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SIGNS OF FOURIER COEFFICIENTS OF TWO CUSP FORMS OF DIFFERENT WEIGHTS

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Abstract. We investigate sign changes of Fourier coefficients of cusp forms of different weights.

1. Introduction

Signs of Fourier coefficients or Hecke eigenvalues of cusp forms in one and several variables have been recently studied in various aspects; for a survey see for example [1].

In this note we will investigate the signs of the Fourier coefficients $a(n)$ and $b(n)$ $(n \in \mathbf{N})$ of two non-zero cusp forms f and g of different weights k_1 and k_2 at least 2, respectively, and level N. If $a(n)$ and $b(n)$ are totally real algebraic numbers for all n and $a(1) = b(1) = 1$, then we will show that up to the action of a Galois automorphism, infinitely many of the $a(n)$ have the same sign (respectively opposite sign) as the corresponding $b(n)$. The main ingredients in the proof are the analytic properties of the Rankin-Selberg zeta function attached to f and g, a classical theorem of Landau on Dirichlet series with non-negative coefficients and the "bounded denominators" argument in the theory of modular forms.

As an amusing and rather immediate corollary one obtains that the generating function of the numbers $h(n)a(n)\,(n \geq 1)$ never is a cusp form of any even weight ≥ 2 and any level. Here h is any function on the positive integers that takes rational values, is of polynomial growth and such that $h(1) = 1$, $h(n) > 0$ for $n \gg 1$ and $h(n) \gg n^c$ for some $c > 0$ whenever n is large. (We in fact shall prove a slightly more general statement.) Of course, this statement is believed to be morally true by anyone, but a priori it seems not so clear how to produce a formally correct and simple proof of it.

Notation. For $z \in \mathcal{H}$ the complex upper half-plane, we set $q := e^{2\pi i z}$.

For an integer k and N a natural number we denote by $S_k(N)$ the space of cusp forms of weight k on the group $\Gamma_0(N)$ consisting of matrices in $SL_2(\mathbf{Z})$ with lower left component divisible by N. We will always suppose that $k \geq 2$.

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For $f, g \in S_k(N)$ we let

$$\langle f, g\rangle = \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)}y^{k-2}dxdy \qquad (z = x+iy)$$

be the Petersson scalar product of f and g.

If $f(z) = \sum_{n\geq 1} a(n)q^n \in S_k(N)$ and $\rho \in Gal(\overline{\mathbf{Q}}/\mathbf{Q})$, we set $f^\rho(z) := \sum_{n\geq 1} a(n)^\rho q^n$. Then $f^\rho \in S_k(N)$, as is well known. Indeed, this follows from the fact that $S_k(N)$ has a basis consisting of functions with rational Fourier coefficients [3, Thm. 3.5.2].

If s is a complex number, we denote by σ its real part.

2. Statement of results

Theorem. *Let $f(z) = \sum_{n\geq 1} a(n)q^n \in S_{k_1}(N)$ and $g(z) = \sum_{n\geq 1} b(n)q^n \in S_{k_2}(N)$ with $k_1 \neq k_2$ and suppose that $a(n)$ and $b(n)$ are totally real algebraic numbers for all $n \geq 1$ and f and g are normalized, i.e. $a(1) = b(1) = 1$. Let $\epsilon \in \{\pm 1\}$. Then there exists $\rho \in Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ and a sequence of natural numbers $(n_\nu)_{\nu\in\mathbf{N}}$ such that $a(n_\nu) \neq 0,\ b(n_\nu) \neq 0$ and $sign\ a(n_\nu)^\rho = \epsilon\, sign\ b(n_\nu)^\rho$ for all ν.*

Remark. Note that the assumptions of the Theorem are satisfied if f and g are normalized Hecke eigenforms in the corresponding subspaces of newforms of level N.

Corollary. *Let $f(z) = \sum_{n\geq 1} a(n)q^n \in S_k(N)$ with $a(n)$ totally real for all $n \geq 1$ and $a(1) = 1$. Let $h : \mathbf{N} \to \mathbf{R}$ be a function that is of polynomial growth and takes totally real, totally positive algebraic numbers for $n \gg 1$. Assume further that $h(1) = 1$ and that there exists $\rho_0 \in Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ such that $h(n)^{\rho_0} \gg n^c$ for some $c > 0$ whenever n is large. Then the series*

$$\sum_{n\geq 1} h(n)a(n)q^n$$

is not a cusp form in $S_\ell(M)$, for any $\ell \geq 2$ and any $M \in \mathbf{N}$.

3. Proofs

Proof of Theorem. The space $S_k(N)$ has a basis of functions each of which is obtained (by applying the standard U- and V-operators in Atkin-Lehner theory) from a unique normalized Hecke eigenform that is a newform of exact level a divisor of N. Furthermore, the field obtained from $\mathbf{Q}$ by adjoining the Fourier coefficients of such a Hecke eigenform is a number field, i.e. is of finite degree over $\mathbf{Q}$, and is totally real. If for any $\rho \in Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ we write f^ρ and g^ρ in terms of these corresponding bases, the coefficients in these bases are real algebraic numbers due to our assumption on $a(n)$ and $b(n)$; hence

$$K_{f,g} := \mathbf{Q}(\{a(n), b(n)\}_{n\geq 1})$$

is also a totally real number field.

We let G be the set of embeddings of $K_{f,g}$ over $\mathbf{Q}$ into $\mathbf{R}$.

We will only treat the case $\epsilon = -1$; the other case works in a similar way, mutatis mutandis. Since G is finite it is sufficient to show that there exists a sequence $(n_\nu)_{\nu\in\mathbf{N}}$ in $\mathbf{N}$ and for each n_ν there exists an element $\rho_\nu \in G$ such that $(a(n_\nu)b(n_\nu))^{\rho_\nu} < 0$.

Assume that this would not be true. Then

$$(a(n)b(n))^\rho \geq 0$$

for all n large enough, say for $n \geq n_0$ and all $\rho \in G$.

Let $p_1, \ldots, p_r$ be the different prime numbers less than n_0 and put

$$M := p_1 \ldots p_r.$$

Since $((a(1)b(1))^\rho = 1 > 0$ for all $\rho \in G$, we conclude that

$$(a(n)b(n))^\rho \geq 0$$

for all n with $gcd(n, M) = 1$ and all $\rho \in G$. In particular, if

$$c(n) := tr_{K_{f,g}/\mathbf{Q}}(a(n)b(n)) \qquad (n \in \mathbf{N}),$$

then $c(n) \geq 0$ for all n with $gcd(n, M) = 1$.

Let us denote by f_M^ρ and g_M^ρ the series obtained from f^ρ and g^ρ, respectively, by restricting the summation to those n with $gcd(n, M) = 1$. Then, as is well-known, $f_M^\rho \in S_{k_1}(NM^2)$ and $g_M^\rho \in S_{k_2}(NM^2)$.

Let

$$R_{f_M^\rho, g_M^\rho}(s) := \sum_{n \geq 1, gcd(n,M)=1} a(n)^\rho b(n)^\rho n^{-s} \qquad (\sigma \gg 1)$$

be the Rankin-Selberg Dirichlet series attached to f_M^ρ and g_M^ρ and suppose without loss of generality that $k_1 > k_2$. If we set

$$R^*_{f_M^\rho, g_M^\rho}(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k_2+1)\zeta_{NM^2}(2s-(k_1+k_2)+2)R_{f_M^\rho, g_M^\rho}(s) \quad (\sigma \gg 1),$$

where

$$\zeta_{NM^2}(s) := \prod_{p|NM^2}(1-p^{-s}) \cdot \zeta(s),$$

then, as is well-known, $R^*_{f_M^\rho, g_M^\rho}(s)$ extends to an *entire* function on $\mathbf{C}$. Indeed, by the Rankin-Selberg method one has the integral representation

$$R_{f_M^\rho, g_M^\rho}(s) = \int_{\Gamma_0(A)\backslash\mathcal{H}} f_M^\rho(z)\overline{g_M^\rho(z)}E^*_{k_1-k_2}(z; s-k_1+1)y^{k_1-2}dxdy \qquad (z = x+iy).$$

Here $A := NM^2$ and for k a non-negative integer we have put

$$E_k^*(z; s) := \pi^{-s}\Gamma(s+k)E_k(z; s),$$

where

$$E_k(z; s) := \zeta(2s+k) \sum_{\gamma = \left(\begin{smallmatrix} \cdot & \cdot \\ c & d \end{smallmatrix}\right) \in \Gamma_0(A)_\infty \backslash \Gamma_0(A)} \frac{y^s}{(cz+d)^k |cz+d|^{2s}} \quad (z \in \mathcal{H}; s \in \mathbf{C}, \sigma \gg 1)$$

is the non-holomorphic Eisenstein series of weight $k > 0$ and level A for the cusp $i\infty$ and

$$\Gamma_0(A)_\infty = \left\{ \pm 1 \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbf{Z} \right\}.$$

Note that for $k > 0$ the function $E_k^*(z; s)$ extends to an entire function in s; cf. e.g. [2], Cor. 7.2.11, p. 286, with χ resp. ψ the trivial character modulo 1 resp. modulo A and z replaced by Az in the notation there.

Note that for $k = 0$ the Eisenstein series has a pole at $s = 1$ (loc. cit.), and so in the equal weight case $k_1 = k_2$ the function $R_{f_M^\rho, g_M^\rho}(s)$ has a possible pole at $s = k_1$ of residue essentially equal to $\langle f_M^\rho, g_M^\rho \rangle$. Hence our results probably would have some extension also to the case $k_1 = k_2$ if in addition in this case one assumes some orthogonality conditions for f and g along with all their conjugates.

We put

$$R(s) := \sum_{n\geq 1, gcd(n,M)=1} c(n)n^{-s} \qquad (\sigma \gg 1)$$

and

$$R^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k_2+1)\zeta_{NM^2}(2s-(k_1+k_2)+2)R(s) \qquad (\sigma \gg 1).$$

Then $R^*(s)$ extends holomorphically to $\mathbf{C}$.

Denote the coefficients of the Dirichlet series

$$\zeta_{NM^2}(2s-(k_1+k_2)+2)R(s) \qquad (\sigma \gg 1)$$

by $e(n)\,(n \in \mathbf{N})$. Since the coefficients of $R(s)$ are non-negative, we have $e(n) \geq 0$ for all $n \in \mathbf{N}$. (Note that if we only had $c(n) \geq 0$ for all but a finite number of n, then in general we could not conclude that $e(n) \geq 0$ for all but a finite number of n.)

By a well-known and classical theorem of Landau we now find that

$$\sum_{n\geq 1} e(n)n^{-s} \qquad (\sigma \gg 1) \tag{1}$$

must either have a singularity at the real point of its abscissa of convergence or must converge for all $s \in \mathbf{C}$. Since (1) has holomorphic continuation to $\mathbf{C}$, the first alternative is excluded.

In particular we conclude that

$$e(n) \ll_A n^A \tag{2}$$

for all negative A.

We now invoke the "bounded denominators" argument [3, Thm. 3.5.2]. Since $a(n)$ and $b(n)$ are algebraic for all n, there exist non-zero integers D_1 and D_2 such that $D_1a(n)$ and $D_2b(n)$ are in $\mathcal{O}$, the ring of algebraic integers of $\overline{\mathbf{Q}}$, for all $n \geq 1$. It follows that $(D_1a(n))^\rho, (D_2b(n))^\rho \in \mathcal{O}$; hence $(D_1D_2a(n)b(n))^\rho \in \mathcal{O}$ for all $\rho \in G$ and $n \in \mathbf{N}$. From this we conclude that $D_1D_2c(n) \in \mathbf{Z}$ for all n; hence also $D_1D_2e(n) \in \mathbf{Z}$ for all n.

From (2) (say with $A = -1$) we now find that $e(n) = 0$ for $n \gg 1$. But

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s-k_2+2)\sum_{n\geq 1} e(n)n^{-s} \qquad (\sigma \gg 1)$$

extends to an entire function as we noted above. Since $\Gamma(s)\Gamma(s-k_2+2)$ has poles at $s = 0, -1, -2, \ldots$ we see that the Dirichlet polynomial $\sum_{n\geq 1} e(n)n^{-s}\,(s \in \mathbf{C})$ must vanish at these points, and we obtain a linear system of equations for the $e(n)\,(n \ll 1)$ whose determinant is a Vandermonde determinant, hence non-zero. It follows that $e(n) = 0$; hence $c(n) = 0$ for all $n \geq 1$, which contradicts $c(1) = |G|$.

This concludes the proof of the Theorem. $\square$

Proof of Corollary. We put

$$g(z) := \sum_{n\geq 1} h(n)a(n)q^n \qquad (z \in \mathcal{H})$$

and suppose that $g \in S_\ell(M)$ for some $\ell \geq 2$ and $M \in \mathbf{N}$. First assume that $\ell \neq k$. Then by the Theorem with $\epsilon = -1$ and N replaced by $lcm(N, M)$, there exist $\rho \in Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ and $n \in \mathbf{N}$, n arbitrarily large, such that $a(n) \neq 0$ and

$$sign\ (h(n)a(n))^\rho = -sign\ a(n)^\rho.$$

Since for n large, $h(n)^\rho > 0$, we obtain a contradiction.

Now suppose that $\ell = k$. Since $g \in S_k(M)$, also $g^{\rho_0} \in S_k(M)$, and by Deligne's bound we have

$$(h(n)a(n))^{\rho_0} \ll_\epsilon n^{(k-1)/2+\epsilon} \qquad (\epsilon > 0).$$

Taking $\epsilon = c$ and observing our assumption $h(n)^{\rho_0} \gg n^c$, it follows that

$$a(n)^{\rho_0} \ll n^{(k-1)/2}.$$

On the other hand, by a classical and well-known result of Rankin one has

$$lim\ sup_{n\to\infty} \frac{|a(n)|^{\rho_0}}{n^{(k-1)/2}} = \infty,$$

a contradiction. This proves our claim. $\square$

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