}{2} \log 3 > \lambda/2$. We may therefore assume that $\beta_1(M) = 2$. Hence the quotient of $H_1(M, \mathbb{Z})$ by its torsion subgroup is a free abelian group L of rank 2. We let $h: \pi_1(M) \rightarrow L$ denote the natural homomorphism.

We distinguish two cases. First consider the case in which M contains a non-trivial closed geodesic C of some length $\ell < \lambda$ such that the conjugacy class represented by C is contained in the kernel of h . Let T denote the maximal embedded tube about C . According to [3, Corollary 10.5] we have $\operatorname{Vol} T \geq V(\lambda)$. If ρ denotes the radius of T , this gives

$$\pi \ell \sinh^2 \rho = \operatorname{Vol} T \geq V(\lambda) = \pi \lambda \sinh^2 R(\lambda) > \pi \ell \sinh^2 R(\lambda),$$

and hence $\rho > R(\lambda)$.

Thus the second alternative of the proposition holds in this case.

We now turn to the case in which no non-trivial closed geodesic of length $< \lambda$ represents a conjugacy class contained in the kernel of h . Since M is closed, there are only a finite number $n \geq 0$ of conjugacy classes in $\pi_1(M)$ that are represented by closed geodesics of length $< \lambda$. Let $\gamma_1, \dots, \gamma_n$ be elements belonging to these n conjugacy classes. Then $\bar{\gamma}_i = h(\gamma_i)$ is a non-trivial element of L for $i = 1, \dots, n$. Since L is a free abelian group of rank 2, there exists, for each $i \in \{1, \dots, n\}$, a homomorphism ϕ_i of L onto \mathbb{Z} such that $\phi_i(\bar{\gamma}_i) = 0$. Because $\bar{\gamma}_i \neq 0$, the epimorphism ϕ_i is unique up to sign.

The epimorphism $\phi_i \circ h: \pi_1(M) \rightarrow \mathbb{Z}$ corresponds to a primitive element of $H^1(M; \mathbb{Z})$, whose Poincaré dual in $H_2(M; \mathbb{Z})$ we shall denote by c_i . Since the set $\mathcal{N} \subset H_2(M; \mathbb{Z})$ given by the hypothesis of the theorem is infinite, there is an element c of \mathcal{N} which is distinct from $\pm c_i$ for $i = 1, \dots, n$. Since $c \in \mathcal{N}$ it follows

from the hypothesis that there is a connected incompressible surface $S \subset M$ which represents the homology class c and is not a fibroid.

We now apply Theorem A of [5], which asserts that if S is a connected non-fibroid incompressible surface in a closed, orientable hyperbolic 3-manifold M , and if λ is any positive number, then either (i) M contains a non-trivial closed geodesic of length $< \lambda$ which is homotopic in M to a closed curve in $M - S$ or (ii) M contains a hyperbolic ball of radius $\lambda/2$. In the present situation, with λ chosen as above, we claim that alternative (i) of the conclusion of Theorem A of [5] cannot hold.

Indeed, suppose that C is a non-trivial closed geodesic of length $< \lambda$ with the properties stated in (i). Since C has length $< \lambda$, the conjugacy class represented by C contains γ_i for some $i \in \{1, \dots, n\}$. Since C is homotopic to a closed curve in $M - S$, it follows that the image of γ_i in $H_1(M; \mathbb{Z})$ has homological intersection number 0 with c . Thus if $\psi: \pi_1(M) \rightarrow \mathbb{Z}$ is the homomorphism corresponding to the Poincaré dual of c , we have $\psi(\gamma_i) = 0$. Now since L is the quotient of $H_1(M)$ by its torsion subgroup, ψ factors as $\phi \circ h$, where ϕ is some homomorphism from L to \mathbb{Z} . Since c is primitive, ψ is surjective, and hence so is ϕ . But we have $\phi(\bar{\gamma}_i) = \psi(\gamma_i) = 0$. In view of the uniqueness that we observed above for ϕ_i , it follows that $\phi = \pm\phi_i$, so that $\psi = \pm\phi_i \circ h$ and hence $c = \pm c_i$. This contradicts our choice of c .

Hence (ii) must hold. This means that the maximal injectivity radius of M is at least $\lambda/2$. Thus the first alternative of the proposition holds in this case. \square

Proposition 2. *Let M be a closed hyperbolic 3-manifold. Suppose that there is an infinite subset \mathcal{N} of $H_2(M; \mathbb{Z})$ such that every element of \mathcal{N} is represented by some connected, incompressible surface which is not a fibroid. Then the maximal injectivity radius of M exceeds 0.32798.*

Proof. We set $\lambda = 2 \times 0.32798 = 0.65596$.

According to Proposition 1, either the maximal injectivity radius of M is at least $\lambda/2$ — so that the conclusion of the theorem holds — or M contains a closed geodesic C of length at most λ such that the maximal tube about C has volume at least $V(\lambda)$, where $V(\lambda)$ is defined by (2). In the latter case, if R denotes the radius of T , we have

$$R \geq R(\lambda) = 0.806787 \dots$$

Now according to [7, Proposition 4.1], the maximal injectivity radius of M is bounded below by

$$\operatorname{arcsinh} \left(\frac{\tanh R}{2} \right) > \operatorname{arcsinh} \left(\frac{\tanh 0.806787}{2} \right) > 0.32799.$$

This gives the conclusion of the theorem in this case. \square

Theorem 3. *Let M be a closed hyperbolic 3-manifold. Suppose that there is an infinite set \mathcal{N} of primitive elements of $H_2(M; \mathbb{Z})$ such that no element of \mathcal{N} is represented by a (connected) fibroid. Then the maximal injectivity radius of M exceeds 0.32798.*

Proof. If $\pi_1(M)$ has a non-abelian free quotient, then by [6, Theorem 1.3], the maximal injectivity radius of M is at least $\frac{1}{2} \log 3 = 0.549 \dots$. Now suppose that $\pi_1(M)$ has no non-abelian free quotient. If \mathcal{N} is the set given by the hypothesis of Theorem 3, it now follows from [6, Proposition 2.1] that every element of \mathcal{N} is represented by a connected incompressible surface, which by hypothesis cannot

be a fibroid. Thus \mathcal{N} has the properties stated in the hypothesis of Proposition 2. The latter result therefore implies that the maximal injectivity radius of M exceeds 0.32798. \square

If M is a 3-manifold whose first Betti number is at least 2, then $H_2(M; \mathbb{Z})$ has infinitely many primitive elements. If a non-trivial element of $H_2(M; \mathbb{Z})$ is represented by a connected surface, it must be primitive, since it has intersection number 1 with a class in $H_1(M; \mathbb{Z})$. Hence Theorem 3 implies:

Corollary 4. *Let M be a closed hyperbolic 3-manifold. Suppose that the first Betti number of M is at least 2 and that M contains no non-separating fibroid. Then the maximal injectivity radius of M exceeds 0.32798.*

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