BICOMMUTANTS OF REDUCED UNBOUNDED OPERATOR ALGEBRAS

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Abstract. The unbounded bicommutant \( (\mathfrak{M}_{E'})''_{wc} \) of the reduction of an O*-algebra \( \mathfrak{M} \) via a given projection \( E' \) weakly commuting with \( \mathfrak{M} \) is studied, with the aim of finding conditions under which the reduction of a GW*-algebra is a GW*-algebra itself. The obtained results are applied to the problem of the existence of conditional expectations on O*-algebras.

1. Introduction and preliminaries

If \( \mathfrak{M} \) is a von Neumann algebra and \( E' \) a projection on the commutant \( \mathfrak{M}' \) of \( \mathfrak{M} \), then it is well known that the reduced algebra \( \mathfrak{M}_{E'} = \{XE' : X \in \mathfrak{M}\} \) is again a von Neumann algebra. The notion of a von Neumann algebra has been generalized to unbounded operator algebras by introducing several classes of O*-algebras, such as GW*-algebras, EW*-algebras, etc. It is then natural and important for applications to pose the question if the reduced algebra of a GW*-algebra via a projection \( E' \) picked in its weak bounded commutant is again a GW*-algebra. In order to answer this question we undertake a more general study of the reduction process for O*-algebras. We begin with reviewing the basic definitions and properties of O*-algebras needed in this paper and refer to [5, 6] for more details.

Let \( \mathcal{H} \) be a Hilbert space with inner product \( \langle \cdot | \cdot \rangle \) and \( \mathcal{D} \) a dense subspace of \( \mathcal{H} \). We denote by \( \mathcal{L}^1(\mathcal{D}) \) the set of all linear operators \( X \) defined in \( \mathcal{D} \) such that \( XD \subset \mathcal{D} \), the domain \( D(X^*) \) of its adjoint contains \( \mathcal{D} \) and \( X^*D \subset \mathcal{D} \). Then \( \mathcal{L}^1(\mathcal{D}) \) is an *-algebra with the usual operations: \( X + Y, \alpha X, XY \) and the involution \( X \mapsto X^\dagger := X^*|D \). A *-subalgebra of \( \mathcal{L}^1(\mathcal{D}) \) is called an O*-algebra on \( \mathcal{D} \). In this paper we will assume that an O*-algebra always contains the identity \( I \).

Let \( \mathfrak{M} \) be an O*-algebra on \( \mathcal{D} \). The graph topology \( t_{\mathfrak{M}} \) on \( \mathcal{D} \) is the locally convex topology defined by the family \( \{\| \cdot \|_X : X \in \mathfrak{M}\} \) of seminorms: \( \|\xi\|_X = \|X\xi\| \), \( \xi \in \mathcal{D} \). If the locally convex space \( \mathcal{D}[t_{\mathfrak{M}}] \) is complete, then \( \mathfrak{M} \) is said to be closed. In particular, if \( \mathcal{D}[t_{\mathfrak{M}}] \) is a Fréchet space, then \( \mathcal{D} \) is called the Fréchet domain of \( \mathfrak{M} \). More in general, we denote by \( \bar{\mathcal{D}}(\mathfrak{M}) \) the completion of the locally convex space \( \mathcal{D}[t_{\mathfrak{M}}] \) and put

\[ \bar{X} := X|\bar{\mathcal{D}}(\mathfrak{M}) \quad \text{and} \quad \widehat{\mathfrak{M}} := \{\bar{X} : X \in \mathfrak{M}\}. \]
Then \( \widetilde{\mathcal{D}}(\mathcal{M}) = \bigcap_{X \in \mathcal{M}} D(\overline{X}) \) and \( \widetilde{\mathcal{M}} \) is a closed \( O^* \)-algebra on \( \widetilde{\mathcal{D}}(\mathcal{M}) \) which is called the closure of \( \mathcal{M} \), since it is the smallest closed extension of \( \mathcal{M} \). We next recall the notion of self-adjointness of \( \mathcal{M} \). If \( \mathcal{D} = \mathcal{D}^*(\mathcal{M}) := \bigcap_{X \in \mathcal{M}} D(X^*) \), then \( \mathcal{M} \) is said to be self-adjoint. If \( \overline{\mathcal{D}}(\mathcal{M}) = \mathcal{D}^*(\mathcal{M}) \), then \( \mathcal{M} \) is said to be essentially self-adjoint. It is clear that
\[
\mathcal{D} \subset \overline{\mathcal{D}}(\mathcal{M}) \subset \mathcal{D}^*(\mathcal{M}),
\]
\[
X \subset \overline{X} \subset X^*, \ \forall X \in \mathcal{M}.
\]
The weak commutant \( \mathcal{M}''_w \) of \( \mathcal{M} \) is defined by
\[
\mathcal{M}''_w = \{ C \in \mathcal{B}(\mathcal{H}) : \langle C X \xi | \eta \rangle = \langle C \xi | X^\dagger \eta \rangle, \ \forall X \in \mathcal{M}, \ \forall \xi, \eta \in \mathcal{D} \},
\]
where \( \mathcal{B}(\mathcal{H}) \) is the *-algebra of all bounded linear operators on \( \mathcal{H} \). Then \( \mathcal{M}''_w \) is a weak-operator closed *-invariant subspace of \( \mathcal{B}(\mathcal{H}) \) (but it is not, in general, a von Neumann algebra) and \( (\mathcal{M}''_w)' = \mathcal{M}''_w \). If \( \mathcal{M}'_w \mathcal{D} \subset \mathcal{D} \), as happens for self-adjoint \( \mathcal{M} \), then \( \mathcal{M}'_w \) is a von Neumann algebra. In this paper we will use the following bicommutants of \( \mathcal{M} \):
\[
(\mathcal{M}_w')' = \{ A \in \mathcal{B}(\mathcal{H}) : AC = CA, \ \forall C \in \mathcal{M}_w \},
\]
\[
\mathcal{M}''_w := (\mathcal{M}_w')'_c = \{ X \in \mathcal{L}^1(\mathcal{D}) : \langle C X \xi | \eta \rangle = \langle \xi | C X^\dagger \eta \rangle, \ \forall C \in \mathcal{M}_w, \ \forall \xi, \eta \in \mathcal{D} \}.
\]
Then, \( (\mathcal{M}_w')' \) is a von Neumann algebra on \( \mathcal{H} \) and \( \mathcal{M}''_w \) is a \( \tau_{w^*} \)-closed \( O^* \)-algebra on \( \mathcal{D} \) such that \( \mathcal{M} \subset \mathcal{M}''_w \) and \( (\mathcal{M}_w')'_c = (\mathcal{M}_w')_w \), where the strong*-topology \( \tau_{w^*} \) is defined by the family \( \{ p^*_\xi(\cdot) : \xi \in \mathcal{D} \} \) of seminorms:
\[
p^*_\xi(X) := \| X \xi \| + \| X^\dagger \xi \|, \ \ \ X \in \mathcal{L}^1(\mathcal{D}).
\]
Furthermore, if \( \mathcal{M}_w \mathcal{D} \subset \mathcal{D} \), then
\[
\mathcal{M}''_w = \{ X \in \mathcal{L}^1(\mathcal{D}) : \overline{X} \ \text{is affiliated with} \ (\mathcal{M}_w')' = \overline{(\mathcal{M}_w')'|D^{w^*}} \cap \mathcal{L}^1(\mathcal{D}) \}.
\]
An \( O^* \)-algebra \( \mathcal{M} \) on \( \mathcal{D} \) is called a GW*-algebra if \( \mathcal{M}''_w \mathcal{D} \subset \mathcal{D} \) and \( \mathcal{M} = \mathcal{M}''_w \).

Let \( \mathcal{M} \) be a closed \( O^* \)-algebra on \( \mathcal{D} \) such that \( \mathcal{M}_w \mathcal{D} \subset \mathcal{D} \) and let \( \mathcal{E} \) be a projection in the von Neumann algebra \( \mathcal{M}_w \). Let \( \mathcal{M}_{E'} \) denote the reduced algebra \( \mathcal{M}_{E'} = \{ X E' : X \in \mathcal{M} \} \). Then it is known [1] that
\[
(\mathcal{M}_{E'})'_w = (\mathcal{M}_w')_{E'},
\]
\[
((\mathcal{M}_{E'})'_w)' = ((\mathcal{M}_w')_w)'_{E'}.
\]
But the following question is open.

**Question.** Does the equality \( (\mathcal{M}_{E'})''_w = (\mathcal{M}_w''_{E'}) \) hold?

If the answer to this question is affirmative, then the reduced algebra \( \mathcal{M}_{E'} \) of a given GW*-algebra is again a GW*-algebra. This question will be considered in Section 2, where we will show that if \( \mathcal{D} \) is the Fréchet domain of \( \mathcal{M} \) and the linear span \( \langle \mathcal{M}_w',E' \rangle \) of the \( \mathcal{M} \)-invariant subspace \( \mathcal{M}_w E' \mathcal{D} \) is essentially self-adjoint, then \( (\mathcal{M}_{E'})''_w = (\mathcal{M}_w''_{E'}) \). An \( \mathcal{M} \)-invariant subspace \( \mathcal{M} \) of \( \mathcal{D} \) is called essentially self-adjoint if the \( O^* \)-algebra \( \mathcal{M} | \mathcal{M} := \{ X | \mathcal{M} : X \in \mathcal{M} \} \) of \( \mathcal{M} \) is essentially self-adjoint.

Furthermore, we will show that, if the graph topology \( t_{\mathcal{M}} \) is defined by a sequence \( \{ T_n \} \) of essentially self-adjoint operators of \( \mathcal{M} \), whose spectral projections leave the domain \( \mathcal{D} \) invariant, then again \( (\mathcal{M}_{E'})''_w = (\mathcal{M}_w''_{E'}) \).
In Section 3 we shall apply the results of Section 2 to the study of conditional expectations for O*-algebras.

2. Main results

Let $\mathfrak{M}$ be a closed O*-algebra on $D$ such that $\mathfrak{M}'D \subset D$ and let $E'$ be a projection in the von Neumann algebra $\mathfrak{M}'$. In this section we look for conditions for the equality $(\mathfrak{M}_w'E')_{\text{wc}} = (\mathfrak{M}_w'E')_{\text{wc}}'$ to hold. Since $(\mathfrak{M}_w'E')_{\text{wc}} = (\mathfrak{M}_w'E')_{\text{wc}}'$, it follows that

\[(\mathfrak{M}_w'E')_{\text{wc}} \subset ((\mathfrak{M}_w'E')_{\text{wc}})' = (\mathfrak{M}_w'E')_{\text{wc}}'.\]

Hence we need only to prove the converse inclusion: $(\mathfrak{M}_w'E')_{\text{wc}}' \subset (\mathfrak{M}_w'E')_{\text{wc}}$. For any $X \in (\mathfrak{M}_w'E')_{\text{wc}}'$, we put

$$(M)_{\text{wc}} = (\mathfrak{M}_w'E')_{\text{wc}}.$$

Then we have the following.

**Lemma 2.1.** $X \in \mathcal{L}(\mathcal{E})$ and $((\mathfrak{M}_w'E')_{\text{wc}}')_{\text{wc}} := \{X : X \in (\mathfrak{M}_w'E')_{\text{wc}}'\}$ is an O*-algebra on $\mathcal{E}$.

**Proof.** For any $X \in (\mathfrak{M}_w'E')_{\text{wc}}'$, $C, K_j \in \mathfrak{M}_w'$ and $\xi, \zeta, \eta \in D$ we have

\[
\left\langle X \left( \sum_k C_k E' \xi_k + (I-Z)\eta \right) \right| \sum_j K_j E' \zeta_j + (I-Z)\eta \right\rangle
= \sum_{k,j} \left\langle C_k X E' \xi_k \left| K_j E' \zeta_j \right. \right\rangle
= \sum_{k,j} \left\langle E' K_j^* C_k E' X E' \xi_k \left| E' \zeta_j \right. \right\rangle
= \sum_{k,j} \left\langle E' K_j^* C_k E' \xi_k \left| X^\dagger E' \zeta_j \right. \right\rangle
= \left\langle \sum_k C_k E' \xi_k \left| \sum_j K_j X^\dagger E' \zeta_j \right. \right\rangle
= \left\langle \sum_k C_k E' \xi_k \left| \sum_j K_j E' \zeta_j + (I-Z)\eta \right. \right\rangle,
\]

which implies that $X$ is well-defined and $(X^\dagger)_c = (X^\dagger)_c$. It is clear that $X$ is a linear operator on $\mathcal{E}$. Hence $X \in \mathcal{L}(\mathcal{E})$. Moreover, as is easily seen, $(XY)_c = X_c Y_c$. Hence, $e(\mathfrak{M}_w'E') := ((\mathfrak{M}_w'E')_{\text{wc}}')_{\text{wc}}$ is an O*-algebra on $\mathcal{E}$. $\square$
Since $e(M_E')$ is an $O^*$-algebra on $E$, its closure is an $O^*$-algebra on $E$. It is easily seen that $\tilde{E} = (M''_wE'D')' \cap (M_E')' \oplus (I - Z)D$. On the other hand, $D = ZD \oplus (I - Z)D$; hence, if $(M''_wE'D')' \cap (M_E')' = ZD$, we can extend any $X \in (M_E')''_{wc}$ to $D$.

**Theorem 2.2.** Let $M$ be a closed $O^*$-algebra on $D$ such that $M'' \subseteq D$ and let $E'$ be a projection in the von Neumann algebra $M''_w$. Suppose that $\tilde{E}((M_E')''_{wc}) = D$ or, equivalently, $(M''_wE'D')' \cap (M_E')' = ZD$. Then, $(M''_wE') = (M''_w)_{wc}$.

**Proof.** By the assumption, for every $X \in (M_E')''_{wc}$, $X_e$ extends to an operator $\tilde{X}_e$ on $D$; that is, $\tilde{X}_e = \tilde{X}_e^*D$ and $(\tilde{X}_e)_{E'} = X$. Furthermore, we have $\tilde{X}_e \in M''_w$. Indeed, for every $C \in (M''_w)_{wc}$, we have

$$CX_e\left(\sum_k C_k X'\xi_k + (I - Z)\eta\right) = C \sum_k C_k X'\xi_k = \sum_k C C_k X'\xi_k = X_e\left(\sum_k C_k X'\xi_k + (I - Z)\eta\right),$$

for every $C_k \in (M''_w)_{wc}$ and $C_k \xi_k, \eta \in D$. Hence, $C X_e = X_e C$ on $E$ and so $C \tilde{X}_e = \tilde{X}_e C$. Thus $\tilde{X}_e \in M''_w$ and $X = (\tilde{X}_e)_{E'} \in (M''_w)_{E'}$. In conclusion, $(M_E')''_{wc} \subseteq (M''_w)_{E'}$ and by (1) the equality follows.

The next step consists, of course, in looking for situations where the conditions of Theorem 2.2 are satisfied. We begin with the following.

**Lemma 2.3.** Let $M$ and $E'$ be as above. Suppose that

- (i) for every $X \in (M_E')''_{wc}$ there exists an element $Y$ of $M''_{wc}$ such that

$$\|X_e\| \leq \|Y\|, \quad \forall \xi \in E$$

(equivalently, $X_e^*X_e \leq Y^*Y$ on $E$);

- (ii) $(M''_wE'D')' \cap (M_E')' = ZD$.

Then, $(M_E')''_{wc} = (M''_w)_{E'}$.

**Proof.** The conditions (i) and (ii) immediately imply that

$$ZD \subseteq (M''_wE'D')' \cap (M_E')'.$$

Hence

$$(M''_wE'D')' \cap (M_E')' = ZD.$$

By Theorem 2.2, we get $(M_E')''_{wc} = (M''_w)_{E'}$.

**Remark 2.4.** A comment is in order for condition (ii) of Lemma 2.3. Let us in fact consider the following statements:

- (i) The $M''_w$-invariant subspace $(M''_wE'D')$ is essentially self-adjoint for $M''_w$.

- (ii) $(M''_wE'D')' \cap (M_E')' = ZD$.

Then (i) $\Rightarrow$ (ii). In particular if $M$ is self-adjoint, then the two conditions are equivalent. For the proof we refer to [5].
Theorem 2.5. Let $\mathcal{M}$ be a closed $O^*$-algebra $\mathcal{D}$ with $\mathcal{D}$ a Fréchet domain of $\mathcal{M}$. Assume that $\mathcal{M}_{\mathcal{D}} \subseteq \mathcal{D}$. Let $E'$ be a projection in $\mathcal{M}_{\mathcal{D}}$ and assume that $(\mathcal{M}_{\mathcal{D}} E' \mathcal{D})$ is essentially self-adjoint for $\mathcal{M}_{\mathcal{D}}$. Then, $(\mathcal{M}_{\mathcal{D}} E' \mathcal{D})' = (\mathcal{M}_{\mathcal{D}} E' \mathcal{D})''$.

Proof. Since $\mathcal{D}[t_{\mathcal{D}}]$ is a Fréchet space, the topology $t_{\mathcal{D}}$ is defined by a sequence $\{T_n\}$ of elements of $\mathcal{M}$ satisfying

$$\|\xi\| \leq \|T_1 \xi\| \leq \|T_2 \xi\|, \ldots, \forall \xi \in \mathcal{D}.$$ 

Any $X \in \mathcal{L}(\mathcal{D})$ is a closed linear operator on the Fréchet space $\mathcal{D}[t_{\mathcal{D}}]$ into the Hilbert space $\mathcal{H}$, which implies, by the closed graph theorem, that the topology $t_{\mathcal{D}}$ is equivalent to the graph topology $t_{\mathcal{L}(\mathcal{D})}$ defined by $\mathcal{L}(\mathcal{D})$. This in turn implies that for every $X \in \mathcal{L}(\mathcal{D})$ there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that

$$\|X \xi\| \leq \gamma \|T_{n_0} \xi\|, \forall \xi \in \mathcal{D}. \quad (2)$$ 

Now take an arbitrary $X \in (\mathcal{M}_{\mathcal{D}} E' \mathcal{D})''$. Since $XE' \in \mathcal{L}(\mathcal{D})$, it follows from (2) that

$$\|XE' \xi\| \leq \gamma \|T_{n_0} \xi\|, \forall \xi \in \mathcal{D}. \quad (3)$$ 

Since $\mathcal{M}_{\mathcal{D}}$ is a von Neumann algebra, for every $C \in \mathcal{M}_{\mathcal{D}}$ and $\xi \in \mathcal{D}$, we have

$$\|X_c(CE' \xi)\|^2 = \|CXE' \xi\|^2 = \langle (E'C^*CE'X) XE' \xi \rangle = \|(E'C^*CE')^{1/2}XE' \xi\|^2 \leq \gamma^2 \|T_{n_0} (E'C^*CE')^{1/2} \xi\|^2 \leq \gamma^2 \|CE'T_{n_0} \xi\|^2 = \gamma^2 \|T_{n_0} CE' \xi\|^2.$$ 

Hence, condition (i) of Lemma 2.3 is satisfied. By the same lemma we then get the equality $(\mathcal{M}_{\mathcal{D}} E' \mathcal{D})' = (\mathcal{M}_{\mathcal{D}} E' \mathcal{D})''$. \hfill $\square$

Lemma 2.6. Let $\mathcal{M}$ be a closed $O^*$-algebra on a Fréchet domain $\mathcal{D}$ in $\mathcal{H}$ such that $t_{\mathcal{M}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators of $\mathcal{M}$ whose spectral projections belong to $\mathcal{M}$.

Let $\mathcal{M}$ be an $\mathcal{M}$-invariant subspace of $\mathcal{D}$ such that the projection $P_{\mathcal{M}}$ onto $\overline{\mathcal{M}}$ belongs to $\mathcal{M}_{\mathcal{D}}$.

Then $\mathcal{M}$ is essentially self-adjoint for $\mathcal{M}$ or, equivalently, $P_{\mathcal{M}} \mathcal{D} = \overline{\mathcal{M}}_{\mathcal{D}}$.

Proof. First we observe that $\overline{\mathcal{M}}_{\mathcal{D}} \subseteq P_{\mathcal{M}} \mathcal{D}$. Hence we only need to show that the converse inclusion also holds. Take an arbitrary $\xi \in \mathcal{D}$. Then there exists a sequence $\{\xi_n\}$ in $\mathcal{M}$ such that $\lim_{n \to \infty} \|\xi_n - P_{\mathcal{M}} \xi\| = 0$. By (4), for every $X \in \mathcal{M}$, there exists $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that

$$\|X \xi\| \leq \gamma \|T_{n_0} \xi\|, \forall \xi \in \mathcal{D}. \quad (4)$$ 

By the assumption, $\overline{T_{n_0}}$ is self-adjoint. Let $\overline{T_{n_0}} = \int_{-\infty}^{\infty} \lambda dE_{T_{n_0}}(\lambda)$ be the spectral resolution of $\overline{T_{n_0}}$. We put $E_{k} := E_{T_{n_0}} (k) - E_{T_{n_0}} (-k), k \in \mathbb{N}$. Then, by assumption $E_k \mathcal{D} \subseteq \mathcal{M}$, $\forall k \in \mathbb{N}$ and so, $E_k \xi_n \in \mathcal{M}$, $\forall k, n \in \mathbb{N}$ and $\lim_{n \to \infty} \|E_k \xi_n - E_k P_{\mathcal{M}} \xi\| = 0$. Furthermore, by (4) we have

$$\|XE_k \xi_n - XE_k \xi_m\| \leq \gamma \|T_{n_0} E_k (\xi_n - \xi_m)\| \to 0, \text{ as } n, m \to \infty.$$
Hence $E_k P_M \xi \in D(X^\dagger \| M)$. Furthermore,
\[
\| X E_k P_M \xi - X P_M \xi \| \leq \gamma \| T_{n_0} E_k P_M \xi - T_{n_0} P_M \xi \|
= \gamma \| P_M (E_k T_{n_0} \xi - T_{n_0} \xi) \|
\leq \gamma \| E_k T_{n_0} \xi - T_{n_0} \xi \| \to 0 \text{ as } k \to \infty,
\]
which implies that $P_M \xi \in D(X^\dagger \| M)$. Hence,
\[
P_M \xi \in \bigcap_{X \in \mathfrak{M}} D(X^\dagger \| M) = \mathfrak{M}^{\text{inv}}.
\]
Thus $P_M \mathcal{D} = \mathfrak{M}^{\text{inv}}$. This is equivalent to $\mathcal{M}$ being essentially self-adjoint for $\mathfrak{M}$, since $\mathfrak{M}$ is clearly self-adjoint.

By Theorem 2.5 and Lemma 2.6, we have the following.

**Theorem 2.7.** Let $\mathfrak{M}$ be a closed $O^*$-algebra such that $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators of $\mathfrak{M}$ whose spectral projections leave the domain $\mathcal{D}$ invariant. Let $E'$ be a projection in $\mathfrak{M}_w'$. Then $(\mathfrak{M}_E')''_{\text{we}} = (\mathfrak{M}_w)_{\text{we}} E'$.

*Proof.* It can be shown that $\mathfrak{M}$ is a self-adjoint $O^*$-algebra on the Fréchet domain $\mathcal{D}$, so that $\mathfrak{M}_w' \mathcal{D} \subset \mathcal{D}$, $t_{\mathfrak{M}_w'} = t_{\mathfrak{M}}$ and $\mathfrak{M}_{\text{we}}$ is a self-adjoint $O^*$-algebra on the Fréchet domain $\mathcal{D}$. Furthermore
\[
P_{(\mathfrak{M}_w' E' \mathcal{D})} = Z \in \mathfrak{M}_w' \cap (\mathfrak{M}_w)' = (\mathfrak{M}_{\text{we}})' \cap (\mathfrak{M}_{\text{we}})'.
\]
Moreover, since the spectral projections of the operators $T_n$ leave the domain $\mathcal{D}$ invariant, they automatically belong to $\mathfrak{M}_{\text{we}}'$. Hence, the self-adjoint $O^*$-algebra $\mathfrak{M}_{\text{we}}'$ and the $\mathfrak{M}_{\text{we}}'$-invariant subspace $(\mathfrak{M}_w' E' \mathcal{D})$ of $\mathcal{D}$ satisfy all the conditions in Lemma 2.6, and so $Z \mathcal{D} = (\mathfrak{M}_w' E' \mathcal{D})_{\text{we}}$. Then, by Theorem 2.5 $(\mathfrak{M}_E')''_{\text{we}} = (\mathfrak{M}_{\text{we}})' E'$.\]

**Corollary 2.8.** Let $T$ be an essentially self-adjoint operator in $\mathcal{H}$ and $\mathfrak{M}$ be a self-adjoint $O^*$-algebra on $\mathcal{D}^\infty (\mathcal{T}) := \bigcap_{n \in \mathbb{N}} D(\mathcal{T}^n)$, containing $T$. Let $E'$ be a projection in $\mathfrak{M}_w'$. Then $(\mathfrak{M}_E')''_{\text{we}} = (\mathfrak{M}_{\text{we}})' E'$.

*Proof.* The spectral projections $E_T(\lambda)$, $\lambda \in \mathbb{R}$, of $\mathcal{T}$ satisfy:
\begin{itemize}
    \item $E_T(\lambda) \in (\mathfrak{M}_w)', \ \forall \lambda \in \mathbb{R}$;
    \item $E_T(\lambda) \mathcal{H} \subset \mathcal{D}^\infty (\mathcal{T})$, $\forall \lambda \in \mathbb{R}$.
\end{itemize}
The statement then follows from Theorem 2.7.\]

Apart from GW*-algebras, another unbounded generalization of von Neumann algebras is provided by the notion of extended W*-algebras, shortly EW*-algebras, defined as follows: A closed $O^*$-algebra $\mathfrak{M}$ on $\mathcal{D}$ is said to be an EW*-algebra if $(I + X^\dagger X)^{-1}$ exists in $\mathfrak{M}_b := \{ A \in \mathfrak{M} : A \in \mathcal{B}(\mathcal{H}) \}$, for every $X \in \mathfrak{M}$ and $\mathfrak{M}_b := \{ A \in \mathfrak{M} : A \in \mathfrak{M}_b \}$ is a von Neumann algebra [2]. It is easily shown that every EW*-algebra on a Fréchet domain satisfies the conditions of Theorem 2.7 (recall that every symmetric element of an EW*-algebra is essentially self-adjoint). Hence we have the following.

**Corollary 2.9.** Let $\mathfrak{M}$ be an EW*-algebra on the Fréchet domain $\mathcal{D}$ in Hilbert space $\mathcal{H}$ and $E'$ a projection in $\mathfrak{M}_w'$. Then, $(\mathfrak{M}_E')''_{\text{we}} = (\mathfrak{M}_{\text{we}})' E'$.\]
3. Applications

In this section we show how to use the results of Section 2 in the analysis of the existence of conditional expectations for O*-algebras, which were first studied in [3, 4].

Let \( \mathcal{M} \) be a given O*-algebra on \( D \) in \( \mathcal{H} \) with a strongly cyclic vector \( \xi_0 \). Here \( \xi_0 \in D \) is said to be strongly cyclic for \( \mathcal{M} \) if \( \mathcal{M}\xi_0 = D \). With no loss of generality we will assume that \( \|\xi_0\| = 1 \). Let \( \mathcal{N} \) be an O*-subalgebra of \( \mathcal{M} \). A map \( \mathcal{E} \) of \( \mathcal{M} \) onto \( \mathcal{N} \) is said to be a conditional expectation of \( (\mathcal{M}, \xi_0) \) w.r.t. \( \mathcal{N} \) if it satisfies the following conditions:

(i) \( \mathcal{E}(X) = \mathcal{E}(X') \), \( \forall X \in \mathcal{M} \), and \( \mathcal{E}(A) = A, \forall A \in \mathcal{N} \);
(ii) \( \mathcal{E}(XA) = \mathcal{E}(X)A \) and \( \mathcal{E}(AX) = A\mathcal{E}(X) \), \( \forall X \in \mathcal{M} \), \( \forall A \in \mathcal{N} \);
(iii) \( \omega_{\xi_0}(\mathcal{E}(X)) = \omega_{\xi_0}(X) \), \( \forall X \in \mathcal{M} \), where \( \omega_{\xi_0} \) is a state on \( \mathcal{M} \) defined as \( \omega_{\xi_0}(X) = \langle X\xi_0 | \xi_0 \rangle \), \( X \in \mathcal{M} \).

In the case of von Neumann algebras, Takesaki [7] characterized the existence of conditional expectations for O*-algebras, which were first studied in [3, 4]. Ogi, Takakura and the second author considered a linear map \( \mathcal{E} \) of \( \mathcal{M} \) onto \( \mathcal{N} \) satisfying the above conditions (i), (ii) and (iii) on \( D(\mathcal{E}) \). Such a map is called an unbounded conditional expectation of \( (\mathcal{M}, \xi_0) \) w.r.t. \( \mathcal{N} \), and it was shown that there exists the largest unbounded conditional expectation \( \mathcal{E}_\mathcal{N} \) of \( (\mathcal{M}, \xi_0) \) w.r.t. \( \mathcal{N} \). Furthermore, the existence of conditional expectation was characterized, using Takesaki’s result in the case of von Neumann algebras. But a deeper analysis is needed since, at that stage, the problem was not solved even in the case of GW*-algebras. One of the reasons is that the reduction of a GW*-algebra is not necessarily a GW*-algebra, in contrast with the case of von Neumann algebras. For \( \mathcal{E}_\mathcal{N} \) to be a conditional expectation of \( (\mathcal{M}, \xi_0) \) w.r.t. \( \mathcal{N} \) (i.e., everywhere defined on \( \mathcal{M} \)), the following result given in [4, Corollary 6.2] holds.

**Lemma 3.1.** Let \( \mathcal{M} \) be a closed O*-algebra on \( D \) in \( \mathcal{H} \) such that \( \mathcal{M}\xi_0D \subset D \). Let \( \xi_0 \) be a strongly cyclic and separating vector, in the sense that \( \mathcal{M}\xi_0 = \mathcal{H} \). Suppose that \( \mathcal{N} \) is a closed O*-subalgebra of \( \mathcal{M} \) satisfying

(i) \( \mathcal{M}\xi_0D \subset D \);
(ii) \( \mathcal{N}\xi_0 \) is essentially self-adjoint for \( \mathcal{N} \);
(iii) \( \Delta_\xi_0^{it}(\mathcal{M}\xi_0')/\Delta_\xi_0^{-it} = (\mathcal{M}\xi_0')', \forall t \in \mathbb{R} \), where \( \Delta_\xi_0^{it} \) is the modular operator of the left Hilbert algebra \( (\mathcal{M}\xi_0')/\xi_0 \);
(iv) \( \mathcal{N}P_\mathcal{H} \) is a GW*-algebra on \( P_\mathcal{H}D \), where \( P_\mathcal{H} \) is a projection of \( \mathcal{H} \) onto \( \mathcal{M}\xi_0 \).

Then \( \mathcal{E}_\mathcal{N} \) is a conditional expectation of \( (\mathcal{M}, \xi_0) \) w.r.t. \( \mathcal{N} \).

By Lemma 3.1 and Theorem 2.5 we deduce the following.

**Theorem 3.2.** Let \( \mathcal{M} \) be a closed O*-algebra on \( D \) in \( \mathcal{H} \) with a strongly cyclic and separating vector \( \xi_0 \) such that \( \mathcal{M}\xi_0D \subset D \) and let \( \mathcal{N} \) be an O*-subalgebra of \( \mathcal{M} \). Suppose that \( \mathcal{N} \) is a GW*-algebra on a Fréchet domain \( D \) satisfying

(i) \( \mathcal{M}\xi_0 \) and \( (\mathcal{M}\xi_0P_\mathcal{N}D) \) are essentially self-adjoint for \( \mathcal{N} \);
(ii) \( \Delta_\xi_0^{it}(\mathcal{M}\xi_0')/\Delta_\xi_0^{-it} = (\mathcal{M}\xi_0')', \forall t \in \mathbb{R} \).

Then \( \mathcal{E}_\mathcal{N} \) is a conditional expectation of \( (\mathcal{M}, \xi_0) \) w.r.t. \( \mathcal{N} \).

By Lemma 3.1 and Theorem 2.7 we deduce the following.
Theorem 3.3. Let $(\mathcal{M}, \xi_0)$ be as in Theorem 3.2 and $\mathcal{A}$ an $O^*$-subalgebra of $\mathcal{M}$. Suppose $\mathcal{A}$ is a $GW^*$-algebra on $\mathcal{D}$ satisfying:

(i) $t_{\mathcal{A}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators in $\mathcal{A}$ whose spectral projections leave the domain $\mathcal{D}$ invariant;

(ii) $\Delta_{\xi_0}^{-it}(\mathcal{M}_w')\Delta_{\xi_0}^{it} = (\mathcal{M}_w')'$, $\forall t \in \mathbb{R}$.

Then $\mathcal{E}_{\mathcal{A}}$ is a conditional expectation of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{A}$.

Proof. By Lemma 2.6, $\mathcal{A}\xi_0$ and $(\mathcal{M}_w P_{\mathcal{A}} \mathcal{D})$ are essentially self-adjoint for $\mathcal{A}$; by Theorem 3.2 it follows that $\mathcal{E}_{\mathcal{A}}$ is a conditional expectation of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{A}$. □

Corollary 3.4. Let $(\mathcal{M}, \xi_0)$ be as in Theorem 3.2 and $\mathcal{A}$ an $O^*$-subalgebra of $\mathcal{M}$. Suppose $\mathcal{A}$ is a $GW^*$-algebra on $\mathcal{D}^{\infty}(T)$, where $T$ is an essentially self-adjoint operator in $\mathcal{A}$, and $\Delta_{\xi_0}^{-it}(\mathcal{M}_w')\Delta_{\xi_0}^{it} = (\mathcal{M}_w')'$, $\forall t \in \mathbb{R}$. Then, $\mathcal{E}_{\mathcal{A}}$ is a conditional expectation of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{A}$.

Corollary 3.5. Let $(\mathcal{M}, \xi_0)$ be as in Theorem 3.2 and $\mathcal{A}$ an $O^*$-subalgebra of $\mathcal{M}$. Suppose $\mathcal{A}$ is an $EW^*$-algebra on a Fréchet domain $\mathcal{D}$ satisfying $\Delta_{\xi_0}^{-it}(\mathcal{M}_w')\Delta_{\xi_0}^{it} = (\mathcal{M}_w')'$, $\forall t \in \mathbb{R}$. Then $\mathcal{E}_{\mathcal{A}}$ can be extended to a conditional expectation $\mathcal{E}_{\mathcal{A}}$ of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{A}$.

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