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SYMMETRY OF INTEGRAL EQUATIONS ON BOUNDED DOMAINS

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ABSTRACT. In this paper, we will investigate the symmetry of both domains and solutions of integral equations on bounded domains via the method of moving planes.

1. INTRODUCTION AND THE MAIN RESULT

Let n be a positive integer and $\Omega \subset R^n$ be a bounded domain. We will consider the following integral equations:

$$\begin{cases} u(x) = A \displaystyle\int_\Omega \frac{|u|^p(y)}{|x-y|^{n-\alpha}} dy + B, \\ u = \beta \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where p, α, β, A and B are constants satisfying

$$p, A > 0, \quad \beta, B \geq 0 \ \text{ and } \ 1 < \alpha < n. \tag{1.2}$$

Regarding the right-hand side of (1.1) as an integral operator of u, if $p \leq 1$, then it is easy to see that the operator is bounded from $L^1(\Omega)$ into $L^1(\Omega)$. Actually,

$$\begin{aligned} \int_\Omega \Big| A \int_\Omega \frac{|u|^p(y)}{|x-y|^{n-\alpha}} dy + B \Big| dx &\leq \frac{Ad^\alpha n\omega_n}{\alpha} \int_\Omega |u|^p(y) dy + B|\Omega| \\ &\leq \frac{Ad^\alpha n\omega_n}{\alpha} |\Omega|^{1-p} \int_\Omega |u|(y) dy + B|\Omega|, \end{aligned}$$

where d denotes the diameter of Ω and ω_n the volume of the unit ball in R^n. If $p > 1$, from the Hardy-Littlewood-Sobolev inequality, it follows that the operator is bounded from $L^q(\Omega)$ into $L^q(\Omega)$ for any $q > \max\{\frac{n}{n-\alpha}, \frac{n}{\alpha}(p-1)\}$. Actually, let $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{r}$ and then $1 < r$ and $pr < q$, which are implied by $q > \frac{n}{n-\alpha}$ and $q > \frac{n}{\alpha}(p-1)$, respectively. It follows from the Hardy-Littlewood-Sobolev inequality

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and the Hölder inequality that

$$\int_\Omega \Big| A \int_\Omega \frac{|u|^p(y)}{|x-y|^{n-\alpha}} dy + B \Big|^q dx \le 2^{q-1} \Big\{ A^q \Big(\int_\Omega |u|^{pr}(y) dy \Big)^{\frac{q}{r}} + B^q |\Omega| \Big\}$$

$$\le 2^{q-1} \Big\{ A^q |\Omega|^{1-\frac{pr}{q}} \Big(\int_\Omega |u|^q(y) dy \Big)^p + B^q |\Omega| \Big\}.$$

Therefore, it is natural to assume that

$$u \in L^q(\Omega), \quad \text{where } \begin{cases} q = 1 \text{ if } p \le 1; \\ \\ q > \max\{\frac{n}{n-\alpha}, \frac{n}{\alpha}(p-1)\} \text{ if } p > 1. \end{cases} \tag{1.3}$$

Equation (1.1) arises from minimizing some energy functionals. For example, as $\alpha = 2$, it is the Euler-Lagrange equation for the energy functional of a barotropic self-gravitating gas contained in the domain Ω, where $|u|^{\frac{1}{p}}$ is the density of the gas and β is the pressure outside Ω (cf. [4]). In this case, it is also equivalent to the stationary compressible Navier-Stokes equations of the gas (cf. [5]).

Our purpose in this paper is to apply the method of moving planes to investigate the symmetry not only of solutions u but also of domains Ω of (1.1). The method of moving planes has been extensively used to study the symmetry of solutions of differential equations. Here we only mention the original papers [2] and [3]. Some other important results in this direction are listed in the references of [1]. To the best of the authors' knowledge, [1] is the first approach to study the symmetry of integral equations via the method of moving planes, where Chen, Li and Ou obtained the symmetry of solutions u of (1.1) as $\Omega = R^n$ and $p = \frac{n+\alpha}{n-\alpha}$ and also classified the solutions. As for integral equations on bounded domains, we will obtain the symmetry of both solutions and domains.

Our main result is the following:

Theorem 1.1. *Suppose $\Omega \subset R^n$ is a C^1 bounded domain and (1.1), (1.2) and (1.3) hold. If $u > 0$ in Ω, then Ω is a ball and u is radially symmetric and monotone decreasing about the center of the ball.*

Remark 1.2. (i) From the proof of Theorem 1.1, we will see that Theorem 1.1 still holds if in (1.1) $|u|^p$ is replaced by $|f(u)|$ provided f is positive and increasing on $[0, \infty)$ and there exist positive constants p and C such that

$$\begin{cases} (i) \;\; f(u) \le C(1 + u^p) \;\text{ for any }\; u \ge 0, \;\text{ and} \\ \\ (ii) \;\; f(u) - f(v) \le C(u^p - v^p) \;\text{ for any }\; 0 \le v \le u. \end{cases}$$

(ii) It is clear that (1.1) implies the following differential equations:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = A u^p \;\text{ in }\; \Omega, \\ \\ u = \beta \;\text{ on }\; \partial\Omega; \end{cases} \tag{1.4}$$

however the inverse is not true since (1.4) implies indeed

$$u(x) = A \int_\Omega \frac{|u|^p(y)}{|x-y|^{n-\alpha}} dy + \tilde{B}(x),$$

where $\tilde{B}(x)$ usually cannot be a constant. To regain (1.1) from (1.4), we need some extra conditions; that is, the system should be overdetermined. Serrin [3] studied these kinds of problems and proved that Ω must be a ball. From this point of view, we see that integral equations usually give us more information than differential equations. Ströhmer [5] solved (1.1) with u and Ω as unknowns under the constraint that $\int_\Omega |u|^{\frac{1}{p}}$ equals a given positive number.

(iii) In (1.2), we assume that $1 < \alpha < n$ instead of $0 < \alpha < n$, a more general assumption for α (cf. [1]), since $1 < \alpha$ implies $u \in C^1(\bar{\Omega})$ (cf. Lemma 2.2), which is needed in our proof of Theorem 1.1.

2. A preliminary result

In this section, we show that our assumptions in Theorem 1.1 imply $u \in C^1(\bar{\Omega})$. We first prove a simple lemma.

Lemma 2.1. *Suppose $w \in L^r(\Omega)$ with $1 \le r < \infty$ and*

$$v(x) = A\int_\Omega \frac{|w|(y)}{|x-y|^{n-\alpha}}dy + B. \tag{2.1}$$

Then $v \in W^{1,s}(\Omega)$, where $\frac{1}{s} + \frac{\alpha-1}{n} = \frac{1}{r}$ if $1 \le r < \frac{n}{\alpha-1}$, and s can be any real number greater than 1 if $r \ge \frac{n}{\alpha-1}$.

Proof. Without loss of generality, we assume that $w \in C_0^\infty(\Omega)$. Then we have

$$\partial_{x_i} v(x) = -A(n-\alpha)\int_\Omega \frac{|w|(y)(x_i-y_i)}{|x-y|^{n-\alpha+2}}dy.$$

It follows that

$$|\partial_{x_i} v(x)| \le A(n-\alpha)\int_\Omega \frac{|w|(y)}{|x-y|^{n-\alpha+1}}dy.$$

If $1 \le r < \frac{n}{\alpha-1}$, from the Hardy-Littlewood-Sobolev inequality, we have

$$||Dv||_{L^s(\Omega)} \le A(n-\alpha)C||w||_{L^r(\Omega)}, \tag{2.2}$$

where $\frac{1}{s} + \frac{\alpha-1}{n} = \frac{1}{r}$ and $C > 0$ is a constant depending only on n, s and α. If $r \ge \frac{n}{\alpha-1}$, for any $s > \frac{n}{n-\alpha+1}$, let $\frac{1}{s} + \frac{\alpha-1}{n} = \frac{1}{r'}$ and then $1 < r' < \frac{n}{\alpha-1} \le r$. From the Hardy-Littlewood-Sobolev inequality and the Hölder inequality,

$$||Dv||_{L^s(\Omega)} \le A(n-\alpha)C||w||_{L^{r'}(\Omega)} \le A(n-\alpha)C|\Omega|^{\frac{1}{r'}-\frac{1}{r}}||w||_{L^r(\Omega)}, \tag{2.3}$$

where $C > 0$ is a constant depending only on n, s and α.

In view of (2.1) (recall that d is the diameter of Ω and ω_n is the volume of the unit ball in R^n),

$$||v||_{L^1(\Omega)} \le \frac{Ad^\alpha n\omega_n}{\alpha}||w||_{L^1(\Omega)} + B|\Omega| \le \frac{Ad^\alpha n\omega_n}{\alpha}|\Omega|^{1-\frac{1}{r}}||w||_{L^r(\Omega)} + B|\Omega|.$$

Combining this with (2.2) and (2.3), using Sobolev imbedding (note that Ω is bounded), we obtain the conclusion clearly. □

Lemma 2.2. $u \in C^1(\bar{\Omega})$.

Proof. We first prove $u \in C(\bar{\Omega})$, which is sufficient to show $u \in L^{q'}(\Omega)$ for some q' satisfying $\frac{q'}{p} \ge \frac{n}{\alpha-1}$. Actually, since $u^p \in L^{\frac{q'}{p}}(\Omega)$, from (1.1) and Lemma 2.1 with $s = n+1$, we obtain $u \in W^{1,n+1}(\Omega) \subset C(\bar{\Omega})$. Therefore we only need to consider the case that $\frac{q}{p} < \frac{n}{\alpha-1}$. Let $\frac{1}{s} + \frac{\alpha-1}{n} = \frac{p}{q}$ and from Lemma 2.1, we deduce $u \in W^{1,s}(\Omega)$. Using Sobolev embedding and the standard 'bootstrap' method, we will arrive at $u \in L^{q'}(\Omega)$, where q' satisfies $\frac{q'}{p} \ge \frac{n}{\alpha-1}$.

Now we turn to show $u \in C^1(\bar{\Omega})$. Fix $\eta \in C_0^\infty(R)$ satisfying $0 \le \eta \le 1$ and $\eta(t) = 0$ as $|t| \le 1$, and $\eta(t) = 1$ as $|t| \ge 2$. Define for any $\epsilon > 0$,

$$u_\epsilon = A \int_\Omega \frac{\eta_\epsilon |u|^p(y)}{|x-y|^{n-\alpha}} dy + B, \quad \eta_\epsilon = \eta(|x-y|/\epsilon).$$

From $u \in C(\bar{\Omega})$ and $\alpha > 1$, we deduce that $\int_\Omega \frac{|u|^p(y)(x_i - y_i)}{|x-y|^{n-\alpha+2}} dy$ is integrable and

$$u_\epsilon \to u, \quad \partial_{x_i} u_\epsilon \to -A(n-\alpha) \int_\Omega \frac{|u|^p(y)(x_i - y_i)}{|x-y|^{n-\alpha+2}} dy$$

uniformly in Ω as $\epsilon \to 0$. Hence $u \in C^1(\bar{\Omega})$ and $\partial_{x_i} u = -A(n-\alpha) \int_\Omega \frac{|u|^p(y)(x_i - y_i)}{|x-y|^{n-\alpha+2}} dy$. $\square$

3. Proof of Theorem 1.1

We use the method of moving planes to prove Theorem 1.1. Choose any direction and rotate the coordinate system if it is necessary such that the x_1-axis is parallel to it. For any $\lambda \in R$, set

$$T_\lambda = \{(x_1, x_2, \ldots, x_n) : x_1 = \lambda\}$$

to be the hyperplane vertical to the x_1-axis. For λ being sufficiently large, since Ω is bounded, we have that the hyperplane T_λ is disjoint from $\bar{\Omega}$. Let the hyperplane T_λ move continuously toward Ω, i.e. decrease λ continuously, until it begins to intersect $\bar{\Omega}$. Let λ_0 be such a λ; that is,

$$\lambda_0 = \max\{\lambda : T_\lambda \cap \bar{\Omega} \ne \emptyset\}.$$

From that moment on, at every stage T_λ will cut off from Ω an open cap:

$$\Sigma_\lambda = \{(x_1, x_2, \ldots, x_n) \in \Omega : x_1 > \lambda\}.$$

For any $x = (x_1, x_2, \ldots, x_n)$, set $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ to be the reflection of x about the hyperplane T_λ. Set

$$\Sigma'_\lambda = \{x^\lambda \in \Sigma_\lambda\}$$

to be the reflection of Σ_λ about T_λ. Since Ω is C^1, at the beginning, Σ'_λ will remain in Ω, at least until one of the following occurs:

$$\tag{3.1} \begin{cases} (i)\ \ \Sigma'_\lambda \ \text{ is internally tangent to } \ \partial\Omega \ \text{ at some point } \ \tilde{z} \ \text{ not on } \ T_\lambda; \\ \\ (ii)\ \ T_\lambda \ \text{ is orthogonal to the boundary of } \ \Omega \ \text{ at some point } \ \hat{z}. \end{cases}$$

Denote by λ_1 the first value of λ such that T_λ reaches one of the above positions.

In what follows, we will show that

$$\tag{3.2} \begin{cases} (i)\ \ u(x^\lambda) > u(x) \ \text{ for any } \ \lambda_1 < \lambda < \lambda_0 \ \text{ and } \ x \in \Sigma_\lambda; \\ \\ (ii)\ \ \Sigma_{\lambda_1} \cup (\Omega \cap T_{\lambda_1}) \cup \Sigma'_{\lambda_1} = \Omega; \\ \\ (iii)\ \ u(x^{\lambda_1}) = u(x) \ \text{ for any } \ x \in \Sigma_{\lambda_1}. \end{cases}$$

Since x_1 can be any direction, Theorem 1.1 is a direct consequence of (3.2).

For any $\lambda_1 < \lambda < \lambda_0$ and $x, y \in \Sigma_\lambda$, since $|x-y^\lambda| = |x^\lambda - y|$ and $|x^\lambda - y^\lambda| = |x-y|$, from (1.1), we rewrite u at x and at x^λ as

(3.3)
$$\begin{cases} u(x) = A\int_{\Sigma_\lambda}\Big(\frac{u^p(y)}{|x-y|^{n-\alpha}} + \frac{u^p(y^\lambda)}{|x^\lambda - y|^{n-\alpha}}\Big)dy + A\int_{\Omega_\lambda}\frac{u^p(y)}{|x-y|^{n-\alpha}}dy + C \ \text{ and} \\ u(x^\lambda) = A\int_{\Sigma_\lambda}\Big(\frac{u^p(y)}{|x^\lambda-y|^{n-\alpha}} + \frac{u^p(y^\lambda)}{|x - y|^{n-\alpha}}\Big)dy + A\int_{\Omega_\lambda}\frac{u^p(y)}{|x^\lambda-y|^{n-\alpha}}dy + C \end{cases}$$

respectively, where
$$\Omega_\lambda = \Omega \backslash (\overline{\Sigma_\lambda \cup \Sigma'_\lambda}).$$

We establish (3.2) by the following five lemmas.

Lemma 3.1. *Suppose $\lambda_1 < \lambda < \lambda_0$. If $\lambda_0 - \lambda$ is small enough, then $u(x) \leq u(x^\lambda)$ for any $x \in \Sigma_\lambda$.*

Proof. From (3.3) and $|x^\lambda - y| < |x-y|$ for any $x \in \Sigma_\lambda$ and $y \in \Omega_\lambda$, we deduce
$$u(x) - u(x^\lambda) < A\int_{\Sigma_\lambda}\Big(u^p(y) - u^p(y^\lambda)\Big)\Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}}\Big)dy.$$

Set
$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda : u(x) > u(x^\lambda)\}.$$

From $|x-y| < |x^\lambda - y|$ for any $x \in \Sigma_\lambda$ and $y \in \Sigma_\lambda$, it follows that
$$u(x) - u(x^\lambda) < A\int_{\Sigma_\lambda^-}\Big(u^p(y) - u^p(y^\lambda)\Big)\frac{1}{|x-y|^{n-\alpha}}dy. \tag{3.4}$$

We claim that if $\lambda_0 - \lambda$ is small enough, then (3.4) implies
$$|\Sigma_\lambda^-| = 0. \tag{3.5}$$

Hence $\Sigma_\lambda^- = \emptyset$; that is, $u(x) \leq u(x^\lambda)$ for any $x \in \Sigma_\lambda$.

Actually, if $p \leq 1$, from (3.4), it follows that
$$u(x) - u(x^\lambda) \leq \int_{\Sigma_\lambda^-}\Big(u(y) - u(y^\lambda)\Big)^p\frac{1}{|x-y|^{n-\alpha}}dy.$$

Taking the L^1 norm on Σ_λ^- on both sides, we have
$$||u(x) - u(x^\lambda)||_{L^1(\Sigma_\lambda^-)} \leq A||(u(x) - u(x^\lambda))^p||_{L^1(\Sigma_\lambda^-)}\sup_{y\in\Sigma_\lambda^-}\int_{\Sigma_\lambda^-}\frac{1}{|x-y|^{n-\alpha}}dx.$$

For any $y \in \Sigma_\lambda^-$, by rearrangement,
$$\int_{\Sigma_\lambda^-}\frac{1}{|x-y|^{n-\alpha}}dx \leq n\omega_n\int_0^{d^*} r^{\alpha-1}dr = \frac{n\omega_n(d^*)^\alpha}{\alpha},$$

where $\omega_n(d^*)^n = |\Sigma_\lambda^-|$. Hence
$$||u(x) - u(x^\lambda)||_{L^1(\Sigma_\lambda^-)} \leq \frac{An\omega_n^{1-\frac{\alpha}{n}}}{\alpha}|\Sigma_\lambda^-|^{1-p+\frac{\alpha}{n}}||u(x) - u(x^\lambda)||_{L^1(\Sigma_\lambda^-)}.$$

Let $\lambda_0 - \lambda$ be small enough such that $|\Sigma_\lambda^-|^{1-p+\frac{\alpha}{n}} \leq |\Sigma_\lambda|^{1-p+\frac{\alpha}{n}} \leq \frac{\alpha}{2An\omega_n^{1-\frac{\alpha}{n}}}$. Then
$$||u(x) - u(x^\lambda)||_{L^1(\Sigma_\lambda^-)} \leq \frac{1}{2}||u(x) - u(x^\lambda)||_{L^1(\Sigma_\lambda^-)},$$

which implies (3.5).

If $p > 1$, from (3.4), we have

$$u(x) - u(x^\lambda) < pA \int_{\Sigma_\lambda^-} u^{p-1}(y)\Big(u(y) - u(y^\lambda)\Big)\frac{1}{|x-y|^{n-\alpha}}dy.$$

From the Hardy-Littlewood-Sobolev inequality, it follows that

$$||u(x) - u(x^\lambda)||_{L^r(\Sigma_\lambda^-)} \le C||u^{\frac{n}{\alpha}(p-1)}||^{p-1}_{L^{\frac{n}{\alpha}(p-1)}(\Sigma_\lambda^-)}||u(x) - u(x^\lambda)||_{L^r(\Sigma_\lambda^-)} \tag{3.6}$$

for any $\frac{n}{n-\alpha} < r \le q$, where $C > 0$ is a constant depending only on r, n and α. Let $\lambda_0 - \lambda$ be small enough such that

$$C||u^{\frac{n}{\alpha}(p-1)}||^{p-1}_{L^{\frac{n}{\alpha}(p-1)}(\Sigma_\lambda^-)} \le \frac{1}{2}$$

and then (3.5) follows from (3.6). □

Lemma 3.2. *Suppose $\lambda_1 < \lambda < \lambda_0$ and $u(x) \le u(x^\lambda)$ for any $x \in \Sigma_\lambda$. Then $u(x) < u(x^\lambda)$ for any $x \in \Sigma_\lambda$.*

Proof. From (3.3), it follows that

$$u(x^\lambda) - u(x) = A\int_{\Sigma_\lambda}\Big(u^p(y^\lambda) - u^p(y)\Big)\Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}}\Big)dy$$

$$+A\int_{\Omega_\lambda} u^p(y)\Big(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x-y|^{n-\alpha}}\Big)dy.$$

Since $u(x) \le u(x^\lambda)$ for any $x \in \Sigma_\lambda$ and $\Omega_\lambda \ne \emptyset$ (implied by $\lambda_1 < \lambda < \lambda_0$), we have

$$u(x^\lambda) - u(x) \ge A\int_{\Omega_\lambda} u^p(y)\Big(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x-y|^{n-\alpha}}\Big)dy > 0. \tag{3.7}$$

□

Lemma 3.3. *Suppose $\lambda_1 < \lambda < \lambda_0$ and $u(x) < u(x^\lambda)$ for any $x \in \Sigma_\lambda$. Then there exists $\epsilon > 0$ such that $u(x) \le u(x^{\tilde\lambda})$ for any $x \in \Sigma_{\tilde\lambda}$, where $\tilde\lambda = \lambda - \epsilon > \lambda_1$.*

Proof. For $\delta > 0$ (which will be determined later), let $D \subset \Sigma_\lambda$ be a closed set such that $|\Sigma_\lambda \backslash D| \le \delta$. Let $\epsilon > 0$ be small enough such that

$$\begin{cases} (i)\ \ \tilde\lambda := \lambda - \epsilon > \lambda_1; \\ \\ (ii)\ \ |\Sigma_{\tilde\lambda}\backslash\Sigma_\lambda| \le \delta; \\ \\ (iii)\ \ u(x^{\tilde\lambda}) - u(x) > 0 \text{ for any } x \in D. \end{cases}$$

To achieve (iii), we should notice that $u(x) < u(x^\lambda)$ for any $x \in D$; D is closed and u is continuous. Set

$$\Sigma_{\tilde\lambda}^- = \{x \in \Sigma_{\tilde\lambda} : u(x) > u(x^{\tilde\lambda})\}.$$

Then we see that (3.4) still holds with Σ_λ^- replaced by $\Sigma_{\tilde\lambda}^-$. If $p \le 1$, we choose δ to satisfy $(2\delta)^{1-p+\frac{\alpha}{n}} \le \frac{\alpha}{2An\omega_n^{1-\frac{\alpha}{n}}}$. If $p > 1$, since

$$|\Sigma_{\tilde\lambda}^-| \le |\Sigma_{\tilde\lambda}^-\backslash\Sigma_\lambda| + |\Sigma_\lambda\backslash D| \le 2\delta,$$

we choose δ small enough such that $C||u^{\frac{n}{\alpha}(p-1)}||^{p-1}_{L^{\frac{n}{\alpha}(p-1)}(\Sigma^-_{\tilde{\lambda}})} \le \frac{1}{2}$, where C is the same constant as in (3.6). Then by the same arguments used to show (3.5), we have $\Sigma^-_{\tilde{\lambda}} = \emptyset$. □

Lemma 3.4. *Suppose that as* $\lambda = \lambda_1$ *the first case of* (3.1) *occurs; that is,* Σ'_{λ_1} *is internally tangent to* $\partial\Omega$ *at* $\tilde{z} \notin T_{\lambda_1}$. *If* $u(x) \le u(x^{\lambda_1})$ *for any* $x \in \Sigma_{\lambda_1}$, *then* $\Sigma_{\lambda_1} \cup (\Omega \cap T_{\lambda_1}) \cup \Sigma'_{\lambda_1} = \Omega$.

Proof. Suppose by contradiction that $\Sigma_{\lambda_1} \cup (\Omega \cap T_{\lambda_1}) \cup \Sigma'_{\lambda_1} \ne \Omega$. This is equivalent to $\Omega_{\lambda_1} \ne \emptyset$. Therefore we see that (3.7) still holds with x replaced by $\tilde{z}$, x^λ by $\tilde{z}^{\lambda_1}$ and Ω_λ by Ω_{λ_1} respectively. But $\tilde{z}, \tilde{z}^{\lambda_1} \in \partial\Omega$ and on the boundary u is constant. This is a contradiction. □

Lemma 3.5. *Suppose that as* $\lambda = \lambda_1$ *the second case of* (3.1) *occurs; that is,* T_{λ_1} *is orthogonal to* $\partial\Omega$ *at some point* $\hat{z}$. *If* $u(x) \le u(x^{\lambda_1})$ *for any* $x \in \Sigma_{\lambda_1}$, *then* $\Sigma_{\lambda_1} \cup (\Omega \cap T_{\lambda_1}) \cup \Sigma'_{\lambda_1} = \Omega$.

Proof. From the assumptions, it is easy to see that

$$\partial_{x_1} u(\hat{z}) = 0. \tag{3.8}$$

If $\Sigma_{\lambda_1} \cup (\Omega \cap T_{\lambda_1}) \cup \Sigma'_{\lambda_1} \ne \Omega$ or $\Omega_{\lambda_1} \ne \emptyset$, then there exists a ball $B \subset \Omega_{\lambda_1}$. Let $\{x^m\}_{m=1}^\infty \subset \partial\Sigma_{\lambda_1} \backslash T_{\lambda_1}$ such that $x^m \to \hat{z}$. It follows that $(x^m)^{\lambda_1} \to \hat{z}$. Since B lies on the left of $\hat{z}$, without loss of generality, we assume that B lies on the left of $\{(x^m)^{\lambda_1}\}_{m=1}^\infty$. More precisely, we assume that there exists $\delta > 0$ such that $(x^m)_1^{\lambda_1} - y_1 \ge \delta$ for any $(x^m)^{\lambda_1} = ((x^m)_1^{\lambda_1}, \dots, (x^m)_n^{\lambda_1})$ and $y = (y_1, \dots, y_n) \in B$.

For any $y \in B$, let $\bar{x}^m$ be on the segment from $(x^m)^{\lambda_1}$ to x^m such that

$$\frac{1}{|(x^m)^{\lambda_1} - y|^{n-\alpha}} - \frac{1}{|x^m - y|^{n-\alpha}} = -(n-\alpha)\frac{(\bar{x}^m - y)((x^m)^{\lambda_1} - x^m)}{|\bar{x}^m - y|^{n-\alpha+2}}.$$

Since $(x^m)_1^{\lambda_1} \le \bar{x}_1^m \le x_1^m$, we have

$$(\bar{x}^m - y)(x^m - (x^m)^{\lambda_1}) = (\bar{x}_1^m - y_1)(x_1^m - (x^m)_1^{\lambda_1}) \ge \delta(x_1^m - (x^m)_1^{\lambda_1}) = \delta|x^m - (x^m)^{\lambda_1}|.$$

It is easy to see that the first inequality of (3.7) still holds with x replaced by x^m, x^λ by $(x^m)^{\lambda_1}$ and Ω_λ by Ω_{λ_1} respectively for any m. Combining this with the above two inequalities, we have

$$\begin{aligned} u((x^m)^{\lambda_1}) - u(x^m) &\ge A\int_{\Omega_{\lambda_1}} u^p(y)\Big(\frac{1}{|(x^m)^{\lambda_1} - y|^{n-\alpha}} - \frac{1}{|x^m - y|^{n-\alpha}}\Big)dy \\ &\ge A(n-\alpha)\delta\int_B u^p(y)\frac{|(x^m)^{\lambda_1} - x^m|}{|\bar{x}^m - y|^{n-\alpha+2}}dy; \end{aligned}$$

that is,

$$\liminf_{m\to\infty} \frac{u((x^m)^{\lambda_1}) - u(x^m)}{|(x^m)^{\lambda_1} - x^m|} > 0.$$

This contradicts (3.8). □

Proof of (3.2). From Lemma 3.1, we see that

$$\{\lambda : u(x) \le u(x^\lambda) \ \text{ for any } \ x \in \Sigma_\lambda \ \text{ and } \ \Sigma'_\lambda \subset \Omega\} \ne \emptyset.$$

Let

$$\hat{\lambda} = \inf\{\lambda : u(x) \le u(x^\lambda) \ \text{ for any } \ x \in \Sigma_\lambda \ \text{ and } \ \Sigma'_\lambda \subset \Omega\}.$$

If $\lambda_1 < \hat{\lambda}$, then from Lemma 3.2, we have $u(x) < u(x^{\hat{\lambda}})$ for any $x \in \Sigma_{\hat{\lambda}}$; and then by Lemma 3.3, we have a contradiction since $\hat{\lambda}$ can be decreased a little bit. Therefore $\lambda_1 = \hat{\lambda}$ (by Lemmas 3.4 and 3.5, $\hat{\lambda} \not< \lambda_1$).

(3.2) (i) is a direct consequence of Lemma 3.2. By Lemmas 3.4 and 3.5, we have (3.2) (ii). Choosing $-x_1$ as the x_1-axis, we obtain (3.2) (iii). □

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