

## A FUNDAMENTAL DICHOTOMY FOR JULIA SETS OF A FAMILY OF ELLIPTIC FUNCTIONS

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ABSTRACT. We investigate topological properties of Julia sets of iterated elliptic functions of the form  $g = 1/\wp$ , where  $\wp$  is the Weierstrass elliptic function, on triangular lattices. These functions can be parametrized by  $\mathbb{C} - \{0\}$ , and they all have a superattracting fixed point at zero and three other distinct critical values. We prove that the Julia set of  $g$  is either Cantor or connected, and we obtain examples of each type.

### 1. INTRODUCTION

Quadratic polynomials always have a superattracting component  $F_\infty$  containing infinity, and the location of the finite critical point with respect to  $F_\infty$  provides a characterization of the connectivity of the Julia set. For quadratic polynomials, if the critical point iterates to  $\infty$ , then it must be contained in  $F_\infty$ . Fatou [7] and Julia [9] proved the following fundamental dichotomy in the topology of the Julia set.

**Theorem 1.1.** *If the finite critical point of a quadratic polynomial  $P$  is contained in  $F_\infty$ , then the Julia set of  $P$  is a Cantor set. Otherwise, the Julia set is connected.*

For higher degree polynomials, the presence of additional critical points complicates this dichotomy. For example, Brodin gave an example in [4] of a cubic polynomial with a Cantor Julia set that contained one critical point. Furthermore, the Julia set of a cubic polynomial can be disconnected but not Cantor. A thorough analysis of the connectivity properties of Julia sets of cubic polynomials can be found in [3].

For quadratic rational maps  $R$ , we also observe a dichotomy in the connectivity of the Julia sets: the Julia set of  $R$  is either connected or Cantor [21, 23]. A similar result is true for the meromorphic family  $h_\lambda(z) = \lambda \tan z$ , which has no critical points but exactly two asymptotic values that play a role similar to that of critical values [18].

Connectivity properties of Julia sets of elliptic functions were first studied by the author and Hawkins in [14], but a complete classification is known for only two families, and both of these families always have connected Julia sets. The Julia set of the Weierstrass elliptic  $\wp$ -function on any triangular lattice is always connected

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[14]; this family of functions contains examples where the Fatou set is nonempty as well as examples where the Julia set is the entire sphere. Additional topological properties of Julia sets in this family were studied in [16]. In [11], Hawkins proved that the Weierstrass elliptic function on any rhombic square lattice always has Julia set equal to the entire sphere and hence is connected. Isolated examples of other elliptic functions with connected Julia sets were constructed in [14] as well.

Examples of elliptic functions with Cantor Julia sets first appeared in [14]. In [19], the author investigated elliptic functions of the form  $f = 1/\wp$  on real rectangular lattices, and examples with Cantor Julia sets were found within every equivalence class of real rectangular lattice shape. However, there are many functions within this family whose Julia sets are not well understood.

In this paper, we focus on the family of elliptic functions  $g_\Omega = 1/\wp_\Omega$  on the space of triangular lattices  $\Omega$ . Triangular lattices are those having the property  $e^{2\pi i/3}\Omega = \Omega$ , and the family of maps  $g_\Omega$  can be parametrized by  $\mathbb{C} - \{0\}$ . Since the function  $g_\Omega$  is locally two-to-one, the dynamics in many ways is similar to that of quadratic rational maps. However,  $g_\Omega$  has infinitely many critical points and four distinct critical values. Since the postcritical orbits influence the dynamics so strongly, the functions  $g_\Omega$  can also exhibit dynamical behavior that is distinct from that of quadratic rational maps. For example,  $g_\Omega$  can have three superattracting cycles, where a quadratic rational map can have at most two [20].

The functions  $g_\Omega$  always have a superattracting fixed point at the origin, and the symmetry of the triangular lattice shape combined with algebraic properties of the Weierstrass elliptic function implies that the orbits of the three other distinct critical values are related. Our main result is the following theorem.

**Theorem 1.2.** *Let  $\Omega$  be a triangular lattice. If all four critical values of  $g_\Omega = 1/\wp_\Omega$  are contained in one component of the Fatou set, then the Julia set of  $g_\Omega$  is Cantor. Otherwise, the Julia set is connected.*

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## 2. BACKGROUND ON ELLIPTIC DYNAMICS

We begin with some preliminaries about elliptic functions, the Weierstrass  $\wp$ -function and period lattices.

**2.1. Lattices in the plane.** Let  $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$  such that  $\lambda_2/\lambda_1 \notin \mathbb{R}$ . A lattice  $\Lambda \subset \mathbb{C}$  is defined by  $\Lambda = [\lambda_1, \lambda_2] := \{m\lambda_1 + n\lambda_2 : m, n \in \mathbb{Z}\}$ , noting that two different sets of vectors can generate the same lattice  $\Lambda$ .

We can view  $\Lambda$  as a group acting on  $\mathbb{C}$  by translation, each  $\lambda \in \Lambda$  inducing the transformation of  $\mathbb{C}$ :

$$T_\lambda : z \rightarrow z + \lambda.$$

**Definition 2.1.** A closed, connected subset  $\mathcal{Q}$  of  $\mathbb{C}$  is defined to be a *fundamental region* for  $\Lambda$  if

- (1) for each  $z \in \mathbb{C}$ ,  $\mathcal{Q}$  contains at least one point in the same  $\Lambda$ -orbit as  $z$ ;
- (2) no two points in the interior of  $\mathcal{Q}$  are in the same  $\Lambda$ -orbit.

If  $\mathcal{Q}$  is any fundamental region for  $\Lambda$ , then for any  $s \in \mathbb{C}$ , the set

$$\mathcal{Q} + s = \{z + s : z \in \mathcal{Q}\}$$

is also a fundamental region. If  $\mathcal{Q}$  is a parallelogram we call  $\mathcal{Q}$  a *period parallelogram* for  $\Lambda$ .

Frequently we refer to types of lattices by the shapes of the corresponding period parallelograms. If  $\Lambda$  is a lattice, and  $k \neq 0$  is any complex number, then  $k\Lambda$  is also a lattice defined by taking  $k\lambda$  for each  $\lambda \in \Lambda$ ;  $k\Lambda$  is said to be *similar* to  $\Lambda$ . Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*. In this paper, we focus on triangular lattices, denoted by  $\Omega$  and having the property that  $\varepsilon\Omega = \Omega$ , where  $\varepsilon = e^{2\pi i/3}$ . The period parallelograms of a triangular lattice are formed by two equilateral triangles.

**2.2. Elliptic functions.** We begin with  $f: \mathbb{C} \rightarrow \mathbb{C}_\infty$ , a meromorphic function where  $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$  denotes the Riemann sphere.

An *elliptic function* is a meromorphic function in  $\mathbb{C}$  which is periodic with respect to a lattice  $\Lambda$ . For any  $z \in \mathbb{C}$  and any lattice  $\Lambda$ , the Weierstrass elliptic function is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

It is well known that  $\wp_\Lambda$  is even, is periodic with respect to  $\Lambda$ , and has order 2, with double poles at lattice points.

The Weierstrass elliptic function and its derivative are related by the differential equation

$$(1) \quad \wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3,$$

where  $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} w^{-4}$  and  $g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} w^{-6}$ .

The numbers  $g_2(\Lambda)$  and  $g_3(\Lambda)$  are invariants of the lattice  $\Lambda$  in the following sense: if  $g_2(\Lambda) = g_2(\Lambda')$  and  $g_3(\Lambda) = g_3(\Lambda')$ , then  $\Lambda = \Lambda'$ . Furthermore, given any  $g_2$  and  $g_3$  such that  $g_2^3 - 27g_3^2 \neq 0$  there exists a lattice  $\Lambda$  having  $g_2 = g_2(\Lambda)$  and  $g_3 = g_3(\Lambda)$  as its invariants [6].

The Weierstrass elliptic function and the lattice invariants satisfy the following homogeneity properties.

**Proposition 2.2.** *For any lattice  $\Lambda$  and for any  $m \in \mathbb{C} \setminus \{0\}$ ,*

$$\begin{aligned} \wp_{m\Lambda}(mz) &= m^{-2}\wp_\Lambda(z), \\ g_2(m\Lambda) &= m^{-4}g_2(\Lambda), \\ g_3(m\Lambda) &= m^{-6}g_3(\Lambda). \end{aligned}$$

To construct examples of elliptic functions satisfying the conclusion of our main theorem, we will need the following result. If  $\bar{\Lambda} = \Lambda$ , then we say that  $\Lambda$  is a real lattice. We say that  $\wp_\Lambda$  is real if  $z \in \mathbb{R}$  implies that  $\wp_\Lambda(z) \in \mathbb{R} \cup \{\infty\}$ .

**Theorem 2.3** ([17]). *The following are equivalent:*

- (1)  $\Lambda$  is a real lattice;
- (2)  $\wp_\Lambda$  is a real function;
- (3)  $g_2, g_3 \in \mathbb{R}$ .

We can determine the critical values of the Weierstrass elliptic function on an arbitrary lattice  $\Lambda = [\lambda_1, \lambda_2]$ . Define  $\lambda_3 = \lambda_1 + \lambda_2$ . For  $j = 1, 2, 3$ , notice

that  $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$  for all  $z$ . Taking derivatives of both sides we obtain  $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$ . Substituting  $z = \lambda_j/2$ , we see that

$$(2) \quad \wp'_\Lambda(z) = 0 \text{ when } z = \frac{\lambda_j}{2} + \Lambda,$$

for  $j = 1, 2, 3$ . We use the notation

$$(3) \quad e_1 = \wp_\Lambda\left(\frac{\lambda_1}{2}\right), \quad e_2 = \wp_\Lambda\left(\frac{\lambda_2}{2}\right), \quad e_3 = \wp_\Lambda\left(\frac{\lambda_3}{2}\right)$$

to denote the critical values of  $\wp_\Lambda$ .

In the rest of the paper, we use  $\Omega$  to denote a triangular lattice. The invariants and critical values of  $\wp_\Omega$  on a triangular lattice  $\Omega$  take an especially nice form.

**Proposition 2.4** ([6]). *Let  $\Omega$  be a triangular lattice. Then*

- (1)  $g_2(\Omega) = 0$ .
- (2)  $e_1, e_2, e_3$  all have the same modulus and are cube roots of  $g_3/4$ , so  $e_1 = e^{4\pi i/3}e_3$  and  $e_2 = e^{2\pi i/3}e_3$ .
- (3) If  $g_3 \in \mathbb{R}$ , then  $\wp_\Omega: \mathbb{R} \rightarrow [e_1, \infty]$ .

Among all lattices, those having the most regular period parallelograms are distinguished in many respects. Results on how the lattice shape influences the dynamics of the Weierstrass elliptic function can be found in [10, 11, 12, 13, 14, 15, 16].

**2.3. Background on the dynamics of meromorphic functions.** Let  $f: \mathbb{C} \rightarrow \mathbb{C}_\infty$  be a meromorphic function. The *Fatou set*  $F(f)$  is the set of points  $z \in \mathbb{C}_\infty$  such that  $\{f^n: n \in \mathbb{N}\}$  is defined and normal in some neighborhood of  $z$ . The *Julia set* is the complement of the Fatou set on the sphere,  $J(f) = \mathbb{C}_\infty \setminus F(f)$ . Notice that  $\mathbb{C}_\infty \setminus \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}$  is the largest open set where all iterates are defined. If  $f$  has at least one pole that is not an omitted value, then  $\overline{\bigcup_{n \geq 0} f^{-n}(\infty)}$  has more than two elements. Since  $f(\mathbb{C}_\infty \setminus \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \setminus \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}$ , Montel's theorem implies that

$$(4) \quad J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}.$$

Let  $\text{Crit}(f)$  denote the set of critical points of  $f$ , i.e.,

$$\text{Crit}(f) = \{z: f'(z) = 0\}.$$

If  $z_0$  is a critical point, then  $f(z_0)$  is a *critical value*. The *postcritical set* of  $f$  is

$$P(f) = \overline{\bigcup_{n > 0} f^n(\text{Crit}(f))}.$$

A point  $z_0$  is *periodic* of period  $p$  if there exists a  $p \geq 1$  such that  $f^p(z_0) = z_0$ . We also call the set  $\{z_0, f(z_0), \dots, f^{p-1}(z_0)\}$  a *p-cycle*. The *multiplier* of a point  $z_0$  of period  $p$  is the derivative  $(f^p)'(z_0)$ . A periodic point  $z_0$  is classified as *attracting*, *repelling*, or *neutral* if  $|(f^p)'(z_0)|$  is less than, greater than, or equal to 1 respectively. If  $|(f^p)'(z_0)| = 0$ , then  $z_0$  is called a *superattracting* periodic point.

Suppose  $U$  is a connected component of the Fatou set. We say that  $U$  is *preperiodic* if there exists  $n > m \geq 0$  such that  $f^n(U) = f^m(U)$ , and the minimum of  $n - m = p$  for all such  $n, m$  is the *period* of the cycle. Elliptic functions have a finite number of critical values, and thus it turns out that the classification of periodic

components of the Fatou set is no more complicated than that of rational maps of the sphere. Periodic components of the Fatou set of elliptic functions may be attracting domains, parabolic domains, Siegel disks, or Herman rings. In particular, elliptic functions have no wandering domains or Baker domains [1, 12, 22].

Let  $C = \{U_0, U_1, \dots, U_{p-1}\}$  be a periodic cycle of components of  $F(f)$ . If  $C$  is a cycle of immediate attractive basins or parabolic domains, then  $U_j \cap Crit(f) \neq \emptyset$  for some  $0 \leq j \leq p - 1$ . If  $C$  is a cycle of Siegel Disks or Herman rings, then  $\partial U_j \subset \bigcup_{n \geq 0} f^n(Crit(f))$  for all  $0 \leq j \leq p - 1$ . In particular, critical points are required for any type of preperiodic Fatou component.

### 3. A FAMILY OF EVEN ELLIPTIC FUNCTIONS ON TRIANGULAR LATTICES

In this section, we investigate the basic dynamics of the function  $g_\Omega(z) = 1/\wp_\Omega(z)$  where we restrict our attention to triangular lattices  $\Omega = [\omega, \varepsilon\omega]$ , where  $\varepsilon = e^{2\pi i/3}$ . Using Proposition 2.4, these lattices  $\Omega$  have  $g_2 = 0$  and thus are parametrized by  $g_3 \in \mathbb{C} - \{0\}$ .

Clearly,  $g_\Omega$  is an even, degree two elliptic function with period lattice  $\Omega$ . The derivative of  $g_\Omega$  is  $g'_\Omega(z) = -\wp'_\Omega(z)/(\wp_\Omega(z))^2$ . Using Equation 2,  $\wp'_\Omega$  is zero at any half lattice point  $\omega/2 + \Omega, \varepsilon\omega/2 + \Omega, (\omega + \varepsilon\omega)/2 + \Omega$ . By Equation 3 and Proposition 2.4 (2),  $\wp_\Omega$  is nonzero at any half lattice point, and thus every half lattice point of  $\Omega$  is a critical point of  $g_\Omega$ . Since lattice points  $\Omega$  are double poles of  $\wp_\Omega$ , lattice points are critical points of  $g_\Omega$  as well. Thus the critical set of  $g_\Omega$  is

$$Crit(g_\Omega) = \left\{ 0, \frac{\omega}{2}, \frac{\varepsilon\omega}{2}, \frac{\omega + \varepsilon\omega}{2} \right\} + \Omega.$$

We note that 0 is therefore a superattracting fixed point of  $g_\Omega$ .

Using Equation 3 and Propostion 2.4, if  $v = g_\Omega(\omega/2) = 1/e_1$  is one of the critical values, then  $\varepsilon v$  and  $\varepsilon^2 v$  are also critical values. Thus the four critical values of  $g_\Omega$  are  $\{0, v, \varepsilon v, \varepsilon^2 v\}$ .

The function  $g_\Omega$  has simple poles at the zeros of  $\wp_\Omega$ . Since  $\Omega$  is triangular, these points occur at the center of the equilateral triangles determined by the lattice,  $\pm \frac{1}{3}(\omega - \varepsilon\omega) + \Omega$  [6]. In Figure 1 we show a period parallelogram  $\mathcal{Q} = \{s\omega + t\varepsilon\omega : 0 \leq s, t \leq 1\}$  for the lattice  $\Omega$  with  $g_3(\Omega) = 4$ . Critical points of  $g_\Omega$  in  $\mathcal{Q}$  are marked with filled circles, and the poles of  $g_\Omega$  in  $\mathcal{Q}$  are denoted by open circles.

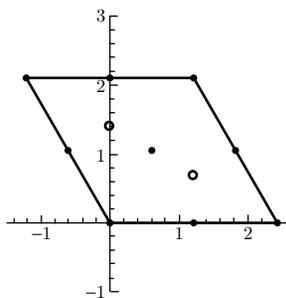


FIGURE 1. Critical points and poles for  $g_\Omega$  when  $g_3(\Omega) = 4$

**3.1. Symmetries.** Julia and Fatou sets of elliptic functions exhibit a number of distinct symmetries. First, we note that the Julia and Fatou sets of an elliptic function on any lattice  $\Lambda$  are periodic with respect to the lattice.

**Theorem 3.1.** *If  $f_\Lambda$  is an elliptic function defined on any lattice  $\Lambda$ , then  $J(f_\Lambda) + \Lambda = J(f_\Lambda)$  and  $F(f_\Lambda) + \Lambda = F(f_\Lambda)$ .*

For even elliptic functions, the Julia and Fatou sets are symmetric with respect to the origin.

**Theorem 3.2.** *If  $f_\Lambda$  is an even elliptic function defined on any lattice  $\Lambda$ , then  $(-1)J(f_\Lambda) = J(f_\Lambda)$  and  $(-1)F(f_\Lambda) = F(f_\Lambda)$ .*

We observe additional symmetry with  $g_\Omega = 1/\wp_\Omega$  on a triangular lattice  $\Omega$ .

**Lemma 3.3.** *If  $\Omega$  is a triangular lattice and  $\varepsilon = e^{2\pi i/3}$ , then*

$$g_\Omega^n(\varepsilon z) = \begin{cases} \varepsilon^2 g_\Omega^n(z) & n \text{ is odd} \\ \varepsilon g_\Omega^n(z) & n \text{ is even} \end{cases}$$

and

$$g_\Omega^n(\varepsilon^2 z) = \begin{cases} \varepsilon g_\Omega^n(z) & n \text{ is odd} \\ \varepsilon^2 g_\Omega^n(z) & n \text{ is even.} \end{cases}$$

*Proof.* We prove the first statement by induction on  $n$ . Since  $\varepsilon\Omega = \Omega$ , applying Proposition 2.2, we have

$$g_\Omega(\varepsilon z) = \frac{1}{\wp_\Omega(\varepsilon z)} = \frac{\varepsilon^2}{\wp_\Omega(z)} = \varepsilon^2 g_\Omega(z)$$

and  $g_\Omega^2(\varepsilon z) = g_\Omega(\varepsilon^2 g_\Omega(z)) = \varepsilon^4 g_\Omega^2(z) = \varepsilon g_\Omega^2(z)$ . Assume the statement is true for  $k = 1, \dots, n$ . If  $n + 1$  is even, we have

$$g_\Omega^{n+1}(\varepsilon z) = g_\Omega(g_\Omega^n(\varepsilon z)) = g_\Omega(\varepsilon^2 g_\Omega^n(z)) = \varepsilon^4 g_\Omega^{n+1}(z) = \varepsilon g_\Omega^{n+1}(z).$$

If  $n + 1$  is odd, we have

$$g_\Omega^{n+1}(\varepsilon z) = g_\Omega(g_\Omega^n(\varepsilon z)) = g_\Omega(\varepsilon g_\Omega^n(z)) = \varepsilon^2 g_\Omega^{n+1}(z).$$

The second statement follows from a similar proof. □

Lemma 3.3 implies that the Julia and Fatou sets exhibit a rotational symmetry.

**Theorem 3.4.** *If  $\Omega$  is a triangular lattice, then  $\varepsilon F(g_\Omega) = F(g_\Omega)$  and  $\varepsilon J(g_\Omega) = J(g_\Omega)$ , where  $\varepsilon$  is a cube root of unity.*

In addition, we also see symmetry with respect to any critical point.

**Proposition 3.5.**  *$J(g_\Omega)$  and  $F(g_\Omega)$  are symmetric with respect to any critical point  $\{0, \omega/2, \varepsilon\omega/2, (\omega + \varepsilon\omega)/2\} + \Omega$ . That is, if  $c$  is any critical point of  $g_\Omega$ , then  $c + z \in J(g_\Omega)$  if and only if  $c - z \in J(g_\Omega)$ . In particular, if  $F_o$  is any component of  $F(g_\Omega)$  that contains a critical point  $c$ , then  $F_o$  is symmetric with respect to  $c$ .*

*Proof.* If  $c = 0$ , then the symmetry is a direct consequence of Theorem 3.2. In general, we have  $g_\Omega(z) = 1/\wp_\Omega(z) = 1/\wp_\Omega(u)$  if  $u \equiv -z \pmod{\Omega}$ . We first consider  $c = \omega/2$ , and we note that  $\omega - c = c$ . Then  $g_\Omega(c + z) = 1/\wp_\Omega(c + z) = 1/\wp_\Omega(-c - z + \omega) = 1/\wp_\Omega(c - z) = g_\Omega(c - z)$  as claimed. The proofs for  $c = \varepsilon\omega/2$  and  $c = (\omega + \varepsilon\omega)/2$  follow similarly. If  $c_{m,n} = c + m\omega + n\varepsilon\omega$ , then  $g_\Omega(c_{m,n} + z) = 1/\wp_\Omega(c_{m,n} + z) = 1/\wp_\Omega(c + z) = 1/\wp_\Omega(c - z) = 1/\wp_\Omega(c_{m,n} - z) = g_\Omega(c_{m,n} - z)$ . □

Since the poles  $\mu$  of  $g_\Omega$  are the centers of the equilateral triangles determined by the lattice  $\Omega$ , Theorem 3.4 and Proposition 3.5 imply the following corollary.

**Corollary 3.6.** *If  $\mu$  is a pole of  $g_\Omega$ , then the Julia and Fatou sets are symmetric with respect to rotation around  $\mu$  by  $2\pi/3$ .*

*Proof.* Let  $\Omega = [\omega, \varepsilon\omega]$ , where  $\varepsilon = e^{2\pi i/3}$ , and let  $s = \frac{1}{3}(\omega - \varepsilon\omega)$  be a pole. Let  $z = s + b$ . Then the rotation of  $z$  around  $s$  by  $2\pi/3$  is  $y = s + \varepsilon b = \varepsilon(z - \omega)$ . Using Theorems 3.1 and 3.4,  $z \in F(g_\Omega)$  if and only if  $\varepsilon(z - \omega) = y \in F(g_\Omega)$ . Using Theorems 3.1 and 3.2, this symmetry passes to all poles  $\mu = \pm s + \Omega$ .  $\square$

#### 4. CANTOR JULIA SETS

The first examples of meromorphic functions with Cantor Julia sets appeared in [5], and examples of elliptic functions with Cantor Julia sets appeared in [14] and [19]. Here, we present criteria which guarantee that the Julia set of  $g_\Omega$  is Cantor.

For a general elliptic function  $f$  on a lattice  $\Lambda$ , the periodicity of  $f$  and  $J(f)$  with respect to  $\Lambda$  gives rise to different possibilities for the connectivity of a component when the Fatou and Julia sets are projected to the torus. If we write

$$\pi_\Lambda : \mathbb{C} \rightarrow \mathbb{C}/\Lambda$$

for the usual toral quotient map, we make the following definition.

**Definition 4.1.** A Fatou component  $F_o$  of the map  $f$  is called a *toral band* if  $F_o$  contains an open subset  $U$  which is simply connected in  $\mathbb{C}$ , but  $\pi_\Lambda(U)$  is not simply connected on  $\mathbb{C}/\Lambda$ . In other words,  $U$  projects down to a topological band around the torus and contains a homotopically nontrivial curve.

We also introduce the notion of a double toral band.

**Definition 4.2.** Suppose that we have an elliptic function  $f$  with period lattice  $\Lambda$ . If we have a component  $W \subset F(f)$  which contains a simple closed loop which forms the boundary of a fundamental region for  $\Lambda$ , then we say  $W$  is a *double toral band*.

Examples of elliptic functions with single and double toral bands were constructed in [14].

Recall that if the Julia set is totally disconnected, then it is called a *Cantor set* since it is homeomorphic to the classical middle thirds Cantor set. In this case there is exactly one Fatou component which is therefore completely invariant, so there is one nonrepelling fixed point.

We will need the following definition of hyperbolicity.

**Definition 4.3.** We say that an elliptic function  $f$  is *hyperbolic* if  $J(f)$  is disjoint from  $P(f)$ .

The following theorem, which was proved in [14], gives a condition which guarantees that the Julia set of an elliptic function  $f$  is Cantor.

**Theorem 4.4.** *If  $W$  is a double toral band for  $f$  that contains all of the critical values and  $f$  is hyperbolic, then  $F(f) = W$  and  $J(f)$  is a Cantor set.*

We now return to the context of the function  $g_\Omega = 1/\wp_\Omega$  on a triangular lattice  $\Omega$ .

**Proposition 4.5.** *Let  $\Omega$  be a triangular lattice and  $g_\Omega = 1/\wp_\Omega$ . If the component  $W \subset F(g_\Omega)$  containing 0 also contains another critical value, then all four critical values lie in  $W$  and  $J(g_\Omega)$  is a Cantor set.*

*Proof.* Suppose the critical value  $v \in W$  where, without loss of generality,  $v = g_\Omega(\omega/2)$ . Since  $W$  is open and connected, it is path connected. Let  $\gamma$  denote a path from 0 to  $v$ . Then  $\varepsilon\gamma$  and  $\varepsilon^2\gamma$  are paths from 0 to the critical values  $\varepsilon v$  and  $\varepsilon^2 v$  contained in the Fatou set by Theorem 3.4, and thus all four critical values lie in  $W$ . Since all critical values approach the superattracting fixed point at 0,  $g_\Omega$  is hyperbolic.

Let  $c_1 = \omega/2$ , so  $v = g_\Omega(c_1)$ . Let  $U_1$  be the component of  $g_\Omega^{-1}(W)$  that contains 0. Then one component of  $g_\Omega^{-1}(\gamma)$  lies in the period parallelogram  $\mathcal{Q} = \{s\omega + t\varepsilon\omega : 0 \leq s, t, < 1\}$  and connects 0 to  $c_1$ . Since  $W$  and  $U_1$  are both Fatou components containing 0, we must have  $W = U_1$ , and thus  $W$  contains 0 and  $c_1$ . By Proposition 3.5,  $W$  is symmetric with respect to  $c_1$ . Thus  $\omega \in W$  and the path  $\delta = \gamma \cup (-\gamma + \omega)$  connects 0 to  $\omega$ . Using Theorems 3.1 and 3.4, the curve  $\delta \cup \varepsilon\delta \cup (\delta + \varepsilon\omega) \cup (\varepsilon\delta + \omega) \subset W$  forms the boundary of a period parallelogram, and thus  $W$  is a double toral band. Since all of the critical values are contained in  $W$  and  $g_\Omega$  is hyperbolic, Theorem 4.4 implies that  $J(g_\Omega)$  is Cantor.  $\square$

## 5. CONNECTED JULIA SETS

In this section, we discuss criteria for the Julia set of  $g_\Omega$  on a triangular lattice  $\Omega$  to be connected. We begin by showing that  $g_\Omega$  has no Herman rings. We then present a condition on the location of the critical values of  $g_\Omega$  which guarantees that the Julia set is connected.

On an arbitrary lattice  $\Lambda$ , the Weierstrass elliptic function  $\wp_\Lambda$  has no Herman rings [13]. We extend this result to the function  $g_\Omega$  on a triangular lattice  $\Omega$ .

**Theorem 5.1.** *If  $\Omega$  is a triangular lattice and  $g_\Omega = 1/\wp_\Omega$ , then  $g_\Omega$  has no cycle of Herman rings.*

*Proof.* Let  $\Omega = [\omega, \varepsilon\omega]$  be a triangular lattice, and suppose that  $g_\Omega$  has a cycle of Herman rings  $\{U_0, U_1, \dots, U_{p-1}\}$  of period  $p$ . Then for any  $i = 0, 1, \dots, p-1$ ,  $(g_\Omega)^p: U_i \rightarrow U_i$  is conjugate to an irrational rotation of the annulus and thus has degree one. The preimages under this conjugacy of the circles  $|\eta| = r$ ,  $1 < r < R$  foliate the annuli with  $(g_\Omega)^p$  forward invariant leaves on which  $(g_\Omega)^p$  is injective. Let  $\gamma$  be a  $(g_\Omega)^p$  invariant leaf of  $U_i$ , and let  $B_\gamma$  denote the bounded component of the complement of  $\gamma$ . Since  $U_i$  is multiply connected, we know that  $B_\gamma$  contains a prepole using Equation 4. Thus, there is a smallest nonnegative number  $n$  such that  $(g_\Omega)^n(\gamma)$  contains a pole  $\mu = \pm\frac{1}{3}(\omega - \varepsilon\omega) + \nu$  with  $\nu \in \Omega$  (a center of an equilateral triangle formed by the lattice) in  $B_{(g_\Omega)^n(\gamma)}$ . Let  $U_j$  denote the Herman ring  $(g_\Omega)^n(U_i)$ .

Since  $U_j$  is homeomorphic to an annulus with the pole  $\mu$  in  $B_{(g_\Omega)^n(\gamma)}$ , Theorem 3.2 implies that  $\varepsilon U_j$  is a Fatou component around another pole  $\varepsilon\mu$ , so there is a lattice point  $l \in \Omega$  such that  $\mu$  is in  $\varepsilon B_{(g_\Omega)^n(\gamma)} + l$ . Therefore both  $U_j$  and  $\varepsilon U_j + l$  are annuli containing simple closed loops  $\gamma$  and  $\varepsilon\gamma + l$  respectively, and each has  $\mu$  in its bounded complement. Furthermore,  $\varepsilon\gamma + l$  is the rotation of  $\gamma$  by  $2\pi/3$  around  $\mu$ . But  $\gamma$  must intersect its rotation around  $\mu$  by  $2\pi/3$ , so there is a point  $z \in \gamma \cap \varepsilon\gamma + l \in U_j$ . Thus if  $z \in U_j$ , then  $e^{-2\pi i/3}z - e^{-2\pi i/3}l \in U_j$ . Using Lemma 3.3 and periodicity,  $g_\Omega(z)$  and  $g_\Omega(e^{-2\pi i/3}z - e^{-2\pi i/3}l) = \varepsilon g_\Omega(z)$  both lie in  $U_{j+1}$ . If  $\delta$

is a path from  $g_\Omega(z)$  to  $\varepsilon g_\Omega(z)$  in  $U_{j+1}$ , then using Theorem 3.4,  $\delta \cup \varepsilon\delta \cup \varepsilon^2\delta$  is a path in  $U_{j+1}$  and thus  $B_{U_{j+1}}$  contains the origin.

We now work with the Herman ring  $U_{j+1}$ , and we let  $\beta$  be a  $(g_\Omega)^p$  invariant leaf of  $U_{j+1}$ . Using Theorem 3.2,  $U_{j+1}$  and  $-U_{j+1}$  are annuli containing simple closed loops  $\beta$  and  $-\beta$  respectively, and each has 0 in its bounded complement. However, there must be a  $z \in \beta \cup -\beta$ , so  $U_{j+1} = -U_{j+1}$ . But  $g_\Omega$  is even, so  $g_\Omega(z) = g_\Omega(-z)$ , which contradicts that  $(g_\Omega)^p$  is degree one on  $U_{j+1}$ .  $\square$

Although there are infinitely many critical points for  $g_\Omega$ , there are exactly 4 critical values and  $g_\Omega$  is locally two-to-one in each fundamental region. The elimination of the possibility of Herman rings simplifies the discussion of the connectivity possibilities of the Julia set.

Theorem 5.2 is based on a similar result for polynomials given by Milnor in [20] and extended to the Weierstrass elliptic function in [14]. Since the proof is a local argument, and both  $\wp_\Omega$  and  $g_\Omega$  are locally two-to-one in each period parallelogram, the proof is identical to that given in [14], and we do not reproduce it here.

**Theorem 5.2.** *Suppose  $\Omega$  is a lattice for which  $g_\Omega$  has no Herman rings and each critical value of  $g_\Omega$  that lies in the Fatou set is the only critical value in that component. Then  $J(g_\Omega)$  is connected. In particular, if each Fatou component contains either zero or one critical value, then  $J(g_\Omega)$  is connected.*

### 6. PROOF OF THE FUNDAMENTAL DICHOTOMY

The connectivity properties of the Julia set of  $g_\Omega$  relate fundamentally to the number of critical values that lie in each Fatou component. The symmetry of the lattice and the algebraic properties of  $g_\Omega$  simplify the possibilities for the location of the critical values in the Fatou set.

**Proposition 6.1.** *If  $\Omega$  is a triangular lattice, every Fatou component of  $g_\Omega$  contains zero, one, or four critical values.*

*Proof.* We show that if there is a Fatou component that contains two critical values, then it contains all four. Let  $W$  denote the superattracting Fatou component containing 0. By Proposition 4.5, if some nonzero critical value  $v$  lies in  $W$ , then all four critical values lie in  $W$ .

Next, suppose two nonzero critical values, say  $v$  and  $\varepsilon v$ , lie in the same component  $V_0$ . Let  $\gamma$  be a path from  $v$  to  $\varepsilon v$  in  $V_0$ . Then  $\varepsilon\gamma$  is a path in  $F(g_\Omega)$  from  $\varepsilon v$  to  $\varepsilon^2 v$  by Theorem 3.4, so  $\varepsilon^2 v \in V_0$ . Since 0 is a superattracting fixed point, all critical values lie in the Fatou set, and thus all periodic Fatou cycles are either attracting or parabolic.

We claim that the superattracting fixed point at 0 is the only nonrepelling cycle. If not, let  $\{U_1, U_2, \dots, U_n\}$  denote a forward invariant cycle of attracting or parabolic Fatou components corresponding to a cycle  $\{p_1, p_2, \dots, p_n\}$ , where  $p_j \neq 0$  for all  $j = 1, \dots, n$ . Suppose  $V_0 = U_j$  is the component containing the three nonzero critical values. Then  $\lim_{k \rightarrow \infty} g_\Omega^{kn}(v) = p_j$  and  $\lim_{k \rightarrow \infty} g_\Omega^{kn}(\varepsilon v) = p_j$ . But by Proposition 3.3,

$$g_\Omega^{kn}(\varepsilon v) = \begin{cases} \varepsilon^2 g_\Omega^{kn}(v) & kn \text{ is odd} \\ \varepsilon g_\Omega^{kn}(v) & kn \text{ is even.} \end{cases}$$

Therefore  $p_j = \varepsilon p_j$  or  $p_j = \varepsilon^2 p_j$ , so  $p_j = 0$ , a contradiction. Thus  $g_\Omega^k(V_0) = W$  for some  $k \geq 0$ .

If  $k = 0$ , then  $V_0 = W$  and we are done, so assume  $k > 0$ . Using Proposition 3.3, for every  $j \geq 0$ ,  $V_j = g_\Omega^j(V_0)$  contains the points  $g_\Omega^j(v)$ ,  $\varepsilon g_\Omega^j(v)$ , and  $\varepsilon^2 g_\Omega^j(v)$ . If  $g_\Omega^j(v) \neq 0$  and  $\gamma_j$  is a path from  $g_\Omega^j(v)$  to  $\varepsilon g_\Omega^j(v)$ , then  $\delta_j = \gamma_j \cup \varepsilon \gamma_j \cup \varepsilon^2 \gamma_j \subset V_j$  forms a loop around the origin.

Since  $\delta_0 \subset V_0$  forms a loop around the origin,  $W$  must be contained within one fundamental region. If not, then there exist  $z, z + \nu \in W$  for some lattice point  $\nu \in \Omega$ . Using Theorem 3.1,  $z + j\nu \in W$  for all  $j \in \mathbb{Z}$ . Since the complement of the loop  $\delta_0$  contains points of the form  $z + j\nu$  in both its bounded and unbounded complements, this implies that  $V_0 \cap W \neq \emptyset$ , so  $V_0 = W$ .

Thus if  $k > 0$ ,  $W$  is contained in one fundamental region. Let  $k_0 > 0$  be the least integer with  $g_\Omega^{k_0}(V_0) = W$ . By Theorem 3.1,  $W + \nu$  is a Fatou component contained in one fundamental region for each  $\nu \in \Omega$ , and  $g_\Omega(W + \nu) = W$ . Since  $g_\Omega^{-1}(0) = \Omega$ , we have that  $g_\Omega^{-1}(W) = \{W + \nu : \nu \in \Omega\}$ . If  $W = V_{k_0} = g_\Omega(V_{k_0-1})$ , then  $V_{k_0-1} = W + \nu$  for some  $\nu \in \Omega$ . But since each component  $W + \nu$  is contained within one fundamental region containing the lattice point  $\nu$ , and  $V_{k_0-1}$  contains a loop around the origin,  $\nu = 0$ . Thus  $V_{k_0-1} = W$ . But  $k_0$  was the least integer with  $V_{k_0} = W$ , contradicting our assumption. Thus  $W = V_0$ , and  $W$  contains all four critical values. □

We note that if  $v$  is attracted to the superattracting fixed point at 0, then either  $v$  lies in  $W$  or there are no other critical values in the component containing  $v$ .

We are now ready to prove the fundamental dichotomy of Julia sets for  $g_\Omega$  on triangular lattices  $\Omega$ .

**Theorem 6.2.** *Let  $\Omega$  be a triangular lattice. If all four critical values of  $g_\Omega = 1/\wp_\Omega$  are contained in one component of the Fatou set, then the Julia set of  $g_\Omega$  is Cantor. Otherwise, the Julia set is connected.*

*Proof.* By Proposition 6.1, every Fatou component of  $g_\Omega$  contains zero, one, or four critical values. If there is a Fatou component containing all four critical values, then  $J(g_\Omega)$  is Cantor by Proposition 4.5. Otherwise, every Fatou component contains zero or one critical value, and then  $J(g_\Omega)$  is connected by Theorem 5.2. □

We conclude by presenting examples that guarantee that the parameter space of  $g_\Omega$  contains functions with connected Julia sets and functions with Cantor Julia sets. We begin by defining the *standard triangular lattice*  $\Omega_1 = [\omega_1, \varepsilon\omega_1]$  as the lattice with invariants  $g_2(\Omega_1) = 0$  and  $g_3(\Omega_1) = 4$ , where we choose the generator  $\omega_1 > 0$ . Using Theorem 2.3,  $\wp_{\Omega_1}$  maps  $\mathbb{R}$  to  $\mathbb{R} \cup \{\infty\}$ , and Proposition 2.4 implies that  $\wp_{\Omega_1}(\omega_1/2) = 1$ . We will use the following lemma, which was proved in [19].

**Lemma 6.3.** *If  $\Omega$  is a triangular lattice with  $g_3 > 0$ , then*

- (1)  $g_\Omega|_{\mathbb{R}} : \mathbb{R} \rightarrow [0, 1/e_1]$  is piecewise monotonic, onto, and periodic with respect to  $\Omega$ .
- (2)  $g_\Omega$  is strictly increasing on  $[0, \omega/2]$  and strictly decreasing on  $[\omega/2, \omega]$ .

**Example 6.4.** An example with a Cantor Julia set.

We prove that  $g_\Omega$  on the lattice  $\Omega = \frac{1}{2}\Omega_1$ , where  $\Omega_1$  is the standard triangular lattice, has a Cantor Julia set. Using Propositions 2.2 and 2.4, we have that  $\wp_\Omega|_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$(5) \quad \wp_\Omega(z) \geq \wp_\Omega\left(\frac{\omega}{2}\right) = \wp_{\frac{1}{2}\Omega_1}\left(\frac{1}{2}\frac{\omega_1}{2}\right) = 2^2 \wp_{\Omega_1}\left(\frac{\omega_1}{2}\right) = 4,$$

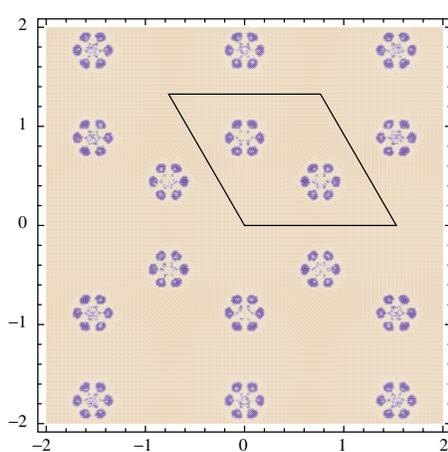


FIGURE 2. A Cantor Julia set

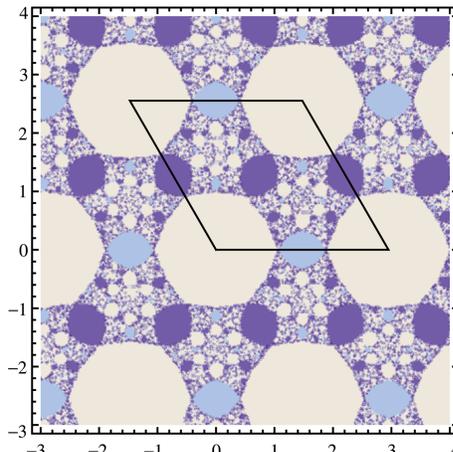


FIGURE 3. A connected Julia set

and  $g_3(\Omega) = 64$ . We claim that  $0 \leq g'_\Omega(z) < 1$  for all  $z \in [0, \omega/2]$ . To prove our claim, we first observe that since  $\wp'_\Omega(z) \leq 0$  on  $[0, \omega/2]$ , we have that  $g'_\Omega = -\wp'_\Omega/(\wp_\Omega)^2 \geq 0$ . Next, to prove that  $g'_\Omega(z) = -\wp'_\Omega(z)/(\wp_\Omega(z))^2 < 1$ , it suffices to prove that  $(\wp'_\Omega(z))^2/(\wp_\Omega(z))^4 < 1$  or, alternatively, that  $(\wp_\Omega(z))^4 - (\wp'_\Omega(z))^2 > 0$ . Using Equations 1 and 5 and Proposition 2.4, we have

$$\begin{aligned} (\wp_\Omega(z))^4 - (\wp'_\Omega(z))^2 &= (\wp_\Omega(z))^4 - 4(\wp_\Omega(z))^3 + g_2(\Omega)\wp_\Omega(z) + g_3(\Omega) \\ &= (\wp_\Omega(z))^3(\wp_\Omega(z) - 4) + 64 > 0 \end{aligned}$$

for all  $z$  in  $[0, \omega/2]$ . Since 0 is a superattracting fixed point and  $0 \leq g'_\Omega(z) < 1$  for all  $z \in [0, \omega/2]$ , the Intermediate Value Theorem implies that 0 is the only fixed point in  $[0, \omega/2]$ . Using Lemma 6.3, 0 is the only fixed point in  $\mathbb{R}$ . Thus all  $x > 0$  iterate to the superattracting fixed point at 0, so  $g_\Omega(\omega/2) > 0$  is in the same Fatou component as 0. Using Proposition 4.5,  $J(g_\Omega)$  is Cantor.

Figure 2 shows an approximation of the Julia set of this function, with a boundary of one period parallelogram outlined in black.

**Example 6.5.** An example with a connected Julia set.

Let  $\Omega_1 = [\omega_1, \varepsilon\omega_1]$  be the standard triangular lattice, and let  $k = \omega_1/2$ . Using the lattice  $\Omega = k\Omega_1$ , Proposition 2.2 implies

$$g_\Omega\left(\frac{\omega}{2}\right) = g_{k\Omega_1}\left(\frac{k\omega_1}{2}\right) = k^2 = \frac{\omega}{2}.$$

Thus  $g_\Omega$  has superattracting fixed points at 0 and  $\omega/2$ , and so  $\omega/2$  must lie in a separate Fatou component from 0. Using Lemma 3.3, we also have a superattracting two-cycle at  $\{\varepsilon\omega/2, \varepsilon^2\omega/2\}$ . By Theorem 6.2,  $J(g_\Omega)$  is connected.

Figure 3 shows the Julia set of this function, with a boundary of one period parallelogram outlined in black. Points attracted to 0 are colored white, points attracted to the superattracting fixed point at  $\omega/2$  are colored grey (light purple), and points attracted to the two-cycle  $\{\varepsilon\omega/2, \varepsilon^2\omega/2\}$  are colored black (dark purple).

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