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A 2×2 LATTICE SPACE-TIME CODE OF THE HIGHEST RANK

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ABSTRACT. For all previous constructions of 2×2 lattice space-time codes with a positive diversity product, the rank $r(2)$ was such that $r(2) \leq 4$. An example of a 2×2 lattice space-time code of rank 5 with a positive diversity product was given by Xing and Li. In this paper, we give an example of a 2×2 lattice space-time code of rank 8 with a positive diversity product. This gives an answer to the open problem set by Xing and Li.

1. Introduction

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ be the sets of all complex numbers, real numbers, integers and positive integers respectively. We denote by $\mathcal{M}_n(\mathbb{C})$ the set of $n \times n$ matrices over $\mathbb{C}$. A lattice space-time code over $\mathbb{C}$ is a set $\mathcal{A}$ consisting of matrices in $\mathcal{M}_n(\mathbb{C})$ such that $\mathcal{A}$ is a free abelian group under the matrix addition. The rank of this group is called the dimension or rank of $\mathcal{A}$. In recent years, the study of space-time codes has generated much interest. Many mathematical areas such as number theory, algebra, combinatorics, etc., have been employed to construct good lattice codes ([1], [2], [5], [6], [7], [8], [11], [9], [12], [13], [14]).

The diversity product of $\mathcal{A}$ is defined by (see [14])

$$\delta(\mathcal{A}) := \inf\{|\det(A-B)| : A, B \in \mathcal{A}, A \neq B\}. \tag{1.1}$$

The normalized diversity product of $\mathcal{A}$ in $\mathcal{M}_n(\mathbb{C})$ is defined by (see [10], [8], [5])

$$d_g = \frac{(\delta(\mathcal{A}))^2}{|\det G| \cdot |\mathcal{L}|^{n/2}} = \frac{(\delta(\mathcal{A}))^2}{\sqrt{|\det g|}}, \tag{1.2}$$

where G is the corresponding generating matrix of the complex lattice $\mathcal{A}$, g is the corresponding real generating matrix for Λ_G, and $|\mathcal{L}|$ denotes the absolute value of the determinant of the 2×2 generating matrix of the two-dimensional real base lattice $\mathcal{L}$.

The criteria for lattice space-time codes are: the rank of $\mathcal{A}$ should be as large as possible, the diversity product of $\mathcal{A}$ should be as large as possible, and the discriminant of $\mathcal{A}$ should be as small as possible. A natural question is: *what is the maximal rank of a lattice space-time code $\mathcal{A}$ such that $\delta(\mathcal{A}) > 0$?*

We define $r(n)$ by (see [14])

$$r(n) := \max\{\text{rank}(\mathcal{A}) : \mathcal{A} \text{ is a lattice in } \mathcal{M}_n(\mathbb{C}), \delta(\mathcal{A}) > 0\}. \tag{1.3}$$

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Determining the exact value for $r(n)$ seems difficult. One obvious lower bound is $r(n) \geq 2n$ (see [1], [2], [7], [8], [9]). Recently, Xing and Li [14] gave an upper bound $r(n) \leq 2n^2$.

The case $n = 2$ is of a particular interest. However, even in this case, we only know that $4 \leq r(n) \leq 8$. No 2×2 lattice space-time codes of rank greater than 4 with a positive diversity product had been constructed until in [14], Xing and Li gave an example of a 2×2 lattice space-time code of rank 5 with a positive diversity product and raised the following question.

Open problem. *Is there a 2×2 lattice space-time code of rank greater than 5 with a positive diversity product?*

In this paper, we give an example of a 2×2 lattice space-time code of rank 8 with a positive diversity product (Section 2). Moreover, we prove that if $\mathcal{A}$ is the lattice generated by A_j $(1 \leq j \leq 8)$ and if the set $\{A_j : \ (1 \leq j \leq 8)\}$ is linearly independent over $\mathbb{R}$, then $\delta(\mathcal{A}) \leq a$, where a is a constant, only dependent on A_j $(1 \leq j \leq 8)$. See Section 3.

2. A 2×2 lattice space-time code of rank 8

Theorem 2.1. *Consider the eight 2×2 matrices over $\mathbb{C}$:*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ A_4 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

and

$$A_5 = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & -\sqrt{q} \end{pmatrix},\ A_6 = \begin{pmatrix} \sqrt{q}\,i & 0 \\ 0 & \sqrt{q}\,i \end{pmatrix},\ A_7 = \begin{pmatrix} 0 & \sqrt{q} \\ \sqrt{q} & 0 \end{pmatrix},\ A_8 = \begin{pmatrix} 0 & \sqrt{q}\,i \\ -\sqrt{q}\,i & 0 \end{pmatrix},$$

where q is a positive integer such that $q \equiv 7 \pmod 8$. Let $\mathcal{A}$ be the lattice generated by $A_1, A_2, A_3, A_4, A_5,\ A_6,\ A_7$ and A_8. Then the rank of $\mathcal{A}$ is 8 and the diversity product of $\mathcal{A}$ is 1.

Proof. First of all, it is clear that $\{A_1, A_2, \ldots, A_8\}$ is linearly independent over $\mathbb{R}$. Let

$$\begin{aligned} A &= x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 + y_1A_5 + y_2A_6 + y_3A_7 + y_4A_8 \\ &= \begin{pmatrix} x_1 + y_1\sqrt{q} + (-x_2 + y_2\sqrt{q})\,i & x_3 + y_3\sqrt{q} + (-x_4 + y_4\sqrt{q})\,i \\ -x_3 + y_3\sqrt{q} + (-x_4 - y_4\sqrt{q})\,i & x_1 - y_1\sqrt{q} + (x_2 + y_2\sqrt{q})\,i \end{pmatrix}. \end{aligned} \tag{2.1}$$

To show that the rank of $\mathcal{A}$ is 8 and the diversity product of $\mathcal{A}$ is at least 1, it is sufficient to prove that for any eight integers $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ with $(x_1, x_2, x_3, x_4,\ y_1, y_2, y_3, y_4) \neq (0, 0, 0, 0, 0, 0, 0, 0)$, we have $|\det(A)| \geq 1$. Without loss of generality, we assume that $\gcd(x_1, x_2, x_3, x_4,\ y_1, y_2, y_3, y_4) = 1$.

From (2.1), we get

$$\begin{aligned} \det(A) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - qy_1^2 - qy_2^2 - qy_3^2 - qy_4^2 \\ +2(x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3)\sqrt{q}i, \end{aligned}$$

and so

$$\begin{aligned} |\det(A)|^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2 - qy_1^2 - qy_2^2 - qy_3^2 - qy_4^2)^2 \\ +4q(x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3)^2. \end{aligned}$$

If one of the two terms $x_1^2+x_2^2+x_3^2+x_4^2-qy_1^2-qy_2^2-qy_3^2-qy_4^2$ and $x_1y_2+x_2y_1+x_3y_4+x_4y_3$ is not zero, then $|\det(A)| \geq \min\{1, 2\sqrt{q}\} = 1$. Therefore we assume that

$$x_1^2+x_2^2+x_3^2+x_4^2-qy_1^2-qy_2^2-qy_3^2-qy_4^2=0 \tag{2.2}$$

and

$$x_1y_2+x_2y_1+x_3y_4+x_4y_3=0. \tag{2.3}$$

Now, equations (2.2) and (2.3) imply that

$$(x_1+y_2)^2+(x_2+y_1)^2+(x_3+y_4)^2+(x_4+y_3)^2=(1+q)(y_1^2+y_2^2+y_3^2+y_4^2). \tag{2.4}$$

It follows that

$$(x_1+y_2)^2+(x_2+y_1)^2+(x_3+y_4)^2+(x_4+y_3)^2 \equiv 0 \pmod 8. \tag{2.5}$$

Since the square of an integer is always $\equiv 0, 1$ or $4 \pmod 8$, then

$$2\,|\,x_1+y_2,\ 2\,|\,x_2+y_1,\ 2\,|\,x_3+y_4,\ 2\,|\,x_4+y_3. \tag{2.6}$$

So x_1 and y_2, x_2 and y_1, x_3 and y_4, x_4 and y_3 are of the same parity. On the other hand, from (2.3), there are three possibilities:

(i) all x_1y_2, x_2y_1, x_3y_4, and x_4y_3 are even;
(ii) two of $x_1y_2, x_2y_1, x_3y_4, x_4y_3$ are odd and the other two are even;
(iii) all x_1y_2, x_2y_1, x_3y_4, and x_4y_3 are odd.

*C*ase (i) We suppose that all x_1y_2, x_2y_1, x_3y_4, and x_4y_3 are even. As $2|x_1+y_2$ and $2|x_1y_2$, then we have $2|x_1$ and $2|y_2$. By the same argument, the other 6 variables are all even. Therefore, with $\gcd(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4)=1$, we obtain a contradiction.

*C*ase (ii) Now we suppose that two of $x_1y_2, x_2y_1, x_3y_4, x_4y_3$ are odd and the other two are even. Without loss of generality, we assume

$$2 \nmid x_1y_2,\ 2 \nmid x_2y_1 \quad \text{and} \quad 2|x_3y_4,\ 2|x_4y_3.$$

Then x_1, x_2, y_1, y_2 are odd, and x_3, x_4, y_3, y_4 are even. From (2.3), we deduce the following congruence:

$$x_1y_2+x_2y_1 \equiv -(x_3y_4+x_4y_3) \equiv 0 \pmod 4.$$

This implies that, by considerations modulo 4, one of x_1y_2, x_2y_1 is 1 and the other is 3, say $x_1y_2 \equiv 1 \pmod 4$ and $x_2y_1 \equiv 3 \pmod 4$. It follows that $x_1+y_2 \equiv 2 \pmod 4$ and $x_2+y_1 \equiv 0 \pmod 4$. Hence, by (2.4) and $2|y_1^2+y_2^2+y_3^2+y_4^2$ we have

$$4+(x_3+y_4)^2+(x_4+y_3)^2 \equiv 0 \pmod{16}.$$

This and the fact that $(x_3+y_4)^2, (x_4+y_3)^2 \equiv 0$ or $4 \pmod{16}$ lead to a contradiction.

Case (iii) Finally, we take all x_1y_2, x_2y_1, x_3y_4, and x_4y_3 to be odd. Since the x_i and y_i are all odd, then equation (2.4) gives us

$$(x_1+y_2)^2+(x_2+y_1)^2+(x_3+y_4)^2+(x_4+y_3)^2 \equiv 0 \pmod{32}.$$

So we obtain

$$\left(\frac{x_1+y_2}{2}\right)^2+\left(\frac{x_2+y_1}{2}\right)^2+\left(\frac{x_3+y_4}{2}\right)^2+\left(\frac{x_4+y_3}{2}\right)^2 \equiv 0 \pmod 8.$$

This implies that $\frac{x_1+y_2}{2}, \frac{x_2+y_1}{2}, \frac{x_3+y_4}{2}$ and $\frac{x_4+y_3}{2}$ are both even.

On the other hand, from equations (2.2) and (2.3) we have

$$(x_1-y_2)^2+(x_2-y_1)^2+(x_3-y_4)^2+(x_4-y_3)^2=(q+1)(y_1^2+y_2^2+y_3^2+y_4^2).$$

It follows that

$$\left(\frac{x_1-y_2}{2}\right)^2+\left(\frac{x_2-y_1}{2}\right)^2+\left(\frac{x_3-y_4}{2}\right)^2+\left(\frac{x_4-y_3}{2}\right)^2\equiv 0 \pmod 8.$$

Therefore, $\frac{x_1-y_2}{2}$, $\frac{x_2-y_1}{2}$, $\frac{x_3-y_4}{2}$, and $\frac{x_4-y_3}{2}$ are also even. Thus we have

$$2 \mid \frac{x_1+y_2}{2}+\frac{x_1-y_2}{2}=x_1.$$

But x_1 is an odd integer. This is impossible.

Combining the above three cases, we have shown that equations (2.2) and (2.3) cannot simultaneously hold. Thus we have $|\det(A)|\geq 1$. This completes the proof. □

We have the following remarks.

Remark 2.2. (1) The condition $q\equiv 7 \pmod 8$ can be replaced by $q=2^{2r}(8t+7)$, for nonnegative integers r and t. To see this, if instead of $q=8t+7$ we consider $2^{2r}(8t+7)$, then we only need to take $2^r y_j$ $(j=1,2,3,4)$ instead of y_j in our proof.

On the other hand, if q doesn't have the form $2^{2r}(8t+7)$, then Theorem 2.1 is not true. In this case, we set $x_1=x_2=x_3=y_3=0$ and $x_4=q$. Thus equation (2.3) is satisfied and equation (2.2) becomes $q^2=q(y_1^2+y_2^2+y_4^2)$, and so we obtain

$$q=y_1^2+y_2^2+y_4^2.$$

The above equation has integer solutions (y_1,y_2,y_4). One can see [3], page 271, or [4], page 133.

(2) From [10], [5] and using equation (1.2), we can compute the discriminant and the normalized diversity product of the lattice $\mathcal{A}$ given by Theorem 1.

Letting $z_1=x_1+y_2\sqrt{q}\,i, z_2=x_2+y_1\sqrt{q}\,i,\ z_3=x_3+y_4\sqrt{q}\,i, z_4=x_4-y_3\sqrt{q}\,i,\ z_1,z_2,z_3,z_4\in\mathbb{Z}[\sqrt{q}\,i]$, from (2.1), we get

$$A=\begin{pmatrix} z_1-i\,z_2 & z_3-i\,z_4\\ -z_3-i\,z_4 & z_1+i\,z_2\end{pmatrix}.$$

The generating matrix G of the lattice $\mathcal{A}$ is $G=\begin{pmatrix} G_1 & 0\\ 0 & -G_2\end{pmatrix}$, where $G_1=\begin{pmatrix} 1 & -i\\ 1 & i\end{pmatrix}$, $G_2=\begin{pmatrix} 1 & -i\\ -1 & -i\end{pmatrix}$, and $|\mathcal{L}|=\sqrt{q}$. Then the discriminant of $\mathcal{A}$ is $d_{\mathcal{A}}=-4$, and the normalized diversity product of $\mathcal{A}$ is $d_g=\frac{1}{4\sqrt{q}}$.

3. An upper bound of any 2×2 lattice space-time code

In this section, we will use a result in Diophantine approximation to get a relative upper bound of $\delta(\mathcal{A})$.

Lemma 3.1. *Let $X \in \mathbb{N}$, $m, n \in \mathbb{N}$, and $n > m \geq 1$. Let $L_i(\vec{x}) = \sum_{j=1}^{n} a_{ij} x_j$, where $\vec{x} = (x_1, \cdots, x_n)$, $a_{ij} \in \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$. If*

$$\sum_{j=1}^{n} |a_{ij}| \leq A_i, i = 1, \cdots, m,$$

then there exists a nonvanishing vector $\vec{x}^{(0)} = (x_{01}, \cdots, x_{0n}) \in \mathbb{Z}^n$, satisfying

$$|L_i(\vec{x}^{(0)})| < A_i X^{1-\frac{n}{m}}, i = 1, \cdots, m,$$

and

$$|x_{0j}| \leq X, j = 1, \cdots, n.$$

Proof. See page 62 of [15]. □

Now we prove the following theorem.

Theorem 3.2. *For any eight 2×2 matrices over $\mathbb{C}$:*

$$A_j = \begin{pmatrix} a_{j1} + a_{j2}\, i & a_{j3} + a_{j4}\, i \\ a_{j5} + a_{j6}\, i & a_{j7} + a_{j8}\, i \end{pmatrix},$$

where $a_{j1}, a_{j2}, a_{j3}, a_{j4}, a_{j5}, a_{j6}, a_{j7}, a_{j8} \in \mathbb{R}$, and $\{A_j \ (1 \leq j \leq 8)\}$ is linearly independent over $\mathbb{R}$. Let $\mathcal{A}$ be the lattice generated by $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ and A_8. Then

$$\delta(\mathcal{A}) \leq \sqrt{(a_1^2 + a_2^2)(a_7^2 + a_8^2)} + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)},$$

where $a_k = \sum_{j=1}^{8} |a_{j_k}|$, $1 \leq k \leq 8$.

Proof. We assume that $x_j \in \mathbb{Z}$ $(1 \leq j \leq 8)$ and let $A = \sum_{j=1}^{8} x_j A_j$. Then

$$A = \begin{pmatrix} \sum_{j=1}^{8} x_j a_{j1} + i \sum_{j=1}^{8} x_j a_{j2} & \sum_{j=1}^{8} x_j a_{j3} + i \sum_{j=1}^{8} x_j a_{j4} \\ \sum_{j=1}^{8} x_j a_{j5} + i \sum_{j=1}^{8} x_j a_{j6} & \sum_{j=1}^{8} x_j a_{j7} + i \sum_{j=1}^{8} x_j a_{j8} \end{pmatrix}.$$

From Lemma 3.1, there exists nonvanishing vector $\vec{x}^{(0)} = (x_{01}, \cdots, x_{08}) \in \mathbb{Z}^8$, satisfying

$$\left| \sum_{j=1}^{8} x_{0j} a_{jk} \right| < a_k X^{1-\frac{8}{4}} = \frac{a_k}{X}, k = 1, \cdots, 4, \tag{3.1}$$

and

$$|x_{0j}| \leq X, j = 1, \cdots, 8,$$

where $a_k = \sum_{j=1}^{n} |a_{jk}|$, $X \in \mathbb{N}$. Thus from (3.1),

$$\begin{aligned} \delta(A) &\leq \left| \det \begin{pmatrix} \frac{1}{X}(a_1 + ia_2) & \frac{1}{X}(a_3 + ia_4) \\ -(a_5 + ia_6)X & (a_7 + ia_8)X \end{pmatrix} \right| = \left| \det \begin{pmatrix} a_1 + ia_2 & a_3 + ia_4 \\ -(a_5 + ia_6) & a_7 + ia_8 \end{pmatrix} \right| \\ &\leq \sqrt{(a_1^2 + a_2^2)(a_7^2 + a_8^2)} + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)}. \end{aligned}$$

This completes the proof of Theorem 3.2. □

Remark 3.3. (1) From Theorem 3.2 and the work done in [10], [5], we have

$$d_g \leq \frac{(\sqrt{(a_1^2 + a_2^2)(a_7^2 + a_8^2)} + \sqrt{(a_3^2 + a_4^2)(a_5^2 + a_6^2)})^2}{\sqrt{|\det g|}}$$

where g is the corresponding real generating matrix for Λ_G.

(2) Recently, Liao, Wang, and Xia ([5]) constructed a full-rate space-time code based on the optimal quadratic extension on $\mathbb{Q}[i]$ as

$$C_{2,2} = \left\{ \begin{pmatrix} z_1 + \exp(\frac{i\pi}{6})z_2 & \rho(z_3 + \exp(\frac{i\pi}{6})z_4) \\ \rho(z_3 + \exp(\frac{i5\pi}{6})z_4) & z_1 + \exp(\frac{i5\pi}{6})z_2 \end{pmatrix} \right\},$$

where $z_1, z_2, z_3, z_4 \in \mathbb{Z}[i]$ and $\rho = \sqrt{1+i}$. They also constructed a full-rate space-time code based on the optimal quadratic extension on $\mathbb{Q}[\zeta_6]$ as

$$C_{2,3} = \left\{ \begin{pmatrix} z_1 + \theta_1 z_2 & \rho(z_3 + \theta_1 z_4) \\ \rho(z_3 + \theta_2 z_4) & z_1 + \theta_2 z_2 \end{pmatrix} \right\},$$

where $\zeta_6 = \exp(\frac{2\pi i}{6})$, $z_1, z_2, z_3, z_4 \in \mathbb{Z}[\zeta_6]$, θ_1, θ_1 are the roots of $x^2 + x + \zeta_6$, and $\rho = \sqrt{1+\zeta_6}$. The normalized diversity product for code $C_{2,2}$ is $d_g = \frac{1}{3\sqrt{2}} \approx 0.2357$ and the normalized diversity product for code $C_{2,3}$ is $d_g = \frac{1}{\frac{3}{4}\sqrt{39}} \approx 0.2135$. These normalized diversity products are both larger than ours obtained in Remark 2.2 (2). In fact, Theorem 3.2 gives a relative upper bound of d_g.

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