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LOWER BOUNDS FOR MOMENTS OF AUTOMORPHIC L-FUNCTIONS OVER SHORT INTERVALS

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ABSTRACT. Let $L(s,\pi)$ be the principal L-function attached to an irreducible unitary cuspidal automorphic representation π of $GL_m(\mathbb{A}_{\mathbb{Q}})$. The aim of the paper is to give a simple method to show the lower bounds of mean value for automorphic L-functions over short intervals.

1. Introduction

An important problem in analytic number theory is to estimate the moments

$$I_k(T) = \int_T^{2T} \left| L\left(\frac{1}{2}+it,\pi\right)\right|^{2k} \mathrm{d}t$$

for all kinds of L-functions $L(s,\pi)$. For the Riemann zeta-function $\zeta(s)$, a conjecture states that the $2k$-th moment should be asymptotic to $C_k T(\log T)^{k^2}$ for a positive constant C_k, where k is a positive real number. The correct upper bound of $I_k(T)$ for $\zeta(s)$ is known only for $k \leq 2$, while the correct lower bound is proved for all rational $k \geq 0$,

$$I_k(T) \gg T(\log T)^{k^2}.$$

Actually, this latter result was proved by Ramachandra [11] for all positive integers k, by Heath-Brown [4] for all positive rational numbers k, and under the Riemann Hypothesis by Ramachandra [10] for all positive real numbers k. In [1] the authors propose conjectures for the full asymptotics of the moments of general L-functions. In particular, the paper provides conjectures for the moments of the Riemann zeta-function, the family of primitive Dirichlet L-functions, quadratic twists of L-functions, and automorphic L-functions attached to cusp forms.

Let π be an irreducible unitary cuspidal automorphic representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ and $s=\sigma+it\in\mathbb{C}$. Then the principal L-function [3] attached to π is given by Euler products of local factors for $\sigma>1$,

$$\begin{aligned} L(s,\pi) &= \prod_{p<\infty} L(s,\pi_p) \\ &= \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s}, \end{aligned}$$

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where

$$L(s,\pi_p)=\prod_{j=1}^{m}\left(1-\alpha_\pi(p,j)p^{-s}\right)^{-1}$$

with $\{\alpha_\pi(p,j):1\le j\le m\}$ being complex Satake parameters of π at the finite place π_p according to the local Langlands correspondence. If p is unramified, its local L-factor is of the form $L(s,\pi_p)=P_p(p^{-s})^{-1}$, where $P_p(x)$ is a polynomial of degree at most m and $P_v(0)=1$. We can write the local factors at ramified places v in the same form with the convention that some of the $\alpha_\pi(p,j)$ may be zero.

In this paper, we give a simple method to obtain lower bounds for integral moments of the principal L-function $L(s,\pi)$ over short intervals. Our main theorem is as follows.

Theorem 1.1. *Let π be an irreducible cuspidal automorphic representation of $GL_m(\mathbb{A}_\mathbb{Q})$ and let k be any positive real number. Then uniformly in σ,*

$$\int_T^{T+H}\left|L\left(\sigma+it,\pi\right)\right|^k \mathrm{d}t \gg H$$

for all $T\ge T_0$ for some sufficiently large T_0, $\sigma\ge 1/2$, and $T\ge H\ge \log^{1+\epsilon}T$ with any $\epsilon>0$.

2. Automorphic L-functions and some lemmas

Let π be an irreducible cuspidal automorphic representation of $GL_m(\mathbb{A}_\mathbb{Q})$. Denote the complete L-function by

$$\Lambda(s,\pi)=q^{\frac{s}{2}}L(s,\pi_\infty)L(s,\pi),$$

where q is an integer called the arithmetic conductor of π and

$$L(s,\pi_\infty)=\prod_{j=1}^{m}\Gamma_\mathbb{R}(s+\mu_j).$$

Here $\Gamma_\mathbb{R}(s)=\pi^{s/2}\Gamma\left(\frac{s}{2}\right)$, and $\{\mu_j:1\le j\le m\}$ are local complex parameters of π at the infinite place ∞.

The following statements collect together analytic facts about principal L-functions which we will use for our proofs.

(**A1**) The Dirichlet series

$$L(s,\pi)=\sum_{n=1}^{\infty}\frac{a_\pi(n)}{n^s}$$

converges absolutely in the half-plane $\Re s>1$, and we have (see [9])

$$\sum_{n\le X}|a_\pi(n)|^2\ll X^{1+\epsilon}. \tag{2.1}$$

(**A2**) The complete L-function $\Lambda(s,\pi)$ has an analytic continuation to the whole complex plane and satisfies the functional equation [3]

$$\Lambda(1-s,\tilde{\pi})=\varepsilon(\pi)\Lambda(s,\pi),$$

where $\varepsilon(\pi)$, a complex number of modulus 1, is the root number, $\tilde{\pi}$ is the contragredient representation of π, and $L_v(s,\tilde{\pi})=\overline{L_v(\bar{s},\pi)}$ for any place v.

(**A3**) $\Lambda(s,\pi)$ is an entire function of order one, bounded in vertical strips with finite width with exponential decay as $|\Im s| \to \infty$; see [2], [3].

(**A4**) The zeros of $\Lambda(s,\pi)$, that is, the nontrivial zeros of $L(s,\pi)$, lie in the open critical strip $0 < \Re s < 1$; see [6]. In particular, $L(s,\pi)$ is nonvanishing in the half-plane $\Re s \geq 1$.

(**A5**) Bounds toward the generalized Ramanujan conjecture, [8], [12], are

$$\begin{aligned} |\alpha_\pi(p,j)| &\leq p^{\frac{1}{2}-\frac{1}{m^2+1}} && \text{if } \pi \text{ is unramified at } p, \\ |\Re \mu_j| &\leq \frac{1}{2} - \frac{1}{m^2+1} && \text{if } \pi \text{ is unramified at } \infty. \end{aligned}$$

We also need the some lemmas. The following result of Littlewood establishes the connection between zeros of an analytic function and its mean-value estimates.

Lemma 2.1. *Let $\phi(s)$ be analytic and nonzero on the rectangle $\mathcal{D}$ with vertices α, β, $\alpha+iT$, and $\beta+iT$, where $\alpha<\beta$. Then*

$$\begin{aligned} 2\pi \sum_{\rho \in \mathcal{D}} \mathrm{Dist}(\rho) &= \int_0^T \log|\phi(\alpha+it)|\mathrm{d}t - \int_0^T \log|\phi(\beta+it)|\mathrm{d}t \\ &\quad + \int_\alpha^\beta \arg\phi(\sigma+iT)\mathrm{d}\sigma - \int_\alpha^\beta \arg\phi(\sigma)\mathrm{d}\sigma, \end{aligned} \tag{2.2}$$

where the sum runs over the zeros ρ of $\phi(s)$ in $\mathcal{D}$, and $\mathrm{Dist}(\rho)$ *is the distance from ρ to the left edge of the rectangle.*

Proof. This is a revised version of the classical Littlewood lemma. See Titchmarsh [13], Sections 9.9 and 9.15. □

Lemma 2.2. (1) *Let $N(T)$ be the number of nontrivial zeros $\rho=\beta+i\gamma$ of $L(s,\pi)$ such that $0<\beta<1$ and $0<\gamma\leq T$. Then*

$$N(T+1)-N(T) \ll \log T. \tag{2.3}$$

(2) *For any $s=\sigma+it$ in the strip $-2\leq\sigma\leq 2$, $|t|\geq t_0$, where t_0 is a fixed positive constant, we have*

$$\frac{L'}{L}(s,\pi) = \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O(\log|t|). \tag{2.4}$$

(3) *With the same notation as in (2), we have*

$$\log L(s,\pi) = \sum_{|t-\gamma|\leq 1} \log(s-\rho) + O(\log|t|). \tag{2.5}$$

Proof. For proofs of (1) and (2), see Iwaniec and Kowalski [5], Proposition 5.7, or Liu and Ye [7], Lemma 4.3. We integrate (2.4) along the straight line from $s=\sigma+it$ to $2+it$. By (2.3), if t is not the ordinate of a zero, then

$$\begin{aligned} \log L(s,\pi) &= \log L(2+it,\pi) + \sum_{|t-\gamma|\leq 1} \log(s-\rho) \\ &\quad - \sum_{|t-\gamma|\leq 1} \log(2+it-\rho) + O(\log|t|) \\ &= \sum_{|t-\gamma|\leq 1} \log(s-\rho) + O(\log|t|). \end{aligned}$$

This proves the lemma. $\square$

3. Proof of Theorem 1.1

We apply Lemma 2.2 with $\phi(s) = L(s, \pi)$, $\beta = 2$, and $\alpha = \sigma_0$ for some $1/2 \leq \sigma_0 \leq 1$. If T is not the ordinate of a zero, then

$$\begin{aligned} 2\pi \sum_{\rho \in \mathcal{D}} \mathrm{Dist}(\rho) &= \int_0^T \log |L(\sigma_0 + it, \pi)| \mathrm{d}t - \int_0^T \log |L(2 + it, \pi)| \mathrm{d}t \\ &\quad + \int_{\sigma_0}^2 \arg L(\sigma + iT, \pi) \mathrm{d}\sigma + K(\sigma_0), \end{aligned} \tag{3.1}$$

where $K(\sigma_0)$ is independent of T.

Lemma 3.1. *If $1/2 \leq \sigma \leq 1$ and T is sufficiently large, then*

$$2\pi \sum_{\rho \in \mathcal{D}} \mathrm{Dist}(\rho) = \int_0^T \log |L(\sigma + it, \pi)| \mathrm{d}t + O(\log T), \tag{3.2}$$

where $\mathcal{D}$ is the rectangle with vertices σ, 2, $\sigma + iT$, and $2 + iT$.

Proof. We define

$$\Lambda_\pi(n) = \sum_{j=1}^m \alpha_\pi^\ell(p, j) \log p$$

if $n = p^\ell$ with ℓ being a positive integer and 0 otherwise. By (**A**5), we have

$$\Lambda_\pi(n) \ll n^{\frac{1}{2} - \frac{1}{m^2+1}} \log n.$$

For $\sigma > 1$, we deduce

$$\begin{aligned} \log L(s, \pi) &= \sum_p \sum_{n=1}^\infty \frac{\Lambda_\pi(p^n)}{(\log p^n) p^{ns}} \\ &= \sum_{n=2}^\infty \frac{\Lambda_\pi(n)}{(\log n) n^s}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^T \log |L(2 + it, \pi)| \mathrm{d}t &= \Re \left(\int_0^T \log L(2 + it, \pi) \mathrm{d}t \right) \\ &= \Re \left(\sum_{n=2}^\infty \frac{\Lambda_\pi(n)}{(\log n) n^2} \frac{n^{-iT} - 1}{-i \log n} \right) \\ &\ll 1. \end{aligned} \tag{3.3}$$

Since $|\arg(\sigma + iT - \rho)| \leq \pi$ for $1/2 \leq \sigma \leq 1$, we deduce by (2.5) and then (2.3) that

$$\begin{aligned} \arg L(\sigma + iT, \pi) &= \Im \left(\log L(\sigma + iT, \pi) \right) \\ &= \sum_{|T - \gamma| \leq 1} \arg(\sigma + iT - \rho) + O(\log T) \\ &\ll \log T. \end{aligned} \tag{3.4}$$

The lemma immediately follows from (3.1), (3.3), and (3.4). □

Proof of Theorem 1.1. According to (3.2), we get

$$2\pi \sum_{\rho \in \mathcal{D}} \mathrm{Dist}(\rho) = \int_T^{T+H} \log|L(\sigma+it,\pi)|\mathrm{d}t + O(\log T), \tag{3.5}$$

where $H = \log^{1+\epsilon} T$ with any fixed $\epsilon > 0$ and $\mathcal{D}$ is the rectangle with vertices $\sigma + iT$, $2+iT$, $\sigma + i(T+H)$, and $2+i(T+H)$. The left-hand side of (3.5) is nonnegative. Hence, there exists a constant $C > 0$ such that

$$\int_T^{T+H} \log|L(\sigma+it,\pi)|\mathrm{d}t \geq -C\log T,$$

where $1/2 \leq \sigma \leq 1$. Now recall that if $a < b$, $f(t) \geq 0$ for $a \leq t \leq b$ and $f(t)$ is a continuous real function on the interval $[a,b]$, then

$$\frac{1}{b-a}\int_a^b \log f(t)\mathrm{d}t \leq \log\left(\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right),$$

which is an easy consequence of the inequality between the arithmetic and geometric means of nonnegative numbers. Taking $f(t) = |\log L(\sigma+it,\pi)|^k$ for any fixed positive real number k, we get

$$\begin{aligned}
\log\left(\frac{1}{H}\int_T^{T+H} |L(\sigma+it,\pi)|^k\mathrm{d}t\right) &\geq \frac{1}{H}\int_T^{T+H} \log|L(\sigma+it,\pi)|^k\mathrm{d}t \\
&= \frac{k}{H}\int_T^{T+H} \log|L(\sigma+it,\pi)|\mathrm{d}t \\
&\geq -Ck\frac{\log T}{H}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_T^{T+H} |L(\sigma+it,\pi)|^k\mathrm{d}t &\geq H\exp\left(-Ck\frac{\log T}{H}\right) \\
&= H\left(1+O\left(\frac{\log T}{H}\right)\right) \\
&\gg H,
\end{aligned}$$

for $\log^{1+\epsilon} T \leq H \leq T$ with any $\epsilon > 0$. This completes the proof of Theorem 1.1. □

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