A 2 × 2 LATTICE SPACE-TIME CODE OF THE HIGHEST RANK

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(Communicated by Wen-Ching Winnie Li)

ABSTRACT. For all previous constructions of 2 × 2 lattice space-time codes with a positive diversity product, the rank \( r(2) \) was such that \( r(2) \leq 4 \). An example of a 2 × 2 lattice space-time code of rank 5 with a positive diversity product was given by Xing and Li. In this paper, we give an example of a 2 × 2 lattice space-time code of rank 8 with a positive diversity product. This gives an answer to the open problem set by Xing and Li.

1. INTRODUCTION

Let \( \mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N} \) be the sets of all complex numbers, real numbers, integers and positive integers respectively. We denote by \( M_n(\mathbb{C}) \) the set of \( n \times n \) matrices over \( \mathbb{C} \). A lattice space-time code over \( \mathbb{C} \) is a set \( \mathcal{A} \) consisting of matrices in \( M_n(\mathbb{C}) \) such that \( \mathcal{A} \) is a free abelian group under the matrix addition. The rank of this group is called the dimension or rank of \( \mathcal{A} \). In recent years, the study of space-time codes has generated much interest. Many mathematical areas such as number theory, algebra, combinatorics, etc., have been employed to construct good lattice codes (1, 2, 5, 6, 11, 9, 12, 13, 14).

The diversity product of \( \mathcal{A} \) is defined by (see 14)

\[
\delta(\mathcal{A}) := \inf\{|\det(A - B)| : A, B \in \mathcal{A}, A \neq B\}.
\]

The normalized diversity product of \( \mathcal{A} \) in \( M_n(\mathbb{C}) \) is defined by (see 10, 8, 5)

\[
d_g = \frac{(\delta(A))^2}{|\det G| \cdot |\mathcal{L}|^{n/2}} = \frac{(\delta(A))^2}{\sqrt{|\det g|}},
\]

where \( G \) is the corresponding generating matrix of the complex lattice \( \mathcal{A} \), \( g \) is the corresponding real generating matrix for \( \Lambda_G \), and \( |\mathcal{L}| \) denotes the absolute value of the determinant of the 2 × 2 generating matrix of the two-dimensional real base lattice \( \mathcal{L} \).

The criteria for lattice space-time codes are: the rank of \( \mathcal{A} \) should be as large as possible, the diversity product of \( \mathcal{A} \) should be as large as possible, and the discriminant of \( \mathcal{A} \) should be as small as possible. A natural question is: what is the maximal rank of a lattice space-time code \( \mathcal{A} \) such that \( \delta(\mathcal{A}) > 0 \)?

We define \( r(n) \) by (see 14)

\[
r(n) := \max\{\text{rank}(\mathcal{A}) : \mathcal{A} \text{ is a lattice in } M_n(\mathbb{C}), \delta(\mathcal{A}) > 0\}.
\]
Determining the exact value for \( r(n) \) seems difficult. One obvious lower bound is \( r(n) \geq 2n \) (see [1], [2], [7], [8], [9]). Recently, Xing and Li [14] gave an upper bound \( r(n) \leq 2n^2 \).

The case \( n = 2 \) is of a particular interest. However, even in this case, we only know that \( 4 \leq r(n) \leq 8 \). No \( 2 \times 2 \) lattice space-time codes of rank greater than 4 with a positive diversity product had been constructed until in [14], Xing and Li gave an example of a \( 2 \times 2 \) lattice space-time code of rank 5 with a positive diversity product and raised the following question.

**Open problem.** *Is there a \( 2 \times 2 \) lattice space-time code of rank greater than 5 with a positive diversity product?*

In this paper, we give an example of a \( 2 \times 2 \) lattice space-time code of rank 8 with a positive diversity product (Section 2). Moreover, we prove that if \( \mathcal{A} \) is the lattice generated by \( A_j \ (1 \leq j \leq 8) \) and if the set \( \{ A_j : (1 \leq j \leq 8) \} \) is linearly independent over \( \mathbb{R} \), then \( \delta(\mathcal{A}) \leq a \), where \( a \) is a constant, only dependent on \( A_j \ (1 \leq j \leq 8) \). See Section 3.

2. A \( 2 \times 2 \) lattice space-time code of rank 8

**Theorem 2.1.** Consider the eight \( 2 \times 2 \) matrices over \( \mathbb{C} \): 

\[
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
A_5 = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & -\sqrt{q} \end{pmatrix}, \quad A_6 = \begin{pmatrix} \sqrt{q}i & 0 \\ 0 & -\sqrt{q}i \end{pmatrix}, \quad A_7 = \begin{pmatrix} 0 & \sqrt{q} \\ \sqrt{q} & 0 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 0 & \sqrt{q}i \\ -\sqrt{q}i & 0 \end{pmatrix},
\]

where \( q \) is a positive integer such that \( q \equiv 7 \mod 8 \). Let \( \mathcal{A} \) be the lattice generated by \( A_1, A_2, A_3, A_4, A_5, A_6, A_7 \) and \( A_8 \). Then the rank of \( \mathcal{A} \) is 8 and the diversity product of \( \mathcal{A} \) is 1.

**Proof.** First of all, it is clear that \( \{ A_1, A_2, \ldots, A_8 \} \) is linearly independent over \( \mathbb{R} \). Let 

\[
A = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 + y_1A_5 + y_2A_6 + y_3A_7 + y_4A_8
\]

(2.1) 

\[
\begin{pmatrix}
  x_1 + y_1 \sqrt{q} + (-x_2 + y_2 \sqrt{q})i \\
  x_3 + y_3 \sqrt{q} + (-x_4 + y_4 \sqrt{q})i \\
  -x_3 + y_3 \sqrt{q} + (-x_4 - y_4 \sqrt{q})i \\
  x_1 - y_1 \sqrt{q} + (x_2 + y_2 \sqrt{q})i
\end{pmatrix}
\]

To show that the rank of \( \mathcal{A} \) is 8 and the diversity product of \( \mathcal{A} \) is at least 1, it is sufficient to prove that for any eight integers \( x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \) with \( (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \neq (0, 0, 0, 0, 0, 0, 0, 0) \), we have \( |\det(A)| \geq 1 \). Without loss of generality, we assume that \( \gcd(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = 1 \).

From (2.1), we get 

\[
\det(A) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - qy_1^2 - qy_2^2 - qy_3^2 - qy_4^2 + 2(x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3)\sqrt{q},
\]

and so 

\[
|\det(A)|^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2 - qy_1^2 - qy_2^2 - qy_3^2 - qy_4^2)^2 + 4q(x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3)^2.
\]
If one of the two terms $x_1^2 + x_2^2 + x_3^2 + x_4^2 = q_1^2 - q_2^2 - q_3^2 - q_4^2$ and $x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3$ is not zero, then $|\det(A)| \geq \min\{1, 2, \sqrt{q_1}\} = 1$. Therefore we assume that
\begin{equation}
(2.2) \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = q_1^2 - q_2^2 - q_3^2 - q_4^2 = 0
\end{equation}

and
\begin{equation}
(2.3) \quad x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3 = 0.
\end{equation}

Now, equations (2.2) and (2.3) imply that
\begin{equation}
(2.4) \quad (x_1 + y_2)^2 + (x_2 + y_1)^2 + (x_3 + y_4)^2 + (x_4 + y_3)^2 = (1 + q)(y_1^2 + y_2^2 + y_3^2 + y_4^2).
\end{equation}

It follows that
\begin{equation}
(2.5) \quad (x_1 + y_2)^2 + (x_2 + y_1)^2 + (x_3 + y_4)^2 + (x_4 + y_3)^2 \equiv 0 \pmod{8}.
\end{equation}

Since the square of an integer is always 0, 1 or 4 (mod 8), then
\begin{equation}
(2.6) \quad 2 \mid x_1 + y_2, 2 \mid x_2 + y_1, 2 \mid x_3 + y_4, 2 \mid x_4 + y_3.
\end{equation}

So $x_1$ and $y_2$, $x_2$ and $y_1$, $x_3$ and $y_4$, $x_4$ and $y_3$ are of the same parity. On the other hand, from (2.3), there are three possibilities:

(i) all $x_1 y_2, x_2 y_1, x_3 y_4$, and $x_4 y_3$ are even;
(ii) two of $x_1 y_2, x_2 y_1, x_3 y_4, x_4 y_3$ are odd and the other two are even;
(iii) all $x_1 y_2, x_2 y_1, x_3 y_4$, and $x_4 y_3$ are odd.

Case (i) We suppose that all $x_1 y_2, x_2 y_1, x_3 y_4$, and $x_4 y_3$ are even. As $2 \mid x_1 + y_2$ and $2 \mid x_2 y_1$, then we have $2 \mid x_1$ and $2 \mid y_2$. By the same argument, the other 6 variables are all even. Therefore, with $\gcd(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = 1$, we obtain a contradiction.

Case (ii) Now we suppose that two of $x_1 y_2, x_2 y_1, x_3 y_4, x_4 y_3$ are odd and the other two are even. Without loss of generality, we assume
\begin{equation}
2 \mid x_1 y_2, 2 \nmid x_2 y_1 \quad \text{and} \quad 2 \mid x_3 y_4, 2 \nmid x_4 y_3.
\end{equation}

Then $x_1, x_2, y_1, y_2$ are odd, and $x_3, x_4, y_3, y_4$ are even. From (2.3), we deduce the following congruence:
\begin{equation}
x_1 y_2 + x_2 y_1 \equiv -(x_3 y_4 + x_4 y_3) \equiv 0 \pmod{4}.
\end{equation}

This implies that, by considerations modulo 4, one of $x_1 y_2, x_2 y_1$ is 1 and the other is 3, say $x_1 y_2 \equiv 1 \pmod{4}$ and $x_2 y_1 \equiv 3 \pmod{4}$. It follows that $x_1 + y_2 \equiv 2 \pmod{4}$ and $x_2 + y_1 \equiv 0 \pmod{4}$. Hence, by (2.1) and $2 | y_1^2 + y_2^2 + y_3^2 + y_4^2$ we have
\begin{equation}
4 + (x_3 + y_4)^2 + (x_4 + y_3)^2 \equiv 0 \pmod{16}.
\end{equation}

This and the fact that $(x_3 + y_4)^2, (x_4 + y_3)^2 \equiv 0$ or 4 (mod 16) lead to a contradiction.

Case (iii) Finally, we take all $x_1 y_2, x_2 y_1, x_3 y_4$, and $x_4 y_3$ to be odd. Since the $x_i$ and $y_i$ are all odd, then equation (2.1) gives us
\begin{equation}
(x_1 + y_2)^2 + (x_2 + y_1)^2 + (x_3 + y_4)^2 + (x_4 + y_3)^2 \equiv 0 \pmod{32}.
\end{equation}

So we obtain
\begin{equation}
\left(\frac{x_1 + y_2}{2}\right)^2 + \left(\frac{x_2 + y_1}{2}\right)^2 + \left(\frac{x_3 + y_4}{2}\right)^2 + \left(\frac{x_4 + y_3}{2}\right)^2 \equiv 0 \pmod{8}.
\end{equation}

This implies that $\frac{x_1 + y_2}{2}, \frac{x_2 + y_1}{2}, \frac{x_3 + y_4}{2}$, and $\frac{x_4 + y_3}{2}$ are both even.
On the other hand, from equations (2.2) and (2.3) we have
\[(x_1 - y_2)^2 + (x_2 - y_1)^2 + (x_3 - y_4)^2 + (x_4 - y_3)^2 = (q + 1)(y_1^2 + y_2^2 + y_3^2 + y_4^2)\].
It follows that
\[\left(\frac{x_1 - y_2}{2}\right)^2 + \left(\frac{x_2 - y_1}{2}\right)^2 + \left(\frac{x_3 - y_4}{2}\right)^2 + \left(\frac{x_4 - y_3}{2}\right)^2 \equiv 0 \pmod{8}.\]
Therefore, \(x_1 - y_2, x_2 - y_1, x_3 - y_4\), and \(x_4 - y_3\) are also even. Thus we have
\[2 \mid \frac{x_1 + y_2}{2} + \frac{x_1 - y_2}{2} = x_1.\]
But \(x_1\) is an odd integer. This is impossible.

Combining the above three cases, we have shown that equations (2.2) and (2.3) cannot simultaneously hold. Thus we have \(|\det(A)| \geq 1\). This completes the proof. \(\square\)

We have the following remarks.

Remark 2.2. (1) The condition \(q \equiv 7 \pmod{8}\) can be replaced by \(q = 2^{2r}(8t + 7)\), for nonnegative integers \(r\) and \(t\). To see this, if instead of \(q = 8t + 7\) we consider \(2^{2r}(8t + 7)\), then we only need to take \(2^r y_j (j = 1, 2, 3, 4)\) instead of \(y_j\) in our proof.

On the other hand, if \(q\) doesn’t have the form \(2^{2r}(8t + 7)\), then Theorem 2.1 is not true. In this case, we set \(x_1 = x_2 = x_3 = y_3 = 0\) and \(x_4 = q\). Thus equation (2.3) is satisfied and equation (2.2) becomes \(q^2 = q(y_1^2 + y_2^2 + y_4^2)\), and so we obtain
\[q = y_1^2 + y_2^2 + y_4^2.\]
The above equation has integer solutions \((y_1, y_2, y_4)\). One can see \[5\], page 271, or \[4\], page 133.

(2) From \(10\), \[5\] and using equation (2.2), we can compute the discriminant and the normalized diversity product of the lattice \(A\) given by Theorem 1.

Letting \(z_1 = x_1 + y_2\sqrt{q}i, z_2 = x_2 + y_1\sqrt{q}i, z_3 = x_3 + y_4\sqrt{q}i, z_4 = x_4 - y_3\sqrt{q}i, z_1, z_2, z_3, z_4 \in \mathbb{Z}[\sqrt{q}i]\), from (2.1), we get
\[A = \begin{pmatrix} z_1 - i z_2 & z_3 - i z_4 \\ -z_3 - i z_4 & z_1 + i z_2 \end{pmatrix}.\]

The generating matrix \(G\) of the lattice \(A\) is \(G = \begin{pmatrix} G_1 & 0 \\ 0 & -G_2 \end{pmatrix}\), where
\[G_1 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, G_2 = \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}, \text{ and } |\mathcal{L}| = \sqrt{q}.\]
Then the discriminant of \(A\) is \(d_A = -4\), and the normalized diversity product of \(A\) is \(d_g = \frac{1}{\sqrt{q}}\).

3. An upper bound of any \(2 \times 2\) lattice space-time code

In this section, we will use a result in Diophantine approximation to get a relative upper bound of \(\delta(A)\).
Lemma 3.1. Let $X \in \mathbb{N}$, $m, n \in \mathbb{N}$, and $n > m \geq 1$. Let $L_i(\bar{x}) = \sum_{j=1}^{n} a_{ij} x_j$, where $\bar{x} = (x_1, \ldots, x_n)$, $a_{ij} \in \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$. If
\[
\sum_{j=1}^{n} |a_{ij}| \leq A_i, i = 1, \ldots, m,
\]
then there exists a nonvanishing vector $\bar{x}^{(0)} = (x_{01}, \ldots, x_{0n}) \in \mathbb{Z}^n$, satisfying
\[
|L_i(\bar{x}^{(0)})| < A_i X^{1-\frac{8}{m}}, i = 1, \ldots, m,
\]
and
\[
|x_{0j}| \leq X, j = 1, \ldots, n.
\]

Proof. See page 62 of [15]. \hfill \Box

Now we prove the following theorem.

Theorem 3.2. For any eight $2 \times 2$ matrices over $\mathbb{C}$:
\[
A_j = \begin{pmatrix} a_{j1} + a_{j2} & a_{j3} + a_{j4} \\ a_{j5} + a_{j6} & a_{j7} + a_{j8} \end{pmatrix},
\]
where $a_{j1}, a_{j2}, a_{j3}, a_{j4}, a_{j5}, a_{j6}, a_{j7}, a_{j8} \in \mathbb{R}$, and $\{ A_j (1 \leq j \leq 8) \}$ is linearly independent over $\mathbb{R}$. Let $A$ be the lattice generated by $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ and $A_8$. Then
\[
\delta(A) \leq \sqrt{(a_1^2 + a_2^2)(a_3^2 + a_4^2) + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)}},
\]
where $a_k = \sum_{j=1}^{8} |a_{jk}|$, $1 \leq k \leq 8$.

Proof. We assume that $x_j \in \mathbb{Z} (1 \leq j \leq 8)$ and let $A = \sum_{j=1}^{8} x_j A_j$. Then
\[
A = \begin{pmatrix} \sum_{j=1}^{8} x_j a_{j1} + i \sum_{j=1}^{8} x_j a_{j2} & \sum_{j=1}^{8} x_j a_{j3} + i \sum_{j=1}^{8} x_j a_{j4} \\ \sum_{j=1}^{8} x_j a_{j5} + i \sum_{j=1}^{8} x_j a_{j6} & \sum_{j=1}^{8} x_j a_{j7} + i \sum_{j=1}^{8} x_j a_{j8} \end{pmatrix}.
\]
From Lemma 3.1 there exists nonvanishing vector $\bar{x}^{(0)} = (x_{01}, \ldots, x_{08}) \in \mathbb{Z}^8$, satisfying
\[
(3.1) \quad \left| \sum_{j=1}^{8} x_{0j} a_{jk} \right| < a_k X^{1-\frac{8}{m}} = \frac{a_k}{X}, k = 1, \ldots, 4,
\]
and
\[
|x_{0j}| \leq X, j = 1, \ldots, 8,
\]
where $a_k = \sum_{j=1}^{n} |a_{jk}|, X \in \mathbb{N}$. Thus from (3.1),
\[
\delta(A) \leq \left| \det \begin{pmatrix} \frac{1}{a_1 + a_2} & \frac{1}{a_3 + a_4} \\ \frac{-(a_5 + ia_6)X}{a_7 + ia_8} & \frac{-(a_7 + ia_8)X}{a_1 + a_2} \end{pmatrix} \right| = \left| \det \begin{pmatrix} a_1 + ia_2 & a_3 + ia_4 \\ -(a_5 + ia_6) & a_7 + ia_8 \end{pmatrix} \right|
\]
\[
\leq \sqrt{(a_1^2 + a_2^2)(a_3^2 + a_4^2) + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)}}.
\]
This completes the proof of Theorem 3.2. \hfill \Box

Remark 3.3. (1) From Theorem 3.2 and the work done in [11], [5], we have
\[
d_g \leq \frac{\sqrt{(a_1^2 + a_2^2)(a_3^2 + a_4^2) + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)^2}}}{\sqrt{|\det g|}}
\]
where $g$ is the corresponding real generating matrix for $\Lambda_g$. 

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Recently, Liao, Wang, and Xia \cite{Liao2005} constructed a full-rate space-time code based on the optimal quadratic extension on $\mathbb{Q}[i]$ as
\[ C_{2,2} = \left\{ \left( \begin{array}{c} z_1 + \exp\left(\frac{1}{6}i\pi\right)z_2 \\ \rho(z_3 + \exp\left(\frac{1}{6}i\pi\right)z_4) \end{array} \right), \right. \]
where $z_1, z_2, z_3, z_4 \in \mathbb{Z}[i]$ and $\rho = \sqrt{1+i}$. They also constructed a full-rate space-time code based on the optimal quadratic extension on $\mathbb{Q}[\zeta_6]$ as
\[ C_{2,3} = \left\{ \left( \begin{array}{c} z_1 + \theta_1 z_2 \\ \rho(z_3 + \theta_1 z_4) \end{array} \right), \right. \]
where $\zeta_6 = \exp\left(\frac{2\pi i}{3}\right)$, $z_1, z_2, z_3, z_4 \in \mathbb{Z}[\zeta_6]$, $\theta_1$ and $\theta_2$ are the roots of $x^2 + x + 1$, and $\rho = \sqrt{1+\zeta_6}$. The normalized diversity product for code $C_{2,2}$ is $d_g = \frac{1}{3\sqrt{2}} \approx 0.2357$ and the normalized diversity product for code $C_{2,3}$ is $d_g = \frac{1}{4\sqrt{3}} \approx 0.2135$. These normalized diversity products are both larger than ours obtained in Remark 2.2 (2). In fact, Theorem 3.2 gives a relative upper bound of $d_g$.

4. Acknowledgements

The authors thank Prof. Xiang-Gen Xia, Department of Electrical and Computer Engineering, University of Delaware, for pointing out to them his papers that help to define the normalized diversity product. They are also grateful to the anonymous referee for insightful and valuable comments that helped to improve the manuscript. The first and second authors were supported by the Natural Science Foundation of the Science and Technology Department of Sichuan Province and the Education Department of Sichuan Province. The third author was supported in part by Purdue University North Central.

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