

A 2×2 LATTICE SPACE-TIME CODE OF THE HIGHEST RANK

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ABSTRACT. For all previous constructions of 2×2 lattice space-time codes with a positive diversity product, the rank $r(2)$ was such that $r(2) \leq 4$. An example of a 2×2 lattice space-time code of rank 5 with a positive diversity product was given by Xing and Li. In this paper, we give an example of a 2×2 lattice space-time code of rank 8 with a positive diversity product. This gives an answer to the open problem set by Xing and Li.

1. INTRODUCTION

Let \mathbb{C} , \mathbb{R} , \mathbb{Z} , \mathbb{N} be the sets of all complex numbers, real numbers, integers and positive integers respectively. We denote by $\mathcal{M}_n(\mathbb{C})$ the set of $n \times n$ matrices over \mathbb{C} . A lattice space-time code over \mathbb{C} is a set \mathcal{A} consisting of matrices in $\mathcal{M}_n(\mathbb{C})$ such that \mathcal{A} is a free abelian group under the matrix addition. The rank of this group is called the dimension or rank of \mathcal{A} . In recent years, the study of space-time codes has generated much interest. Many mathematical areas such as number theory, algebra, combinatorics, etc., have been employed to construct good lattice codes ([1], [2], [5], [6], [7], [8], [11], [9], [12], [13], [14]).

The diversity product of \mathcal{A} is defined by (see [14])

$$(1.1) \quad \delta(\mathcal{A}) := \inf\{|\det(A - B)| : A, B \in \mathcal{A}, A \neq B\}.$$

The normalized diversity product of \mathcal{A} in $\mathcal{M}_n(\mathbb{C})$ is defined by (see [10], [8], [5])

$$(1.2) \quad d_g = \frac{(\delta(\mathcal{A}))^2}{|\det G| \cdot |\mathcal{L}|^{n/2}} = \frac{(\delta(\mathcal{A}))^2}{\sqrt{|\det g|}},$$

where G is the corresponding generating matrix of the complex lattice \mathcal{A} , g is the corresponding real generating matrix for Λ_G , and $|\mathcal{L}|$ denotes the absolute value of the determinant of the 2×2 generating matrix of the two-dimensional real base lattice \mathcal{L} .

The criteria for lattice space-time codes are: the rank of \mathcal{A} should be as large as possible, the diversity product of \mathcal{A} should be as large as possible, and the discriminant of \mathcal{A} should be as small as possible. A natural question is: *what is the maximal rank of a lattice space-time code \mathcal{A} such that $\delta(\mathcal{A}) > 0$?*

We define $r(n)$ by (see [14])

$$(1.3) \quad r(n) := \max\{\text{rank}(\mathcal{A}) : \mathcal{A} \text{ is a lattice in } \mathcal{M}_n(\mathbb{C}), \delta(\mathcal{A}) > 0\}.$$

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Determining the exact value for $r(n)$ seems difficult. One obvious lower bound is $r(n) \geq 2n$ (see [1], [2], [7], [8], [9]). Recently, Xing and Li [14] gave an upper bound $r(n) \leq 2n^2$.

The case $n = 2$ is of a particular interest. However, even in this case, we only know that $4 \leq r(n) \leq 8$. No 2×2 lattice space-time codes of rank greater than 4 with a positive diversity product had been constructed until in [14], Xing and Li gave an example of a 2×2 lattice space-time code of rank 5 with a positive diversity product and raised the following question.

Open problem. *Is there a 2×2 lattice space-time code of rank greater than 5 with a positive diversity product?*

In this paper, we give an example of a 2×2 lattice space-time code of rank 8 with a positive diversity product (Section 2). Moreover, we prove that if \mathcal{A} is the lattice generated by A_j ($1 \leq j \leq 8$) and if the set $\{A_j : (1 \leq j \leq 8)\}$ is linearly independent over \mathbb{R} , then $\delta(\mathcal{A}) \leq a$, where a is a constant, only dependent on A_j ($1 \leq j \leq 8$). See Section 3.

2. A 2×2 LATTICE SPACE-TIME CODE OF RANK 8

Theorem 2.1. *Consider the eight 2×2 matrices over \mathbb{C} :*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

and

$$A_5 = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & -\sqrt{q} \end{pmatrix}, A_6 = \begin{pmatrix} \sqrt{q}i & 0 \\ 0 & \sqrt{q}i \end{pmatrix}, A_7 = \begin{pmatrix} 0 & \sqrt{q} \\ \sqrt{q} & 0 \end{pmatrix}, A_8 = \begin{pmatrix} 0 & \sqrt{q}i \\ -\sqrt{q}i & 0 \end{pmatrix},$$

where q is a positive integer such that $q \equiv 7 \pmod{8}$. Let \mathcal{A} be the lattice generated by $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ and A_8 . Then the rank of \mathcal{A} is 8 and the diversity product of \mathcal{A} is 1.

Proof. First of all, it is clear that $\{A_1, A_2, \dots, A_8\}$ is linearly independent over \mathbb{R} . Let

$$(2.1) \quad \begin{aligned} A &= x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 + y_1A_5 + y_2A_6 + y_3A_7 + y_4A_8 \\ &= \begin{pmatrix} x_1 + y_1\sqrt{q} + (-x_2 + y_2\sqrt{q})i & x_3 + y_3\sqrt{q} + (-x_4 + y_4\sqrt{q})i \\ -x_3 + y_3\sqrt{q} + (-x_4 - y_4\sqrt{q})i & x_1 - y_1\sqrt{q} + (x_2 + y_2\sqrt{q})i \end{pmatrix}. \end{aligned}$$

To show that the rank of \mathcal{A} is 8 and the diversity product of \mathcal{A} is at least 1, it is sufficient to prove that for any eight integers $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ with $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \neq (0, 0, 0, 0, 0, 0, 0, 0)$, we have $|\det(A)| \geq 1$. Without loss of generality, we assume that $\gcd(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = 1$.

From (2.1), we get

$$\begin{aligned} \det(A) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 - qy_1^2 - qy_2^2 - qy_3^2 - qy_4^2 \\ &\quad + 2(x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3)\sqrt{q}i, \end{aligned}$$

and so

$$\begin{aligned} |\det(A)|^2 &= (x_1^2 + x_2^2 + x_3^2 + x_4^2 - qy_1^2 - qy_2^2 - qy_3^2 - qy_4^2)^2 \\ &\quad + 4q(x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3)^2. \end{aligned}$$

If one of the two terms $x_1^2 + x_2^2 + x_3^2 + x_4^2 - qy_1^2 - qy_2^2 - qy_3^2 - qy_4^2$ and $x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3$ is not zero, then $|\det(A)| \geq \min\{1, 2\sqrt{q}\} = 1$. Therefore we assume that

$$(2.2) \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 - qy_1^2 - qy_2^2 - qy_3^2 - qy_4^2 = 0$$

and

$$(2.3) \quad x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3 = 0.$$

Now, equations (2.2) and (2.3) imply that

$$(2.4) \quad (x_1 + y_2)^2 + (x_2 + y_1)^2 + (x_3 + y_4)^2 + (x_4 + y_3)^2 = (1 + q)(y_1^2 + y_2^2 + y_3^2 + y_4^2).$$

It follows that

$$(2.5) \quad (x_1 + y_2)^2 + (x_2 + y_1)^2 + (x_3 + y_4)^2 + (x_4 + y_3)^2 \equiv 0 \pmod{8}.$$

Since the square of an integer is always $\equiv 0, 1$ or $4 \pmod{8}$, then

$$(2.6) \quad 2 \mid x_1 + y_2, 2 \mid x_2 + y_1, 2 \mid x_3 + y_4, 2 \mid x_4 + y_3.$$

So x_1 and y_2 , x_2 and y_1 , x_3 and y_4 , x_4 and y_3 are of the same parity. On the other hand, from (2.3), there are three possibilities:

- (i) all x_1y_2, x_2y_1, x_3y_4 , and x_4y_3 are even;
- (ii) two of $x_1y_2, x_2y_1, x_3y_4, x_4y_3$ are odd and the other two are even;
- (iii) all x_1y_2, x_2y_1, x_3y_4 , and x_4y_3 are odd.

Case (i) We suppose that all x_1y_2, x_2y_1, x_3y_4 , and x_4y_3 are even. As $2 \mid x_1 + y_2$ and $2 \mid x_1y_2$, then we have $2 \mid x_1$ and $2 \mid y_2$. By the same argument, the other 6 variables are all even. Therefore, with $\gcd(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = 1$, we obtain a contradiction.

Case (ii) Now we suppose that two of $x_1y_2, x_2y_1, x_3y_4, x_4y_3$ are odd and the other two are even. Without loss of generality, we assume

$$2 \nmid x_1y_2, 2 \nmid x_2y_1 \quad \text{and} \quad 2 \mid x_3y_4, 2 \mid x_4y_3.$$

Then x_1, x_2, y_1, y_2 are odd, and x_3, x_4, y_3, y_4 are even. From (2.3), we deduce the following congruence:

$$x_1y_2 + x_2y_1 \equiv -(x_3y_4 + x_4y_3) \equiv 0 \pmod{4}.$$

This implies that, by considerations modulo 4, one of x_1y_2, x_2y_1 is 1 and the other is 3, say $x_1y_2 \equiv 1 \pmod{4}$ and $x_2y_1 \equiv 3 \pmod{4}$. It follows that $x_1 + y_2 \equiv 2 \pmod{4}$ and $x_2 + y_1 \equiv 0 \pmod{4}$. Hence, by (2.4) and $2 \mid y_1^2 + y_2^2 + y_3^2 + y_4^2$ we have

$$4 + (x_3 + y_4)^2 + (x_4 + y_3)^2 \equiv 0 \pmod{16}.$$

This and the fact that $(x_3 + y_4)^2, (x_4 + y_3)^2 \equiv 0$ or $4 \pmod{16}$ lead to a contradiction.

Case (iii) Finally, we take all x_1y_2, x_2y_1, x_3y_4 , and x_4y_3 to be odd. Since the x_i and y_i are all odd, then equation (2.4) gives us

$$(x_1 + y_2)^2 + (x_2 + y_1)^2 + (x_3 + y_4)^2 + (x_4 + y_3)^2 \equiv 0 \pmod{32}.$$

So we obtain

$$\left(\frac{x_1 + y_2}{2}\right)^2 + \left(\frac{x_2 + y_1}{2}\right)^2 + \left(\frac{x_3 + y_4}{2}\right)^2 + \left(\frac{x_4 + y_3}{2}\right)^2 \equiv 0 \pmod{8}.$$

This implies that $\frac{x_1 + y_2}{2}, \frac{x_2 + y_1}{2}, \frac{x_3 + y_4}{2}$ and $\frac{x_4 + y_3}{2}$ are both even.

On the other hand, from equations (2.2) and (2.3) we have

$$(x_1 - y_2)^2 + (x_2 - y_1)^2 + (x_3 - y_4)^2 + (x_4 - y_3)^2 = (q + 1)(y_1^2 + y_2^2 + y_3^2 + y_4^2).$$

It follows that

$$\left(\frac{x_1 - y_2}{2}\right)^2 + \left(\frac{x_2 - y_1}{2}\right)^2 + \left(\frac{x_3 - y_4}{2}\right)^2 + \left(\frac{x_4 - y_3}{2}\right)^2 \equiv 0 \pmod{8}.$$

Therefore, $\frac{x_1 - y_2}{2}$, $\frac{x_2 - y_1}{2}$, $\frac{x_3 - y_4}{2}$, and $\frac{x_4 - y_3}{2}$ are also even. Thus we have

$$2 \mid \frac{x_1 + y_2}{2} + \frac{x_1 - y_2}{2} = x_1.$$

But x_1 is an odd integer. This is impossible.

Combining the above three cases, we have shown that equations (2.2) and (2.3) cannot simultaneously hold. Thus we have $|\det(A)| \geq 1$. This completes the proof. \square

We have the following remarks.

Remark 2.2. (1) The condition $q \equiv 7 \pmod{8}$ can be replaced by $q = 2^{2r}(8t + 7)$, for nonnegative integers r and t . To see this, if instead of $q = 8t + 7$ we consider $2^{2r}(8t + 7)$, then we only need to take $2^r y_j$ ($j = 1, 2, 3, 4$) instead of y_j in our proof.

On the other hand, if q doesn't have the form $2^{2r}(8t + 7)$, then Theorem 2.1 is not true. In this case, we set $x_1 = x_2 = x_3 = y_3 = 0$ and $x_4 = q$. Thus equation (2.3) is satisfied and equation (2.2) becomes $q^2 = q(y_1^2 + y_2^2 + y_4^2)$, and so we obtain

$$q = y_1^2 + y_2^2 + y_4^2.$$

The above equation has integer solutions (y_1, y_2, y_4) . One can see [3], page 271, or [4], page 133.

(2) From [10], [5] and using equation (1.2), we can compute the discriminant and the normalized diversity product of the lattice \mathcal{A} given by Theorem 1.

Letting $z_1 = x_1 + y_2\sqrt{q}i$, $z_2 = x_2 + y_1\sqrt{q}i$, $z_3 = x_3 + y_4\sqrt{q}i$, $z_4 = x_4 - y_3\sqrt{q}i$, $z_1, z_2, z_3, z_4 \in \mathbb{Z}[\sqrt{q}i]$, from (2.1), we get

$$A = \begin{pmatrix} z_1 - iz_2 & z_3 - iz_4 \\ -z_3 - iz_4 & z_1 + iz_2 \end{pmatrix}.$$

The generating matrix G of the lattice \mathcal{A} is $G = \begin{pmatrix} G_1 & 0 \\ 0 & -G_2 \end{pmatrix}$, where

$G_1 = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, $G_2 = \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$, and $|\mathcal{L}| = \sqrt{q}$. Then the discriminant of \mathcal{A} is $d_{\mathcal{A}} = -4$, and the normalized diversity product of \mathcal{A} is $d_g = \frac{1}{4\sqrt{q}}$.

3. AN UPPER BOUND OF ANY 2×2 LATTICE SPACE-TIME CODE

In this section, we will use a result in Diophantine approximation to get a relative upper bound of $\delta(\mathcal{A})$.

Lemma 3.1. Let $X \in \mathbb{N}$, $m, n \in \mathbb{N}$, and $n > m \geq 1$. Let $L_i(\vec{x}) = \sum_{j=1}^n a_{ij}x_j$, where $\vec{x} = (x_1, \dots, x_n)$, $a_{ij} \in \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$. If

$$\sum_{j=1}^n |a_{ij}| \leq A_i, i = 1, \dots, m,$$

then there exists a nonvanishing vector $\vec{x}^{(0)} = (x_{01}, \dots, x_{0n}) \in \mathbb{Z}^n$, satisfying

$$|L_i(\vec{x}^{(0)})| < A_i X^{1-\frac{n}{m}}, i = 1, \dots, m,$$

and

$$|x_{0j}| \leq X, j = 1, \dots, n.$$

Proof. See page 62 of [15]. □

Now we prove the following theorem.

Theorem 3.2. For any eight 2×2 matrices over \mathbb{C} :

$$A_j = \begin{pmatrix} a_{j1} + a_{j2}i & a_{j3} + a_{j4}i \\ a_{j5} + a_{j6}i & a_{j7} + a_{j8}i \end{pmatrix},$$

where $a_{j1}, a_{j2}, a_{j3}, a_{j4}, a_{j5}, a_{j6}, a_{j7}, a_{j8} \in \mathbb{R}$, and $\{A_j (1 \leq j \leq 8)\}$ is linearly independent over \mathbb{R} . Let \mathcal{A} be the lattice generated by $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ and A_8 . Then

$$\delta(\mathcal{A}) \leq \sqrt{(a_1^2 + a_2^2)(a_7^2 + a_8^2)} + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)},$$

where $a_k = \sum_{j=1}^8 |a_{jk}|$, $1 \leq k \leq 8$.

Proof. We assume that $x_j \in \mathbb{Z} (1 \leq j \leq 8)$ and let $A = \sum_{j=1}^8 x_j A_j$. Then

$$A = \begin{pmatrix} \sum_{j=1}^8 x_j a_{j1} + i \sum_{j=1}^8 x_j a_{j2} & \sum_{j=1}^8 x_j a_{j3} + i \sum_{j=1}^8 x_j a_{j4} \\ \sum_{j=1}^8 x_j a_{j5} + i \sum_{j=1}^8 x_j a_{j6} & \sum_{j=1}^8 x_j a_{j7} + i \sum_{j=1}^8 x_j a_{j8} \end{pmatrix}.$$

From Lemma 3.1, there exists nonvanishing vector $\vec{x}^{(0)} = (x_{01}, \dots, x_{08}) \in \mathbb{Z}^8$, satisfying

$$(3.1) \quad \left| \sum_{j=1}^8 x_{0j} a_{jk} \right| < a_k X^{1-\frac{8}{4}} = \frac{a_k}{X}, k = 1, \dots, 4,$$

and

$$|x_{0j}| \leq X, j = 1, \dots, 8,$$

where $a_k = \sum_{j=1}^8 |a_{jk}|$, $X \in \mathbb{N}$. Thus from (3.1),

$$\begin{aligned} \delta(A) &\leq \left| \det \begin{pmatrix} \frac{1}{X}(a_1 + ia_2) & \frac{1}{X}(a_3 + ia_4) \\ -(a_5 + ia_6)X & (a_7 + ia_8)X \end{pmatrix} \right| = \left| \det \begin{pmatrix} a_1 + ia_2 & a_3 + ia_4 \\ -(a_5 + ia_6) & a_7 + ia_8 \end{pmatrix} \right| \\ &\leq \sqrt{(a_1^2 + a_2^2)(a_7^2 + a_8^2)} + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)}. \end{aligned}$$

This completes the proof of Theorem 3.2. □

Remark 3.3. (1) From Theorem 3.2 and the work done in [10], [5], we have

$$d_g \leq \frac{(\sqrt{(a_1^2 + a_2^2)(a_7^2 + a_8^2)} + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)})^2}{\sqrt{|\det g|}}$$

where g is the corresponding real generating matrix for Λ_G .

- (2) Recently, Liao, Wang, and Xia ([5]) constructed a full-rate space-time code based on the optimal quadratic extension on $\mathbb{Q}[i]$ as

$$C_{2,2} = \left\{ \begin{pmatrix} z_1 + \exp(\frac{i\pi}{6})z_2 & \rho(z_3 + \exp(\frac{i\pi}{6})z_4) \\ \rho(z_3 + \exp(\frac{i5\pi}{6})z_4) & z_1 + \exp(\frac{i5\pi}{6})z_2 \end{pmatrix} \right\},$$

where $z_1, z_2, z_3, z_4 \in \mathbb{Z}[i]$ and $\rho = \sqrt{1+i}$. They also constructed a full-rate space-time code based on the optimal quadratic extension on $\mathbb{Q}[\zeta_6]$ as

$$C_{2,3} = \left\{ \begin{pmatrix} z_1 + \theta_1 z_2 & \rho(z_3 + \theta_1 z_4) \\ \rho(z_3 + \theta_2 z_4) & z_1 + \theta_2 z_2 \end{pmatrix} \right\},$$

where $\zeta_6 = \exp(\frac{2\pi i}{6})$, $z_1, z_2, z_3, z_4 \in \mathbb{Z}[\zeta_6]$, θ_1, θ_2 are the roots of $x^2 + x + \zeta_6$, and $\rho = \sqrt{1+\zeta_6}$. The normalized diversity product for code $C_{2,2}$ is $d_g = \frac{1}{3\sqrt{2}} \approx 0.2357$ and the normalized diversity product for code $C_{2,3}$ is $d_g = \frac{1}{\frac{3}{4}\sqrt{39}} \approx 0.2135$. These normalized diversity products are both larger than ours obtained in Remark 2.2 (2). In fact, Theorem 3.2 gives a relative upper bound of d_g .

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