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ESTIMATES FOR UNIMODULAR FOURIER MULTIPLIERS ON MODULATION SPACES

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Abstract. We study the action on modulation spaces of Fourier multipliers with symbols $e^{i\mu(\xi)}$, for real-valued functions μ having unbounded second derivatives. In a simplified form our result reads as follows: if μ satisfies the usual symbol estimates of order $\alpha \geq 2$, or if μ is a positively homogeneous function of degree α, then the corresponding Fourier multiplier is bounded as an operator between the weighted modulation spaces $M^{p,q}_s$ and $M^{p,q}$, for all $1 \leq p, q \leq \infty$ and $s \geq (\alpha - 2)n|1/p - 1/2|$. Here s represents the loss of derivatives. The above threshold is shown to be sharp for *any* homogeneous function μ whose Hessian matrix is non-degenerate at some point.

1. Introduction and statement of the results

A Fourier multiplier in $\mathbb{R}^n$ is formally an operator of the type

$$\sigma(D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(\xi) \hat{f}(\xi)\, d\xi,$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$ is the Fourier transform. The function σ is called the *symbol* of the multiplier. Whereas the action of these operators on L^2 is clear (by Parseval's formula), their behavior in L^p, $p \neq 2$, for several classes of symbols is a fundamental topic in harmonic analysis, with important applications to partial differential equations.

In particular, unimodular Fourier multipliers are defined by symbols of the type $\sigma(\xi) = e^{i\mu(\xi)}$, for real-valued functions μ. They arise when solving the Cauchy problem for dispersive equations. For example, for the solution $u(t,x)$ of the Cauchy problem

$$\begin{cases} i\partial_t u + |\Delta|^{\alpha/2} u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

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$(t,x) \in \mathbb{R} \times \mathbb{R}^n$, we have the formula $u(t,x) = \left(e^{it|D|^\alpha} u_0\right)(x)$. The cases $\alpha = 1, 2, 3$ are of particular interest because they correspond to the (half-)wave equation, the Schrödinger equation and (essentially) the Airy equation, respectively.

Unimodular Fourier multipliers generally do not preserve any Lebesgue space L^p, except for $p = 2$. For example, $e^{i|D|^2}$ is bounded on L^p if and only if $p = 2$ (see [16]). Also, $e^{i|D|^2}$ is bounded on the Besov space $\dot{B}^{p,q}_s$ or $B^{p,q}_s$ if and only if $p = 2$ (see [18, 20]). It is then natural to study boundedness properties on other function spaces arising in Fourier analysis. This was recently done in [1] for the modulation spaces $M^{p,q}$, $1 \leq p, q \leq \infty$. These spaces were first introduced by Feichtinger [10, 11] to measure smoothness of a function or distribution in a way different from Besov spaces, and they are now recognized as a useful tool for studying pseudo-differential operators (see [14, 22, 25]). We recall the precise definition in Section 2 below. Here it suffices to observe that, for heuristic purposes, distributions in $M^{p,q}$ may be regarded as functions which locally have the same regularity as a function in $\mathcal{F}L^q$ (the space of distributions whose Fourier transform is in L^q), but at infinity decay like a function in L^p.

Now, it was proved in [1] that if $0 \leq \alpha \leq 2$, then $e^{i|D|^\alpha}$ is bounded on the modulation spaces $M^{p,q}$ for all $1 \leq p, q \leq \infty$ (for $\alpha = 2$ this was already known from [15, 25, 28]). Moreover, the conclusion extends to symbols $\sigma(\xi) = e^{i\mu(\xi)}$ where μ is a positively homogeneous function of degree $\alpha \in [0, 2]$, smooth away from the origin, or even a smooth function on $\mathbb{R}^n$ whose derivatives of order ≥ 2 are bounded.

More generally, similar results also hold, when $p = q$, for a class of Fourier integral operators whose phases have bounded derivatives of order ≥ 2; see [3, 4, 7]. However for $p \neq q$ a loss of regularity or decay may then occur; see [6] for an analysis of this phenomenon. A similar analysis for the class of Hörmander-type Fourier integral operators [17] was carried out in [8].

The purpose of the present paper is to study multipliers whose prototype is $e^{i|D|^\alpha}$ with $\alpha > 2$. In this case one still expects boundedness, but with a loss of regularity, namely from $M^{p,q}_s$ to $M^{p,q}$, for every sufficiently large s (which represents the loss of derivatives). Here $M^{p,q}_s = \{f \in \mathcal{S}'(\mathbb{R}^d) : (1-\Delta)^{s/2} f \in M^{p,q}\}$ is in fact a Sobolev-like space based on $M^{p,q}$.

Our main result can be stated as follows.

Theorem 1.1. *Let $\alpha > 2$ and let μ be a real-valued function of class $C^{[n/2]+3}$ on $\mathbb{R}^n \setminus \{0\}$ which satisfies*

$$|\partial^\gamma \mu(\xi)| \leq A_\gamma |\xi|^{\epsilon - |\gamma|}, \quad 0 < |\xi| \leq 1,\ |\gamma| \leq [n/2] + 1, \tag{2}$$

for some $\epsilon > 0$, and also satisfies

$$|\partial^\gamma \mu(\xi)| \leq A_\gamma |\xi|^{\alpha - 2}, \quad |\xi| > 1,\ 2 \leq |\gamma| \leq [n/2] + 3. \tag{3}$$

Suppose $1 \leq p$, $q \leq \infty$ and $s \geq (\alpha - 2)n|1/p - 1/2|$. Then the Fourier multiplier operator $e^{i\mu(D)}$ is bounded from $M^{p,q}_s(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$.

Here $[\cdot]$ denotes the integer part of a real number; also, the definition of boundedness which is relevant here requires some subtleties when $p = \infty$ or $q = \infty$; see Section 2. As an example, notice that any real-valued function $\mu(\xi)$ that is homogeneous of degree $\alpha > 2$ and C^∞ on $\mathbb{R}^n \setminus \{0\}$ satisfies the assumptions in Theorem 1.1. The conclusion in Theorem 1.1 should hold, with the same threshold, even for less regular functions μ. More precisely, it should be sufficient that

$(1+|\xi|^2)^{(2-\alpha)/2}\partial^\gamma\mu(\xi)$ belong to $M^{\infty,1}$, for $|\gamma|=2$. We plan to study these issues in greater detail in the future (see [21]).

We will prove in Section 4 that the threshold in Theorem 1.1 is generally sharp. Most interestingly, the following theorem holds.

Theorem 1.2. *Let $\alpha > 2$ and let μ be a real-valued function on $\mathbb{R}^n$ which is homogeneous of degree α and C^∞ on $\mathbb{R}^n \setminus \{0\}$. Suppose there exists a point $\xi_0 \neq 0$ at which the Hessian determinant of μ is not zero. Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, and suppose the Fourier multiplier operator $e^{i\mu(D)}$ is bounded from $M^{p,q}_s(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$. Then $s \ge (\alpha-2)n|1/p-1/2|$.*

A class of examples for Theorem 1.2 is provided by the following remark (see Appendix A for a proof).

Remark 1.3. Let $\alpha \ge 2$ and let μ be a real-valued homogeneous function on $\mathbb{R}^n$ of degree α which is C^∞ on $\mathbb{R}^n \setminus \{0\}$. If $\mu(\xi) \neq 0$ for all $\xi \neq 0$, then there exists a point $\xi_0 \neq 0$ at which the Hessian determinant of μ is not zero.

Theorem 1.2 states, in particular, that the unboundedness on $M^{p,q}$ is due to the presence of some curvature of the graph of μ. Also, this suggests an investigation of the optimal threshold in terms of the number of principal curvatures which are identically zero: if at every point the Hessian matrix of μ has rank at most r, we expect the threshold to be $(\alpha-2)r\,|1/p-1/2|$ (see [21]).

Notice that the above negative result shows that the Cauchy problem (1) is not locally well-posed in any $M^{p,q}$ if $p \neq 2$ and $\alpha > 2$. For positive results in this connection we refer to [1, 2, 5, 27] and the references therein.

The paper is organized as follows. In Section 2, we give the definitions and basic properties of modulation spaces. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively. In Section 5 we present an alternative proof of Theorem 1.2.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, f(x)\, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\, f(\xi)\, d\xi.$$

For $m \in \mathcal{S}'(\mathbb{R}^n)$, we define the Fourier multiplier operator $m(D)$ by

$$m(D)f = \mathcal{F}^{-1}[m\,\widehat{f}] = [\mathcal{F}^{-1}m] * f \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

For the multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$, we write $\partial^\gamma = \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$, where $\partial_j = \partial/\partial\xi_j$ and $\xi = (\xi_1, \dots, \xi_n)$. Given $1 \le p \le \infty$, we denote by p' the conjugate exponent of p (that is, $1/p + 1/p' = 1$). For nonnegative functions u and v defined on a set X, the notation $u(x) \asymp v(x)$ $(x \in X)$ means that there exist positive constants c and C such that $cu(x) \le v(x) \le Cu(x)$ for all $x \in X$. In this notation, we sometimes omit referring to the set X if it is obviously recognized. For $x \in \mathbb{R}$, we write $[x]$ to denote the integer part of x.

We recall the definition of the modulation spaces. Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\psi \subset (-1,1)^n \quad \text{and} \quad \sum_{k\in\mathbb{Z}^n} \psi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4}$$

Then the modulation space $M^{p,q}_s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M^{p,q}_s} = \left(\sum_{k\in\mathbb{Z}^n} (1+|k|)^{sq} \|\psi(D-k)f\|^q_{L^p} \right)^{1/q} < \infty.$$

If $s = 0$, we simply write $M^{p,q}(\mathbb{R}^n)$ instead of $M^{p,q}_0(\mathbb{R}^n)$. It is known that the definition of $M^{p,q}_s(\mathbb{R}^n)$ is independent of the choice of $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4). In the rest of this paper, we shall always use the letter ψ to denote a function that satisfies (4).

It should be observed that this definition is indeed equivalent to that given before Theorem 1.1, as a consequence of [25, Theorem 2.2, Corollary 2.3]. Moreover, the following facts are known: $M^{p,q}_s(\mathbb{R}^n)$ is a Banach space; $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}_s(\mathbb{R}^n)$ if $1 \le p, q < \infty$; and $M^{p_1,q_1}_{s_1}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}_{s_2}(\mathbb{R}^n)$ if $p_1 \le p_2$, $q_1 \le q_2$ and $s_1 \ge s_2$. It is also known that

$$\|f\|_{M^{p,q}_s} \asymp \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_g f(x,\xi)|^p \, dx \right)^{q/p} (1+|\xi|^2)^{sq/2} \, d\xi \right\}^{1/q},$$

where $V_g f$ is the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, defined by

$$V_g f(x,\xi) = \int_{\mathbb{R}^n} f(t)\, \overline{g(t-x)}\, e^{-it\cdot\xi} \, d\xi \qquad \text{for } x, \xi \in \mathbb{R}^n. \tag{5}$$

For these facts and for more details about modulation spaces, see [11], [14, Chapter 11] and [26].

To avoid the situation where $\mathcal{S}(\mathbb{R}^n)$ is not dense in $M^{p,q}_s(\mathbb{R}^n)$ when $p = \infty$ or $q = \infty$, we use the following definition of boundedness of Fourier multiplier operators on modulation spaces: We say that $m(D)$ is bounded from $M^{p,q}_s(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that $\|m(D)f\|_{M^{p,q}} \le C\|f\|_{M^{p,q}_s}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, and we set

$$\|m(D)\|_{\mathcal{L}(M^{p,q}_s, M^{p,q})} = \sup\{\|m(D)f\|_{M^{p,q}} \mid f \in \mathcal{S}(\mathbb{R}^n),\ \|f\|_{M^{p,q}_s} = 1\}.$$

Similarly, for $1 \le p$, $q \le \infty$, we set

$$\|m(D)\|_{\mathcal{L}(L^p, L^q)} = \sup\{\|m(D)f\|_{L^q} \mid f \in \mathcal{S}(\mathbb{R}^n),\ \|f\|_{L^p} = 1\}.$$

We shall simply write $\|m(D)\|_{\mathcal{L}(L^p)}$ for $\|m(D)\|_{\mathcal{L}(L^p, L^p)}$.

We now establish some lemmata which will be used in the sequel.

Lemma 2.1. *There exists a constant c depending only on the dimension n such that*

$$\|\mathcal{F}^{-1} f\|_{L^1} \le c \sum_{|\gamma| \le [n/2]+1} \|\partial^\gamma f\|_{L^2}.$$

Proof. We write $N = [n/2] + 1$. The Cauchy-Schwarz inequality and Plancherel's theorem yield

$$\begin{aligned}\|\mathcal{F}^{-1}f\|_{L^1} &\le \|(1+|x|)^N \mathcal{F}^{-1}f(x)\|_{L^2}\|(1+|x|)^{-N}\|_{L^2}\\ &\asymp \|(1+|x|)^N \mathcal{F}^{-1}f(x)\|_{L^2} \asymp \Big\| \sum_{|\gamma|\le N} |x^\gamma| \mathcal{F}^{-1}f(x)\Big\|_{L^2}\\ &\asymp \sum_{|\gamma|\le N} \|x^\gamma \mathcal{F}^{-1}f(x)\|_{L^2} \asymp \sum_{|\gamma|\le N} \|\partial^\gamma f\|_{L^2}.\end{aligned}$$

□

The proof of Theorem 1.1 relies on the following characterization of the boundedness of Fourier multipliers on weighted modulation spaces.

Lemma 2.2. *Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$, and $m \in \mathcal{S}'(\mathbb{R}^n)$. Then $m(D)$ is bounded from $M^{p,q}_s(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if*

$$\sup_{k\in\mathbb{Z}^n} (1+|k|)^{-s} \|\psi(D-k)m(D)\|_{\mathcal{L}(L^p)} < \infty.$$

Moreover,

$$\|m(D)\|_{\mathcal{L}(M^{p,q}_s, M^{p,q})} \asymp \sup_{k\in\mathbb{Z}^n} (1+|k|)^{-s} \|\psi(D-k)m(D)\|_{\mathcal{L}(L^p)}.$$

Proof. The case $s = 0$ of this theorem is known (see [13, Theorem 17 (1)]). It is also known that

$$(1-\Delta)^{-s/2} = (1+|D|^2)^{-s/2} : M^{p,q} \to M^{p,q}_s$$

is an isomorphism (see [25, Theorem 2.2, Corollary 2.3]). Hence we have

$$\begin{aligned}\|m(D)\|_{\mathcal{L}(M^{p,q}_s, M^{p,q})} &\asymp \|m(D)(1+|D|^2)^{-s/2}\|_{\mathcal{L}(M^{p,q}, M^{p,q})}\\ &\asymp \sup_{k\in\mathbb{Z}} \|\psi(D-k)m(D)(1+|D|^2)^{-s/2}\|_{\mathcal{L}(L^p)}.\end{aligned}$$

Thus, in order to prove the lemma, it is sufficient to prove that

$$\|\psi(D-k)m(D)(1+|D|^2)^{-s/2}\|_{\mathcal{L}(L^p)} \asymp (1+|k|^2)^{-s/2}\|\psi(D-k)m(D)\|_{\mathcal{L}(L^p)}. \tag{6}$$

Take a function $\theta \in \mathcal{S}$ with compact support such that $\theta(\xi) = 1$ on $\operatorname{supp}\psi$. Then

$$\begin{aligned}&\psi(D-k)m(D)(1+|D|^2)^{-s/2}\\ &= \psi(D-k)m(D)(1+|k|^2)^{-s/2}\theta(D-k)(1+|D|^2)^{-s/2}(1+|k|^2)^{s/2}\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\psi(D-k)m(D)(1+|k|^2)^{-s/2}\\ &= \psi(D-k)m(D)(1+|D|^2)^{-s/2}\theta(D-k)(1+|D|^2)^{s/2}(1+|k|^2)^{-s/2}.\end{aligned} \tag{8}$$

Using Lemma 2.1, we easily see that there exists a constant c independent of $k \in \mathbb{Z}^n$ such that

$$\|\mathcal{F}^{-1}[\theta(\xi-k)(1+|\xi|^2)^{\mp s/2}(1+|k|^2)^{\pm s/2}]\|_{L^1} \le c.$$

Hence

$$\begin{aligned}&\|\theta(D-k)(1+|D|^2)^{\mp s/2}(1+|k|^2)^{\pm s/2}\|_{\mathcal{L}(L^p)}\\ &\le \|\mathcal{F}^{-1}[\theta(\xi-k)(1+|\xi|^2)^{\mp s/2}(1+|k|^2)^{\pm s/2}]\|_{L^1} \le c.\end{aligned} \tag{9}$$

Combining (7), (8), and (9), we obtain (6). □

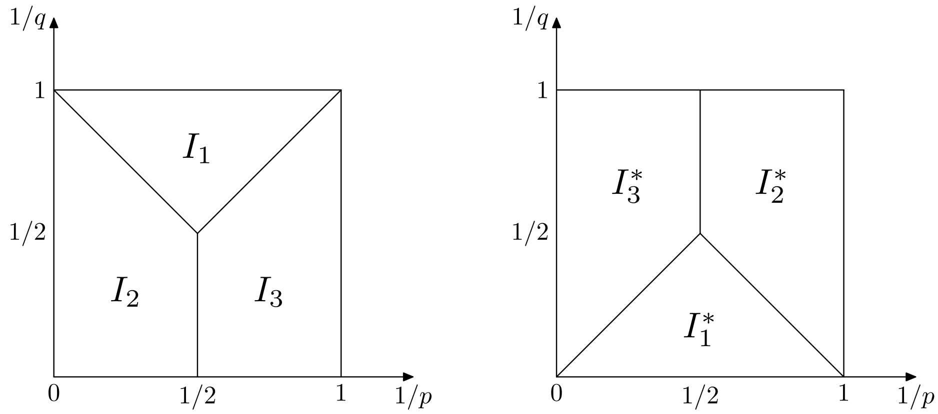

FIGURE 1. The index sets

We now recall the basic complex interpolation result (see, e.g., [12]). Let $\mathcal{M}_s^{p,q}(\mathbb{R}^n)$ be the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ in $M_s^{p,q}(\mathbb{R}^n)$.

Proposition 2.1. *Let* $0 < \theta < 1$, $p_j, q_j \in [1, \infty]$ *and* $s_j \in \mathbb{R}$ *for* $j = 1, 2$. *Set*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1-\theta)s_1 + \theta s_2.$$

Then

$$(\mathcal{M}_{s_1}^{p_1,q_1}(\mathbb{R}^n), \mathcal{M}_{s_2}^{p_2,q_2}(\mathbb{R}^n))_{[\theta]} = \mathcal{M}_s^{p,q}(\mathbb{R}^n).$$

Let us now consider the estimates for the dilation operator on modulation spaces. For $(1/p, 1/q) \in [0,1] \times [0,1]$, we define the subsets

$$\begin{aligned} &I_1 : \ \max(1/p, 1/p') \le 1/q, && I_1^* : \ \min(1/p, 1/p') \ge 1/q, \\ &I_2 : \ \max(1/q, 1/2) \le 1/p', && I_2^* : \ \min(1/q, 1/2) \ge 1/p', \\ &I_3 : \ \max(1/q, 1/2) \le 1/p, && I_3^* : \ \min(1/q, 1/2) \ge 1/p, \end{aligned}$$

as shown in Figure 1.

We introduce the indices

$$\mu_1(p,q) = \begin{cases} -1/p & \text{if} \quad (1/p, 1/q) \in I_1^*, \\ 1/q - 1 & \text{if} \quad (1/p, 1/q) \in I_2^*, \\ -2/p + 1/q & \text{if} \quad (1/p, 1/q) \in I_3^* \end{cases} \tag{10}$$

and

$$\mu_2(p,q) = \begin{cases} -1/p & \text{if} \quad (1/p, 1/q) \in I_1, \\ 1/q - 1 & \text{if} \quad (1/p, 1/q) \in I_2, \\ -2/p + 1/q & \text{if} \quad (1/p, 1/q) \in I_3. \end{cases} \tag{11}$$

Here is the main result about the behaviour of the dilation operator in modulation spaces. Set $U_\lambda f(x) := f(\lambda x)$, $\lambda \neq 0$.

Theorem 2.2 ([24, Theorem 3.1])**.** *Let* $1 \le p, q \le \infty$, *and* $\lambda \neq 0$.

(i) *We have*

$$\|U_\lambda f\|_{M^{p,q}} \le C|\lambda|^{n\mu_1(p,q)} \|f\|_{M^{p,q}}, \qquad \forall\ |\lambda| \ge 1,\ \forall f \in M^{p,q}(\mathbb{R}^n).$$

(ii) *We have*

$$\|U_\lambda f\|_{M^{p,q}} \leq C|\lambda|^{n\mu_2(p,q)}\|f\|_{M^{p,q}}, \qquad \forall\ 0<|\lambda|\leq 1,\ \forall f\in M^{p,q}(\mathbb{R}^n).$$

We also need the following lower bounds for the dilation operator.

Proposition 2.3. *Let $1\leq p,q\leq\infty$, and $\lambda\neq 0$. We have, for some $C>0$,*

$$\|U_\lambda f\|_{M^{p,q}} \geq C|\lambda|^{n\mu_2(p,q)}\|f\|_{M^{p,q}}, \qquad \forall\ |\lambda|\geq 1,\ \forall f\in M^{p,q}(\mathbb{R}^n), \tag{12}$$

and

$$\|U_\lambda f\|_{M^{p,q}} \geq C|\lambda|^{n\mu_1(p,q)}\|f\|_{M^{p,q}}, \qquad \forall\ 0<|\lambda|\leq 1,\ \forall f\in M^{p,q}(\mathbb{R}^n). \tag{13}$$

Proof. The desired estimates follow at once from the ones in Theorem 2.2 applied to $U_{1/\lambda}f$ in place of f. □

3. Sufficient condition for the boundedness of $e^{i\mu(D)}$

The present section is devoted to the proof of Theorem 1.1. We need the following auxiliary results.

Lemma 3.1. *Let m be a bounded function on $\mathbb{R}^n$ with compact support. Suppose that m is of class $C^{[n/2]+1}$ on $\mathbb{R}^n\setminus\{0\}$ and suppose there exists $\epsilon>0$ such that*

$$|\partial^\gamma m(\xi)|\leq C_\gamma|\xi|^{\epsilon-|\gamma|}$$

for $|\gamma|\leq[n/2]+1$. Then $\mathcal{F}^{-1}m\in L^1(\mathbb{R}^n)$.

Proof. Take a $j_0\in\mathbb{Z}$ such that $\operatorname{supp} m\subset\{|\xi|\leq 2^{j_0}\}$. Let $\eta\in\mathcal{S}$ be such that $\operatorname{supp}\eta\subset\{1/2\leq|\xi|\leq 2\}$ and $\sum_{j\in\mathbb{Z}}\eta(\xi/2^j)=1$ for all $\xi\neq 0$. Since $\operatorname{supp}\eta(\cdot/2^j)\subset\{2^{j-1}\leq|\xi|\leq 2^{j+1}\}$, we have

$$m(\xi)=\sum_{j=-\infty}^{j_0}\eta(\xi/2^j)\,m(\xi)=\sum_{j=-\infty}^{j_0}m_j(\xi/2^j),$$

where $m_j(\xi)=\eta(\xi)\,m(2^j\xi)$. Hence,

$$\|\mathcal{F}^{-1}m\|_{L^1}\leq\sum_{j=-\infty}^{j_0}\|2^{jn}(\mathcal{F}^{-1}m_j)(2^j\cdot)\|_{L^1}=\sum_{j=-\infty}^{j_0}\|\mathcal{F}^{-1}m_j\|_{L^1}. \tag{14}$$

Since $\operatorname{supp}\eta\subset\{2^{-1}\leq|\xi|\leq 2\}$, our assumption on the derivatives of m yields

$$\begin{aligned}|\partial^\gamma m_j(\xi)|&=\left|\sum_{\gamma_1+\gamma_2=\gamma}C_{\gamma_1,\gamma_2}(\partial^{\gamma_1}\eta)(\xi)\,2^{j|\gamma_2|}(\partial^{\gamma_2}m)(2^j\xi)\right|\\&\leq\sum_{\gamma_1+\gamma_2=\gamma}C_{\gamma_1,\gamma_2}|(\partial^{\gamma_1}\eta)(\xi)|\,2^{j|\gamma_2|}(C_{\gamma_2}|2^j\xi|^{\epsilon-|\gamma_2|})\leq C_\gamma 2^{j\epsilon}\end{aligned}$$

for all $j\in\mathbb{Z}$ and $|\gamma|\leq[n/2]+1$. Since $\operatorname{supp} m_j\subset\{2^{-1}\leq|\xi|\leq 2\}$, using the above estimate and Lemma 2.1, we see that

$$\|\mathcal{F}^{-1}m_j\|_{L^1}\leq\sum_{|\gamma|\leq[n/2]+1}\|\partial^\gamma m_j\|_{L^2}\leq C2^{j\epsilon}. \tag{15}$$

Now the result follows from (14) and (15). □

The following result is a generalization of [1, Theorem 9].

Lemma 3.2. *Let $\epsilon > 0$. Suppose μ is a real-valued function of class $C^{[n/2]+1}$ on $\mathbb{R}^n \setminus \{0\}$ satisfying*

$$|\partial^\gamma \mu(\xi)| \le C_\gamma |\xi|^{\epsilon - |\gamma|} \tag{16}$$

for $|\gamma| \le [n/2]+1$. Then $\mathcal{F}^{-1}[\eta\, e^{i\mu}] \in L^1(\mathbb{R}^n)$ for each $\eta \in \mathcal{S}(\mathbb{R}^n)$ with compact support.

Proof. Let η be a Schwartz function with compact support. Then by (16) we have

$$|\partial_\xi^\gamma [\eta(\xi)(e^{i\mu(\xi)} - 1)]| \le C_\gamma |\xi|^{\epsilon - |\gamma|}$$

for $|\gamma| \le [n/2]+1$. Hence $\mathcal{F}^{-1}[\eta(e^{i\mu} - 1)] \in L^1$ by Lemma 3.1 and thus

$$\mathcal{F}^{-1}[\eta\, e^{i\mu}] = \mathcal{F}^{-1}[\eta(e^{i\mu} - 1)] + \mathcal{F}^{-1}\eta \in L^1(\mathbb{R}^n).$$

□

Lemma 3.3. *Suppose α and μ satisfy the assumptions of Theorem 1.1. Then there exists a constant C such that*

$$\|\mathcal{F}^{-1}[\psi(\xi - k) e^{i\mu(\xi)}]\|_{L^1} \le C(1 + |k|)^{(\alpha-2)n/2} \tag{17}$$

for all $k \in \mathbb{Z}^n$.

Proof. The estimate (17) for k in a bounded subset of $\mathbb{Z}^n$ readily follows from Lemma 3.2. Thus in the rest of this proof, we assume that $|k|$ is large or, to be precise, that $|k| > 2\sqrt{n}$.

We write

$$\tau_k(\xi) = \mu(\xi + k) - \mu(k) - (\nabla\mu)(k)\cdot\xi, \qquad \phi_k(\xi) = \psi(\xi) e^{i\tau_k(\xi)}.$$

Since the L^1-norm is invariant under translation and modulation, we have

$$\|\mathcal{F}^{-1}[\psi(\xi - k)e^{i\mu(\xi)}]\|_{L^1} = \|\mathcal{F}^{-1}[\psi(\xi)e^{i\mu(\xi+k)}]\|_{L^1} = \|\mathcal{F}^{-1}[\psi(\xi)e^{i\tau_k(\xi)}]\|_{L^1}.$$

Hence the estimate (17) for $|k| > 2\sqrt{n}$ is equivalent to the following:

$$\|\mathcal{F}^{-1}[\psi\, e^{i\tau_k}]\|_{L^1} \le C|k|^{(\alpha-2)n/2}. \tag{18}$$

We shall prove (18). By Taylor's formula, we can write

$$\tau_k(\xi) = 2 \sum_{|\beta|=2} \frac{\xi^\beta}{\beta!} \int_0^1 (1-t)\,(\partial^\beta \mu)(k + t\xi)\,dt. \tag{19}$$

Notice that if $\xi \in (-1,1)^n$ and $0 \le t \le 1$, then $|k + t\xi| \asymp |k|$ and $|(\partial^\nu \mu)(k + t\xi)| \le C_\nu |k|^{\alpha-2}$ for $2 \le |\nu| \le [n/2]+3$. Hence, for $\xi \in (-1,1)^n$ and $|\gamma| \le [n/2]+1$, we use (19) to see that

$$|\partial^\gamma \tau_k(\xi)| = \left| \sum_{|\beta|=2} \sum_{\gamma_1+\gamma_2=\gamma} C_{\beta,\gamma_1,\gamma_2} \partial^{\gamma_1}(\xi^\beta) \int_0^1 (1-t)\, t^{|\gamma_2|}\, (\partial^{\beta+\gamma_2}\mu)(k + t\xi)\,dt \right|$$

$$\le C_\gamma \sum_{|\beta|=2} \sum_{\gamma_1+\gamma_2=\gamma} |\partial^{\gamma_1}(\xi^\beta)| \int_0^1 |k|^{\alpha-2}\,dt \le C_\gamma |k|^{\alpha-2}. \tag{20}$$

This implies that

$$
\begin{aligned}
&|\partial^\gamma(\psi(\xi)\, e^{i\tau_k(\xi)})| \\
&\quad = \left| \sum_{N=0}^{|\gamma|} \sum_{\beta+\nu_1+\cdots+\nu_N=\gamma} C_{\beta,\nu_1,\dots,\nu_N} \left(\partial^\beta \psi(\xi)\right) (\partial^{\nu_1}\tau_k(\xi)) \dots (\partial^{\nu_N}\tau_k(\xi)) e^{i\tau_k(\xi)} \right| \\
&\quad \le C_\gamma \sum_{N=0}^{|\gamma|} \sum_{\beta+\nu_1+\cdots+\nu_N=\gamma} \|\partial^\beta\psi\|_{L^\infty} (C_{\nu_1}|k|^{\alpha-2}) \dots (C_{\nu_N}|k|^{\alpha-2}) \le C_\gamma |k|^{(\alpha-2)|\gamma|}
\end{aligned}
$$

for $|\gamma| \le [n/2]+1$, where we have used the fact that $|k|^{\alpha-2} \ge 1$. Hence

$$
|\partial_\xi^\gamma[\phi_k(\xi/|k|^{(\alpha-2)})]| \le C_\gamma \tag{21}
$$

for $|\gamma| \le [n/2]+1$. Therefore, since $\operatorname{supp} \phi_k(\cdot/|k|^{\alpha-2}) \subset (-|k|^{\alpha-2}, |k|^{\alpha-2})^n$, using Lemma 2.1 and (21) we obtain

$$
\begin{aligned}
&\|\mathcal{F}^{-1}[\psi\, e^{i\tau_k}]\|_{L^1} = \|\mathcal{F}^{-1}[\phi_k]\|_{L^1} = \|\mathcal{F}^{-1}[\phi_k(\cdot/|k|^{\alpha-2})]\|_{L^1} \\
&\le C \sum_{|\gamma|\le[n/2]+1} \|\partial^\gamma[\phi_k(\cdot/|k|^{\alpha-2})]\|_{L^2} \le C|k|^{(\alpha-2)n/2},
\end{aligned}
$$

where $C>0$ is independent of k. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. We write $m(D) = e^{i\mu(D)}$. By virtue of Lemma 2.2, the result follows if we prove the estimate

$$
\|\psi(D-k)m(D)\|_{\mathcal{L}(L^p)} \le c(1+|k|)^{(\alpha-2)n|1/p-1/2|}. \tag{22}
$$

By Lemma 3.3 and Young's inequality, we have

$$
\begin{aligned}
&\|\psi(D-k)m(D)\|_{\mathcal{L}(L^1)} = \|\psi(D-k)m(D)\|_{\mathcal{L}(L^\infty)} \\
&= \|\mathcal{F}^{-1}[\psi(\xi-k)m(\xi)]\|_{L^1} \le c(1+|k|)^{(\alpha-2)n/2}. \qquad (23)
\end{aligned}
$$

By Plancherel's theorem, we have

$$
\|\psi(D-k)m(D)\|_{\mathcal{L}(L^2)} = \|\psi(\xi-k)m(\xi)\|_{L^\infty} = \|\psi\|_{L^\infty} = c. \tag{24}
$$

Now (22) follows from (23) and (24) by interpolation. □

4. Necessary condition for the boundedness of $e^{i\mu(D)}$

In this section we prove Theorem 1.2. We will use the following results.

Lemma 4.1 ([9, 19])**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that g is a C^∞-function on $\mathbb{R}^n$ with $\operatorname{supp} g \subset \Omega$. Suppose that φ is a real-valued C^∞-function on $\mathbb{R}^n$ such that*

(1) *$|\partial^\gamma \varphi| \le A$ on Ω for $|\gamma| \le N$,*
(2) *$|\det \operatorname{Hess} \varphi| \ge 1/A$ on Ω,*

where N is a positive integer determined by the dimension n. Then there exists a constant $C = C(n, g, A)$ such that

$$
\left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\lambda\varphi(\xi))}\, g(\xi)\, d\xi \right| \le C(1+|\lambda|)^{-n/2}
$$

for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$.

Lemma 4.2. *Suppose α and μ satisfy the assumptions of Theorem 1.2. Then there exists a nonempty open cone $\Gamma \subset \mathbb{R}^n$ and positive constants C and B such that*

$$\|\mathcal{F}^{-1}[\psi(\xi)e^{-i\mu(\xi+k)}]\|_{L^\infty} \leq C|k|^{-(\alpha-2)n/2} \tag{25}$$

for $k \in \mathbb{Z}^n \cap \Gamma$ with $|k| > B$.

Proof. We write

$$\tau_k(\xi) = \mu(\xi+k) - \mu(k) - (\nabla\mu)(k)\cdot\xi.$$

Since the L^∞-norm is invariant under translation, we have

$$\begin{aligned}\|\mathcal{F}^{-1}[\psi(\xi)e^{-i\mu(\xi+k)}]\|_{L^\infty} &= \|\mathcal{F}^{-1}[\psi(\xi)e^{-i\tau_k(\xi)}]\|_{L^\infty} \\ &= \sup_{x\in\mathbb{R}^n}\left|\int_{\mathbb{R}^n} e^{i(x\cdot\xi-\tau_k(\xi))}\,\psi(\xi)\,d\xi\right|.\end{aligned}$$

Consider the function $\tau_k(\xi)$ for $\xi \in (-1,1)^n$. Since the second derivatives of μ are homogeneous functions of degree $\alpha-2$, the Hessian matrix of $\tau_k(\xi)$ can be written as

$$\mathrm{Hess}\,[\tau_k(\xi)] = \mathrm{Hess}\,[\mu(\xi+k)] = |\xi+k|^{\alpha-2}(\mathrm{Hess}\,\mu)((\xi+k)/|\xi+k|).$$

Hence the Hessian determinant satisfies

$$\det\mathrm{Hess}\,[\tau_k(\xi)] = |\xi+k|^{n(\alpha-2)}(\det\mathrm{Hess}\,\mu)((\xi+k)/|\xi+k|). \tag{26}$$

By the assumption that $(\det\mathrm{Hess}\,\mu)(\xi_0) \neq 0$ and by the homogeneity of $\det\mathrm{Hess}\,\mu$, we can find a nonempty open cone $\Gamma \subset \mathbb{R}^n$ and sufficiently large positive real numbers A and B such that

$$|(\det\mathrm{Hess}\,\mu)((\xi+k)/|\xi+k|)| \geq 1/A$$

for all $k \in \mathbb{Z}^n \cap \Gamma$ with $|k| > B$ and all $\xi \in (-1,1)^n$. Notice also that $|\xi+k| \asymp |k|$ for $|k|$ large and $\xi \in (-1,1)^n$. Hence by (26) we have

$$|\det\mathrm{Hess}\,[|k|^{-(\alpha-2)}\tau_k(\xi)]| \geq 1/2A \tag{27}$$

for $k \in \mathbb{Z}^n \cap \Gamma$ with $|k| > B$ and for $\xi \in (-1,1)^n$. On the other hand, in the same way as we proved (20), we see that the estimate

$$\left|\partial_\xi^\gamma[|k|^{-(\alpha-2)}\tau_k(\xi)]\right| \leq C \tag{28}$$

holds for $k \in \mathbb{Z}^n$ with $|k| > B$, for $\xi \in (-1,1)^n$, and for $|\gamma| \leq N$, where N is the constant of Lemma 4.1. Recall that $\operatorname{supp}\psi \subset (-1,1)^n$. Therefore, from (27), (28), and Lemma 4.1 with $\Omega = (-1,1)^n$, we obtain

$$\left|\int_{\mathbb{R}^n} e^{i(x\cdot\xi-\tau_k(\xi))}\,\psi(\xi)\,d\xi\right| = \left|\int_{\mathbb{R}^n} e^{i(x\cdot\xi-|k|^{\alpha-2}(\tau_k(\xi)/|k|^{\alpha-2}))}\,\psi(\xi)\,d\xi\right| \leq c|k|^{-(\alpha-2)n/2}$$

for all $k \in \mathbb{Z}^n \cap \Gamma$ with $|k| > B$. □

We can now prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.2 and duality, we have

$$\begin{aligned}\|e^{i\mu(D)}\|_{\mathcal{L}(M_s^{p,q},M^{p,q})} &\asymp \sup_{k\in\mathbb{Z}^n}(1+|k|)^{-s}\|\psi(D-k)e^{i\mu(D)}\|_{\mathcal{L}(L^p)} \\ &= \sup_{k\in\mathbb{Z}^n}(1+|k|)^{-s}\|\psi(D-k)e^{i\mu(D)}\|_{\mathcal{L}(L^{p'})} \asymp \|e^{i\mu(D)}\|_{\mathcal{L}(M_s^{p',q},M^{p',q})},\end{aligned}$$

where $1/p + 1/p' = 1$. Consequently, it is sufficient to consider the case $1 \leq p \leq 2$.

Suppose $1 \le p \le 2$. From Lemma 2.2, it follows that

$$\|\psi(D)e^{i\mu(D+k)}\|_{\mathcal{L}(L^p)} = \|\psi(D-k)e^{i\mu(D)}\|_{\mathcal{L}(L^p)} \le C(1+|k|)^s \tag{29}$$

for all $k \in \mathbb{Z}^n$. Let Γ, C, and B be as in Lemma 4.2. To prove that $s \ge (\alpha - 2)n(1/p - 1/2)$, we shall show the estimate

$$\|\psi(D)e^{-i\mu(D+k)}\|_{\mathcal{L}(L^p,L^{p'})} \le C|k|^{-(\alpha-2)n(1/p-1/2)} \tag{30}$$

for $k \in \mathbb{Z}^n \cap \Gamma$ with $|k| > B$. If this is proved, then the claim follows; in fact, (30) and (29) imply that

$$\begin{aligned}\|\psi^2(D)f\|_{L^{p'}} &= \|\psi(D)e^{-i\mu(D+k)}\psi(D)e^{i\mu(D+k)}f\|_{L^{p'}}\\ &\le C|k|^{-(\alpha-2)n(1/p-1/2)}\|\psi(D)e^{i\mu(D+k)}f\|_{L^p}\\ &\le C|k|^{-(\alpha-2)n(1/p-1/2)+s}\|f\|_{L^p}\end{aligned}$$

for all $f \in \mathcal{S}$ and all $k \in \mathbb{Z}^n \cap \Gamma$ with $|k| > B$, which is possible only when $s \ge (\alpha-2)n(1/p-1/2)$.

Let us prove (30). We have the estimate (25) of Lemma 4.2 for $k \in \mathbb{Z}^n \cap \Gamma$ with $|k| > B$. Consequently, for k in the same region,

$$\|\psi(D)e^{-i\mu(D+k)}\|_{\mathcal{L}(L^1,L^\infty)} = \|\mathcal{F}^{-1}[\psi(\xi)e^{-i\mu(\xi+k)}]\|_{L^\infty} \le c|k|^{-(\alpha-2)n/2}. \tag{31}$$

On the other hand,

$$\|\psi(D)e^{-i\mu(D+k)}\|_{\mathcal{L}(L^2,L^2)} = \|\psi(\xi)\, e^{-i\mu(\xi+k)}\|_{L^\infty} = \|\psi\|_{L^\infty} = c \tag{32}$$

for all $k \in \mathbb{Z}^n$. The estimate (30) now follows from (31) and (32) by interpolation. □

5. An alternative proof of Theorem 1.2

Here we give an alternative proof of Theorem 1.2 based on estimates in [24] for the dilation operator. We need the following formula for the short-time Fourier transform of a character.

Lemma 5.1. *Consider the function $M_\xi 1(x) := e^{ix\cdot\xi}$. Its short-time Fourier transform is given by*

$$V_g(M_\xi 1)(y,\omega) = e^{iy\cdot(\xi-\omega)}\hat{\overline{g}}(\omega-\xi).$$

Proof. This is just an elementary computation from the definition (5). Alternatively, the result also follows from [14, Lemma 3.1.3]. □

Proof of Theorem 1.2. As we already observed, it suffices to prove Theorem 1.2 assuming $1 \le p \le 2$. Moreover, it is sufficient to consider the pairs (p,q) such that $p \le q \le p'$, that is, the shaded triangle T in Figure 2.

Indeed, assume the above and suppose, for contradiction, that $e^{i\mu(D)}$ is bounded from M^{p_0,q_0}_s to M^{p_0,q_0} for some $(1/p_0, 1/q_0)$ outside T such that $1 \le p_0 \le 2$ and $s < (\alpha-2)n\left(\frac{1}{p_0} - \frac{1}{2}\right)$. Then, by interpolating (Proposition 2.1) with the estimate for $(p,q) = (1,2)$ and $s = (\alpha-2)n/2$ (which holds by Theorem 1.1), one would obtain an improved estimate for all points of the segment joining $(1/p_0, 1/q_0)$ and $(1, 1/2)$ inside T, which is not possible.

Therefore, from now on we assume that $p \le q \le p'$, which also implies that $1 \le p \le 2$. By assumption the following estimate holds:

$$\|e^{i\mu(D)}\langle D\rangle^{-s}f\|_{M^{p,q}} \le C\|f\|_{M^{p,q}} \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

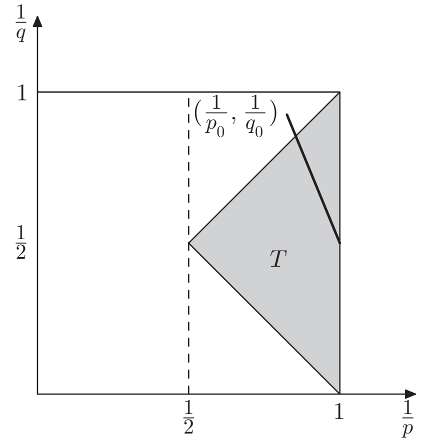

FIGURE 2

We test this estimate on the dilated $U_\lambda f$, $\lambda \geq 1$, where f is a fixed Schwartz function whose Fourier transform is supported in a small neighbourhood $\mathcal{U} \subset \mathbb{R}^n \setminus \{0\}$ of the point ξ_0 in the statement and equals 1 in a neighbourhood of ξ_0. We have $e^{i\mu(D)}\langle D\rangle^{-s}U_\lambda = U_\lambda e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s}$, so that the above estimate becomes

$$\|U_\lambda e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s} f\|_{M^{p,q}} \leq C\|U_\lambda f\|_{M^{p,q}}.$$

As a consequence of the bounds in Theorem 2.2 and Proposition 2.3 we obtain

$$\lambda^{d\mu_2(p,q)}\|e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s} f\|_{M^{p,q}} \leq C\lambda^{n\mu_1(p,q)}\|f\|_{M^{p,q}}. \tag{33}$$

We now show a convenient lower bound for the left-hand side of (33). We see by interchanging integrals and applying Lemma 5.1 that

$$|V_g(e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s} f)(y,\omega)| = (2\pi)^{-n}\Big|\int_{\mathbb{R}^n} e^{iy\cdot\xi + i\lambda^\alpha \mu(\xi)}\hat{\overline{g}}(\omega-\xi)\langle\lambda\xi\rangle^{-s}\hat{f}(\xi)\,d\xi\Big|.$$

Hence a change of variable gives

$$\begin{aligned}(34)\quad &\|V_g(e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-\delta} f)(\cdot,\omega)\|_{L^p}\\ &\qquad = (2\pi)^{-n}\lambda^{\frac{\alpha n}{p}}\left(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{i\lambda^\alpha(y\cdot\xi+\mu(\xi))}\hat{\overline{g}}(\omega-\xi)\langle\lambda\xi\rangle^{-s}\hat{f}(\xi)\,d\xi\Big|^p dy\right)^{\frac{1}{p}}.\end{aligned}$$

Now, let $y_0 = -\nabla\mu(\xi_0)$. Since the Hessian matrix $d^2\mu(\xi_0)$ of μ at ξ_0 is non-degenerate, it follows from the implicit function theorem that there is a neighbourhood $\mathcal{V}$ of y_0 such that, if $\mathcal{U}$ is small enough, the phase $\xi \mapsto y\xi + \mu(\xi)$ has a unique non-degenerate critical point $\xi = \xi(y) \in \mathcal{U}$, for every $y \in \mathcal{V}$. After shrinking $\mathcal{V}$ if necessary, we can suppose that $\hat{f}(\xi(y)) = 1$ for every $y \in \mathcal{V}$. We can also choose a window $g \in \mathcal{S}(\mathbb{R}^n)$, with $\hat{g}(\xi) = 1$ on a large ball, so that $\hat{\overline{g}}(\omega-\xi) = 1$ if $\xi \in \operatorname{supp}\hat{f}$ and, say, $|\omega| \leq 1$.

Hence it follows from the stationary phase theorem (see [23, Proposition 6 and subsequent Note, page 344]) that for $y \in \mathcal{V}$, $|\omega| \le 1$,

$$\begin{aligned}
&|\int_{\mathbb{R}^d} e^{i\lambda^\alpha(y\cdot\xi+\mu(\xi))}\overline{\hat{g}}(\omega-\xi)\langle\lambda\xi\rangle^{-s}\hat{f}(\xi)\,d\xi| \\
&= \lambda^{-s}|\int_{\mathbb{R}^n} e^{i\lambda^\alpha(y\cdot\xi+\mu(\xi))}\lambda^s\langle\lambda\xi\rangle^{-s}\hat{f}(\xi)\,d\xi| \\
&\asymp |\det\left(d^2\mu(\xi(y))\right)|^{-1/2}\langle\lambda\xi(y)\rangle^{-s}\lambda^{-\alpha n/2} + O(\lambda^{-\alpha(n+2)/2-s}),
\end{aligned}$$

where $d^2\mu(\xi(y))$ is the Hessian matrix of μ at the critical point $\xi(y)$ and the constant implicit in the O notation is independent of y and ω. Indeed, here we use the uniform estimates $|\lambda^s\partial^\gamma\langle\lambda\xi\rangle^{-s}| \le C_\gamma$ on the support of $\hat{f}$ ($|\xi| \asymp 1$ there) and the fact that all derivatives of the phase $\xi \mapsto y\cdot\xi+\mu(\xi)$ are uniformly bounded with respect to $y \in \mathcal{V}$. Hence we obtain, for some $C > 0$, that

$$|\int_{\mathbb{R}^n} e^{i\lambda^\alpha(y\cdot\xi+\mu(\xi))}\overline{\hat{g}}(\omega-\xi)\langle\lambda\xi\rangle^{-s}\hat{f}(\xi)\,d\xi| \ge C\lambda^{-\alpha n/2-s} \qquad \text{for } y \in \mathcal{V},\ |\omega| \le 1.$$

By substituting this estimate in (34) we obtain

$$\|V_g(e^{i\mu(\lambda D)}\langle\lambda D\rangle^{-s}f)(\cdot,\omega)\|_{L^p} \ge C\lambda^{\alpha n(1/p-1/2)-s}, \qquad \text{for } |\omega| \le 1.$$

An integration over $|\omega| \le 1$ yields

$$\|e^{i\mu(\lambda D)}\langle\lambda D\rangle^{-s}f\|_{M^{p,q}} \ge C\lambda^{\alpha n(1/p-1/2)-s}.$$

Now, combining this last estimate with (33) and letting $\lambda \to +\infty$ give

$$s \ge \alpha n\,(1/p-1/2) + n(\mu_2(p,q)-\mu_1(p,q)).$$

On the other hand, from (10) and (11) we obtain $\mu_2(p,q)-\mu_1(p,q) = -2\,(1/p-1/2)$ if $p \le q \le p'$, giving the desired threshold for s. □

Appendix A

This appendix is devoted to the proof of the claim in Remark 1.3.

Since the case $n = 1$ is trivial, we assume that $n \ge 2$. In this case μ is of constant sign on $\mathbb{R}^n \setminus \{0\}$. Without loss of generality, we assume that $\mu(\xi) > 0$ for all $\xi \ne 0$. We write

$$\mu(\xi) = |\xi|^\alpha\theta(\xi),$$

where θ is a homogeneous function of degree 0 with $\theta(\xi) > 0$ for $\xi \ne 0$.

The second derivatives of $\mu(\xi)$ are given by

$$\begin{aligned}
\partial_i\partial_j\mu(\xi) =& \alpha|\xi|^{\alpha-2}\delta_{ij}\theta(\xi) + \alpha(\alpha-2)|\xi|^{\alpha-4}\xi_i\xi_j\theta(\xi) \\
&+ (\partial_i|\xi|^\alpha)(\partial_j\theta(\xi)) + (\partial_j|\xi|^\alpha)(\partial_i\theta(\xi)) + |\xi|^\alpha\partial_i\partial_j\theta(\xi).
\end{aligned}$$

There exists a point ξ_0 with $|\xi_0| = 1$ at which the function $\theta(\xi)$ on $\{|\xi| = 1\}$ takes its minimum. Since θ is homogeneous of degree 0, $\theta(\xi_0)$ is the minimum of $\theta(\xi)$ for all $\xi \ne 0$. Hence $(\partial_j\theta)(\xi_0) = 0$ and the symmetric matrix $(\operatorname{Hess}\theta)(\xi_0)$ is nonnegative definite. The Hessian matrix of μ at ξ_0 is given by

$$(\operatorname{Hess}\mu)(\xi_0) = \alpha\theta(\xi_0)E + \alpha(\alpha-2)\theta(\xi_0)\,(\xi_{0,i}\xi_{0,j}) + \theta(\xi_0)(\operatorname{Hess}\theta)(\xi_0),$$

where E is the identity matrix. The first matrix on the right hand side is positive definite since $\alpha\theta(\xi_0) > 0$. The second one is nonnegative definite since so is the matrix $(\xi_{0,i}\xi_{0,j})$ and since $\alpha(\alpha-2)\theta(\xi_0) \ge 0$. The third one is also nonnegative definite since so is $(\text{Hess}\,\theta)(\xi_0)$ and $\theta(\xi_0) > 0$. Hence $(\text{Hess}\,\mu)(\xi_0)$ is positive definite and the determinant is not zero.

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