

BICOMMUTANTS OF REDUCED UNBOUNDED OPERATOR ALGEBRAS

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ABSTRACT. The unbounded bicommutant $(\mathfrak{M}_{E'})''_{wc}$ of the *reduction* of an O^* -algebra \mathfrak{M} via a given projection E' weakly commuting with \mathfrak{M} is studied, with the aim of finding conditions under which the reduction of a GW^* -algebra is a GW^* -algebra itself. The obtained results are applied to the problem of the existence of conditional expectations on O^* -algebras.

1. INTRODUCTION AND PRELIMINARIES

If \mathfrak{M} is a von Neumann algebra and E' a projection on the commutant \mathfrak{M}' of \mathfrak{M} , then it is well known that the *reduced* algebra $\mathfrak{M}_{E'} = \{XE' : X \in \mathfrak{M}\}$ is again a von Neumann algebra. The notion of a von Neumann algebra has been generalized to unbounded operator algebras by introducing several classes of O^* -algebras, such as GW^* -algebras, EW^* -algebras, etc. It is then natural and important for applications to pose the question if the *reduced* algebra of a GW^* -algebra via a projection E' picked in its weak bounded commutant is again a GW^* -algebra. In order to answer this question we undertake a more general study of the reduction process for O^* -algebras. We begin with reviewing the basic definitions and properties of O^* -algebras needed in this paper and refer to [5, 6] for more details.

Let \mathcal{H} be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and \mathcal{D} a dense subspace of \mathcal{H} . We denote by $\mathcal{L}^\dagger(\mathcal{D})$ the set of all linear operators X defined in \mathcal{D} such that $X\mathcal{D} \subset \mathcal{D}$, the domain $D(X^*)$ of its adjoint contains \mathcal{D} and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$ -algebra with the usual operations: $X + Y$, αX , XY and the involution $X \mapsto X^\dagger := X^* \upharpoonright \mathcal{D}$. A $*$ -subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ is called an O^* -algebra on \mathcal{D} . In this paper we will assume that an O^* -algebra always contains the identity I .

Let \mathfrak{M} be an O^* -algebra on \mathcal{D} . The *graph topology* $t_{\mathfrak{M}}$ on \mathcal{D} is the locally convex topology defined by the family $\{\|\cdot\|_X : X \in \mathfrak{M}\}$ of seminorms: $\|\xi\|_X = \|X\xi\|$, $\xi \in \mathcal{D}$. If the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ is complete, then \mathfrak{M} is said to be *closed*. In particular, if $\mathcal{D}[t_{\mathfrak{M}}]$ is a Fréchet space, then \mathcal{D} is called the Fréchet domain of \mathfrak{M} . More in general, we denote by $\widetilde{\mathcal{D}}(\mathfrak{M})$ the completion of the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ and put

$$\widetilde{X} := \overline{X} \upharpoonright \widetilde{\mathcal{D}}(\mathfrak{M}) \quad \text{and} \quad \widetilde{\mathfrak{M}} := \{\widetilde{X} : X \in \mathfrak{M}\}.$$

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Then $\widetilde{\mathfrak{D}}(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} D(\overline{X})$ and $\widetilde{\mathfrak{M}}$ is a closed O^* -algebra on $\widetilde{\mathfrak{D}}(\mathfrak{M})$ which is called the *closure* of \mathfrak{M} , since it is the smallest closed extension of \mathfrak{M} . We next recall the notion of self-adjointness of \mathfrak{M} . If $\mathcal{D} = \mathcal{D}^*(\mathfrak{M}) := \bigcap_{X \in \mathfrak{M}} D(X^*)$, then \mathfrak{M} is said to be *self-adjoint*. If $\widetilde{\mathfrak{D}}(\mathfrak{M}) = \mathcal{D}^*(\mathfrak{M})$, then \mathfrak{M} is said to be *essentially self-adjoint*. It is clear that

$$\begin{aligned} \mathcal{D} &\subset \widetilde{\mathfrak{D}}(\mathfrak{M}) \subset \mathcal{D}^*(\mathfrak{M}), \\ X &\subset \widetilde{X} \subset X^{\dagger*}, \quad \forall X \in \mathfrak{M}. \end{aligned}$$

The *weak commutant* \mathfrak{M}'_w of \mathfrak{M} is defined by

$$\mathfrak{M}'_w = \{C \in \mathcal{B}(\mathcal{H}) : \langle CX\xi | \eta \rangle = \langle C\xi | X^\dagger \eta \rangle, \forall X \in \mathfrak{M}, \forall \xi, \eta \in \mathcal{D}\},$$

where $\mathcal{B}(\mathcal{H})$ is the $*$ -algebra of all bounded linear operators on \mathcal{H} . Then \mathfrak{M}'_w is a weak-operator closed $*$ -invariant subspace of $\mathcal{B}(\mathcal{H})$ (but it is not, in general, a von Neumann algebra) and $(\widetilde{\mathfrak{M}})'_w = \mathfrak{M}'_w$. If $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$, as happens for self-adjoint \mathfrak{M} , then \mathfrak{M}'_w is a von Neumann algebra. In this paper we will use the following bicommutants of \mathfrak{M} :

$$(\mathfrak{M}'_w)' = \{A \in \mathcal{B}(\mathcal{H}) : AC = CA, \forall C \in \mathfrak{M}'_w\},$$

$$\mathfrak{M}''_{wc} := (\mathfrak{M}'_w)'_c = \{X \in \mathcal{L}^\dagger(\mathcal{D}) : \langle CX\xi | \eta \rangle = \langle C\xi | X^\dagger \eta \rangle, \forall C \in \mathfrak{M}'_w, \forall \xi, \eta \in \mathcal{D}\}.$$

Then, $(\mathfrak{M}'_w)'$ is a von Neumann algebra on \mathcal{H} and \mathfrak{M}''_{wc} is a τ_{s^*} -closed O^* -algebra on \mathcal{D} such that $\mathfrak{M} \subset \mathfrak{M}''_{wc}$ and $(\mathfrak{M}''_{wc})'_w = \mathfrak{M}'_w$, where the strong $*$ -topology τ_{s^*} is defined by the family $\{p_\xi^*(\cdot) : \xi \in \mathcal{D}\}$ of seminorms:

$$p_\xi^*(X) := \|X\xi\| + \|X^\dagger \xi\|, \quad X \in \mathcal{L}^\dagger(\mathcal{D}).$$

Furthermore, if $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$, then

$$\mathfrak{M}''_{wc} = \{X \in \mathcal{L}^\dagger(\mathcal{D}) : \overline{X} \text{ is affiliated with } (\mathfrak{M}'_w)'\} = \overline{(\mathfrak{M}'_w)' \upharpoonright \mathcal{D}}^{\tau_{s^*}} \cap \mathcal{L}^\dagger(\mathcal{D}).$$

An O^* -algebra \mathfrak{M} on \mathcal{D} is called a *GW $*$ -algebra* if $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$ and $\mathfrak{M} = \mathfrak{M}''_{wc}$.

Let \mathfrak{M} be a closed O^* -algebra on \mathcal{D} such that $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$ and let E' be a projection in the von Neumann algebra \mathfrak{M}'_w . Let $\mathfrak{M}_{E'}$ denote the *reduced algebra* $\mathfrak{M}_{E'} = \{XE' : X \in \mathfrak{M}\}$. Then it is known [1] that

$$\begin{aligned} (\mathfrak{M}_{E'})'_w &= (\mathfrak{M}'_w)_{E'}, \\ ((\mathfrak{M}_{E'})'_w)' &= ((\mathfrak{M}'_w)')_{E'}. \end{aligned}$$

But the following question is open.

Question. Does the equality $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$ hold?

If the answer to this question is affirmative, then the reduced algebra $\mathfrak{M}_{E'}$ of a given GW $*$ -algebra is again a GW $*$ -algebra. This question will be considered in Section 2, where we will show that if \mathcal{D} is the Fréchet domain of \mathfrak{M} and the linear span $\langle \mathfrak{M}'_w E' \mathcal{D} \rangle$ of the \mathfrak{M} -invariant subspace $\mathfrak{M}'_w E' \mathcal{D}$ is essentially self-adjoint, then $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$. An \mathfrak{M} -invariant subspace \mathcal{M} of \mathcal{D} is called *essentially self-adjoint* if the O^* -algebra $\mathfrak{M} \upharpoonright \mathcal{M} := \{X \upharpoonright \mathcal{M} : X \in \mathfrak{M}\}$ of \mathcal{M} is essentially self-adjoint.

Furthermore, we will show that, if the graph topology $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators of \mathfrak{M} , whose spectral projections leave the domain \mathcal{D} invariant, then again $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$.

In Section 3 we shall apply the results of Section 2 to the study of conditional expectations for O^* -algebras.

2. MAIN RESULTS

Let \mathfrak{M} be a closed O^* -algebra on \mathcal{D} such that $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$ and let E' be a projection in the von Neumann algebra \mathfrak{M}'_w . In this section we look for conditions for the equality $(\mathfrak{M}''_{wc})_{E'} = (\mathfrak{M}'_{E'})''_{wc}$ to hold. Since $(\mathfrak{M}'_{E'})'_w = (\mathfrak{M}'_w)_{E'}$, it follows that

$$(1) \quad (\mathfrak{M}''_{wc})_{E'} \subset ((\mathfrak{M}'_w)_{E'})'_c = (\mathfrak{M}'_{E'})''_{wc}.$$

Hence we need only to prove the converse inclusion: $(\mathfrak{M}'_{E'})''_{wc} \subset (\mathfrak{M}''_{wc})_{E'}$. An element X of $(\mathfrak{M}'_{E'})''_{wc}$ is an operator on $E'\mathcal{D}$, and so to show that $X \in (\mathfrak{M}''_{wc})_{E'}$, we need to extend X to an operator on \mathcal{D} . We will consider this extension problem below.

Let Z denote the *central support* of E' ; i.e., Z is the projection onto the closure of the subspace $\langle \mathfrak{M}'_w E' \mathcal{H} \rangle$. We define $\mathcal{E} := \langle \mathfrak{M}'_w E' \mathcal{D} \rangle \oplus (I - Z)\mathcal{D}$. Then $\langle \mathfrak{M}'_w E' \mathcal{D} \rangle \subset Z\mathcal{D}$ and $\mathcal{E} \subset \mathcal{D}$.

For $X \in (\mathfrak{M}'_{E'})''_{wc}$, we put

$$X_e \left(\sum_k C_k E' \xi_k + (I - Z)\eta \right) := \sum_k C_k X E' \xi_k, \quad C_k \in \mathfrak{M}'_w, \xi_k, \eta \in \mathcal{D}.$$

Then we have the following.

Lemma 2.1. $X_e \in \mathcal{L}^\dagger(\mathcal{E})$ and $((\mathfrak{M}'_{E'})''_{wc})_e := \{X_e : X \in (\mathfrak{M}'_{E'})''_{wc}\}$ is an O^* -algebra on \mathcal{E} .

Proof. For any $X \in (\mathfrak{M}'_{E'})''_{wc}$, $C_k, K_j \in \mathfrak{M}'_w$ and $\xi_k, \zeta_j, \eta, \eta_1 \in \mathcal{D}$ we have

$$\begin{aligned} & \left\langle X_e \left(\sum_k C_k E' \xi_k + (I - Z)\eta \right) \middle| \sum_j K_j E' \zeta_j + (I - Z)\eta_1 \right\rangle \\ &= \sum_{k,j} \langle C_k X E' \xi_k \mid K_j E' \zeta_j \rangle \\ &= \sum_{k,j} \langle E' K_j^* C_k E' X E' \xi_k \mid E' \zeta_j \rangle \\ &= \sum_{k,j} \langle E' K_j^* C_k E' \xi_k \mid X^\dagger E' \zeta_j \rangle \\ &= \left\langle \sum_k C_k E' \xi_k \middle| \sum_j K_j X^\dagger E' \zeta_j \right\rangle \\ &= \left\langle \sum_k C_k E' \xi_k + (I - Z)\eta \middle| (X_e)^\dagger \left(\sum_j K_j E' \zeta_j + (I - Z)\eta_1 \right) \right\rangle, \end{aligned}$$

which implies that X_e is well-defined and $(X_e)^\dagger = (X^\dagger)_e$. It is clear that X_e is a linear operator on \mathcal{E} . Hence $X_e \in \mathcal{L}^\dagger(\mathcal{E})$. Moreover, as is easily seen, $(XY)_e = X_e Y_e$. Hence, $e(\mathfrak{M}'_{E'}) := ((\mathfrak{M}'_{E'})''_{wc})_e$ is an O^* -algebra on \mathcal{E} . \square

Since $e(\mathfrak{M}_{E'})$ is an O^* -algebra on \mathcal{E} , its closure is an O^* -algebra on $\widetilde{\mathcal{E}}$. It is easily seen that $\widetilde{\mathcal{E}} = \overline{\langle \mathfrak{M}'_w E' \mathcal{D} \rangle}^{t_{e(\mathfrak{M}_{E'})}} \oplus (I - Z)\mathcal{D}$. On the other hand, $\mathcal{D} = Z\mathcal{D} \oplus (I - Z)\mathcal{D}$; hence, if $\overline{\langle \mathfrak{M}'_w E' \mathcal{D} \rangle}^{t_{e(\mathfrak{M}_{E'})}} = Z\mathcal{D}$, we can extend any $X \in (\mathfrak{M}_{E'})''_{wc}$ to \mathcal{D} .

Theorem 2.2. *Let \mathfrak{M} be a closed O^* -algebra on \mathcal{D} such that $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$ and let E' be a projection in the von Neumann algebra \mathfrak{M}'_w . Suppose that $\widetilde{\mathcal{E}}((\mathfrak{M}_{E'})''_{wc})_e = \mathcal{D}$ or, equivalently, $\overline{\langle \mathfrak{M}'_w E' \mathcal{D} \rangle}^{t_{e(\mathfrak{M}_{E'})}} = Z\mathcal{D}$. Then, $(\mathfrak{M}''_{wc})_{E'} = (\mathfrak{M}_{E'})''_{wc}$.*

Proof. By the assumption, for every $X \in (\mathfrak{M}_{E'})''_{wc}$, X_e extends to an operator \widetilde{X}_e on \mathcal{D} ; that is, $\widetilde{X}_e = \overline{X_e} \upharpoonright \mathcal{D}$ and $(\widetilde{X}_e)_{E'} = X$. Furthermore, we have $\widetilde{X}_e \in \mathfrak{M}''_{wc}$. Indeed, for every $C \in \mathfrak{M}'_w$, we have

$$\begin{aligned} CX_e \left(\sum_k C_k E' \xi_k + (I - Z)\eta \right) &= C \sum_k C_k X E' \xi_k \\ &= \sum_k C C_k X E' \xi_k \\ &= X_e C \left(\sum_k C_k E' \xi_k + (I - Z)\eta \right), \end{aligned}$$

for every $C_k \in \mathfrak{M}'_w$, $\xi_k, \eta \in \mathcal{D}$. Hence, $CX_e = X_e C$ on \mathcal{E} and so $C\widetilde{X}_e = \widetilde{X}_e C$. Thus $\widetilde{X}_e \in \mathfrak{M}''_{wc}$ and $X = (\widetilde{X}_e)_{E'} \in (\mathfrak{M}''_{wc})_{E'}$. In conclusion, $(\mathfrak{M}_{E'})''_{wc} \subset (\mathfrak{M}''_{wc})_{E'}$ and by (1) the equality follows. \square

The next step consists, of course, in looking for situations where the conditions of Theorem 2.2 are satisfied. We begin with the following.

Lemma 2.3. *Let \mathfrak{M} and E' be as above. Suppose that*

(i) *for every $X \in (\mathfrak{M}_{E'})''_{wc}$ there exists an element Y of \mathfrak{M}''_{wc} such that*

$$\|X_e \xi\| \leq \|Y \xi\|, \quad \forall \xi \in \mathcal{E}$$

(equivalently, $X_e^\dagger X_e \leq Y^\dagger Y$ on \mathcal{E});

(ii) $\overline{\langle \mathfrak{M}'_w E' \mathcal{D} \rangle}^{t_{\mathfrak{M}''_{wc}}} = Z\mathcal{D}$.

Then, $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$.

Proof. The conditions (i) and (ii) immediately imply that

$$Z\mathcal{D} \subset \overline{\langle \mathfrak{M}'_w E' \mathcal{D} \rangle}^{t_{e(\mathfrak{M}_{E'})}}.$$

Hence

$$\overline{\langle \mathfrak{M}'_w E' \mathcal{D} \rangle}^{t_{e(\mathfrak{M}_{E'})}} = Z\mathcal{D}.$$

By Theorem 2.2, we get $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$. \square

Remark 2.4. A comment is in order for condition (ii) of Lemma 2.3. Let us in fact consider the following statements:

(i) The \mathfrak{M}''_{wc} -invariant subspace $\langle \mathfrak{M}'_w E' \mathcal{D} \rangle$ is essentially self-adjoint for \mathfrak{M}''_{wc} .

(ii) $\overline{\langle \mathfrak{M}'_w E' \mathcal{D} \rangle}^{t_{\mathfrak{M}''_{wc}}} = Z\mathcal{D}$.

Then (i) \Rightarrow (ii). In particular if \mathfrak{M} is self-adjoint, then the two conditions are equivalent. For the proof we refer to [5].

Theorem 2.5. *Let \mathfrak{M} be a closed O^* -algebra \mathcal{D} with \mathcal{D} a Fréchet domain of \mathfrak{M} . Assume that $\mathfrak{M}'_{\mathfrak{w}}\mathcal{D} \subset \mathcal{D}$. Let E' be a projection in $\mathfrak{M}'_{\mathfrak{w}}$ and assume that $\langle \mathfrak{M}'_{\mathfrak{w}}E'\mathcal{D} \rangle$ is essentially self-adjoint for $\mathfrak{M}''_{\mathfrak{w}c}$. Then, $(\mathfrak{M}_{E'})''_{\mathfrak{w}c} = (\mathfrak{M}''_{\mathfrak{w}c})_{E'}$.*

Proof. Since $\mathcal{D}[t_{\mathfrak{M}}]$ is a Fréchet space, the topology $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of elements of \mathfrak{M} satisfying

$$\|\xi\| \leq \|T_1\xi\| \leq \|T_2\xi\|, \dots, \quad \forall \xi \in \mathcal{D}.$$

Any $X \in \mathcal{L}^\dagger(\mathcal{D})$ is a closed linear operator on the Fréchet space $\mathcal{D}[t_{\mathfrak{M}}]$ into the Hilbert space \mathcal{H} , which implies, by the closed graph theorem, that the topology $t_{\mathfrak{M}}$ is equivalent to the graph topology $t_{\mathcal{L}^\dagger(\mathcal{D})}$ defined by $\mathcal{L}^\dagger(\mathcal{D})$. This in turn implies that for every $X \in \mathcal{L}^\dagger(\mathcal{D})$ there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that

$$(2) \quad \|X\xi\| \leq \gamma\|T_{n_0}\xi\|, \quad \forall \xi \in \mathcal{D}.$$

Now take an arbitrary $X \in (\mathfrak{M}_{E'})''_{\mathfrak{w}c}$. Since $XE' \in \mathcal{L}^\dagger(\mathcal{D})$, it follows from (2) that

$$(3) \quad \|XE'\xi\| \leq \gamma\|T_{n_0}\xi\|, \quad \forall \xi \in \mathcal{D}.$$

Since $\mathfrak{M}'_{\mathfrak{w}}$ is a von Neumann algebra, for every $C \in \mathfrak{M}'_{\mathfrak{w}}$ and $\xi \in \mathcal{D}$, we have

$$\begin{aligned} \|X_e(CE'\xi)\|^2 &= \|CXE'\xi\|^2 \\ &= \langle E'C^*CE'XE'\xi | XE'\xi \rangle \\ &= \|(E'C^*CE')^{1/2}XE'\xi\|^2 \\ &= \|XE'(E'C^*CE')^{1/2}\xi\|^2 \\ &\leq \gamma^2\|T_{n_0}(E'C^*CE')^{1/2}\xi\|^2 \\ &= \gamma^2\|(E'C^*CE')^{1/2}T_{n_0}\xi\|^2 \\ &= \gamma^2\|CE'T_{n_0}\xi\|^2 \\ &= \gamma^2\|T_{n_0}CE'\xi\|^2. \end{aligned}$$

Hence, condition (i) of Lemma 2.3 is satisfied. By the same lemma we then get the equality $(\mathfrak{M}_{E'})''_{\mathfrak{w}c} = (\mathfrak{M}''_{\mathfrak{w}c})_{E'}$. □

Lemma 2.6. *Let \mathfrak{M} be a closed O^* -algebra on a Fréchet domain \mathcal{D} in \mathcal{H} such that $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators of \mathfrak{M} whose spectral projections belong to \mathfrak{M} .*

Let \mathcal{M} be an \mathfrak{M} -invariant subspace of \mathcal{D} such that the projection $P_{\mathcal{M}}$ onto $\overline{\mathcal{M}}$ belongs to $\mathfrak{M}'_{\mathfrak{w}}$.

Then \mathcal{M} is essentially self-adjoint for \mathfrak{M} or, equivalently, $P_{\mathcal{M}}\mathcal{D} = \overline{\mathcal{M}}^{t_{\mathfrak{M}}}$.

Proof. First we observe that $\overline{\mathcal{M}}^{t_{\mathfrak{M}}} \subset P_{\mathcal{M}}\mathcal{D}$. Hence we only need to show that the converse inclusion also holds. Take an arbitrary $\xi \in \mathcal{D}$. Then there exists a sequence $\{\xi_n\}$ in \mathcal{M} such that $\lim_{n \rightarrow \infty} \|\xi_n - P_{\mathcal{M}}\xi\| = 0$. By (3), for every $X \in \mathfrak{M}$, there exists $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that

$$(4) \quad \|X\xi\| \leq \gamma\|T_{n_0}\xi\|, \quad \forall \xi \in \mathcal{D}.$$

By the assumption, $\overline{T_{n_0}}$ is self-adjoint. Let $\overline{T_{n_0}} = \int_{-\infty}^{\infty} \lambda dE_{T_{n_0}}(\lambda)$ be the spectral resolution of $\overline{T_{n_0}}$. We put $E_k := E_{T_{n_0}}(k) - E_{T_{n_0}}(-k)$, $k \in \mathbb{N}$. Then, by assumption $E_k \upharpoonright \mathcal{D} \in \mathfrak{M}$, $\forall k \in \mathbb{N}$ and so, $E_k\xi_n \in \mathcal{M}$, $\forall k, n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \|E_k\xi_n - E_kP_{\mathcal{M}}\xi\| = 0$. Furthermore, by (4) we have

$$\|XE_k\xi_n - XE_k\xi_m\| \leq \gamma\|T_{n_0}E_k(\xi_n - \xi_m)\| \rightarrow 0, \text{ as } n, m \rightarrow \infty.$$

Hence $E_k P_{\mathcal{M}} \xi \in D(\overline{X \upharpoonright \mathcal{M}})$.

Furthermore,

$$\begin{aligned} \|X E_k P_{\mathcal{M}} \xi - X P_{\mathcal{M}} \xi\| &\leq \gamma \|T_{n_0} E_k P_{\mathcal{M}} \xi - T_{n_0} P_{\mathcal{M}} \xi\| \\ &= \gamma \|P_{\mathcal{M}}(E_k T_{n_0} \xi - T_{n_0} \xi)\| \\ &\leq \gamma \|E_k T_{n_0} \xi - T_{n_0} \xi\| \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

which implies that $P_{\mathcal{M}} \xi \in D(\overline{X \upharpoonright \mathcal{M}})$. Hence,

$$P_{\mathcal{M}} \xi \in \bigcap_{X \in \mathfrak{M}} D(\overline{X \upharpoonright \mathcal{M}}) = \overline{\mathcal{M}}^{t_{\mathfrak{M}}}.$$

Thus $P_{\mathcal{M}} \mathcal{D} = \overline{\mathcal{M}}^{t_{\mathfrak{M}}}$. This is equivalent to \mathcal{M} being essentially self-adjoint for \mathfrak{M} , since \mathfrak{M} is clearly self-adjoint. \square

By Theorem 2.5 and Lemma 2.6 we have the following.

Theorem 2.7. *Let \mathfrak{M} be a closed O^* -algebra such that $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators of \mathfrak{M} whose spectral projections leave the domain \mathcal{D} invariant. Let E' be a projection in \mathfrak{M}'_w . Then $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$.*

Proof. It can be shown that \mathfrak{M} is a self-adjoint O^* -algebra on the Fréchet domain \mathcal{D} , so that $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$, $t_{\mathfrak{M}''_{wc}} = t_{\mathfrak{M}}$ and \mathfrak{M}''_{wc} is a self-adjoint O^* -algebra on the Fréchet domain \mathcal{D} . Furthermore

$$P_{\langle \mathcal{M}'_w E' \mathcal{D} \rangle} = Z \in \mathfrak{M}'_w \cap (\mathfrak{M}'_w)' = (\mathfrak{M}''_{wc})'_w \cap ((\mathfrak{M}''_{wc})'_w)'.$$

Moreover, since the spectral projections of the operators T_n leave the domain \mathcal{D} invariant, they automatically belong to \mathfrak{M}''_{wc} . Hence, the self-adjoint O^* -algebra \mathfrak{M}''_{wc} and the \mathfrak{M}''_{wc} -invariant subspace $\langle \mathcal{M}'_w E' \mathcal{D} \rangle$ of \mathcal{D} satisfy all the conditions in Lemma 2.6, and so $Z \mathcal{D} = \overline{\langle \mathcal{M}'_w E' \mathcal{D} \rangle}^{t_{\mathfrak{M}''_{wc}}}$. Then, by Theorem 2.5, $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$. \square

Corollary 2.8. *Let T be an essentially self-adjoint operator in \mathcal{H} and \mathfrak{M} be a self-adjoint O^* -algebra on $\mathcal{D}^\infty(\overline{T}) := \bigcap_{n \in \mathbb{N}} D(\overline{T}^n)$, containing T . Let E' be a projection in \mathfrak{M}'_w . Then $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$.*

Proof. The spectral projections $E_T(\lambda)$, $\lambda \in \mathbb{R}$, of \overline{T} satisfy:

- $E_T(\lambda) \in (\mathfrak{M}'_w)'$, $\forall \lambda \in \mathbb{R}$;
- $E_T(\lambda) \mathcal{H} \subset \mathcal{D}^\infty(\overline{T})$, $\forall \lambda \in \mathbb{R}$.

The statement then follows from Theorem 2.7. \square

Apart from GW^* -algebras, another unbounded generalization of von Neumann algebras is provided by the notion of *extended W^* -algebras*, shortly EW^* -algebras, defined as follows: A closed O^* -algebra \mathfrak{M} on \mathcal{D} is said to be an EW^* -algebra if $(I + X^\dagger X)^{-1}$ exists in $\mathfrak{M}_b := \{A \in \mathfrak{M} : \overline{A} \in \mathcal{B}(\mathcal{H})\}$, for every $X \in \mathfrak{M}$ and $\overline{\mathfrak{M}}_b := \{\overline{A} \in \mathfrak{M} : A \in \mathfrak{M}_b\}$ is a von Neumann algebra [2]. It is easily shown that every EW^* -algebra on a Fréchet domain satisfies the conditions of Theorem 2.7 (recall that every symmetric element of an EW^* -algebra is essentially self-adjoint). Hence we have the following.

Corollary 2.9. *Let \mathfrak{M} be an EW^* -algebra on the Fréchet domain \mathcal{D} in Hilbert space \mathcal{H} and E' a projection in \mathfrak{M}'_w . Then, $(\mathfrak{M}_{E'})''_{wc} = (\mathfrak{M}''_{wc})_{E'}$.*

3. APPLICATIONS

In this section we show how to use the results of Section 2 in the analysis of the existence of conditional expectations for O^* -algebras, which were first studied in [3, 4].

Let \mathfrak{M} be a given O^* -algebra on \mathcal{D} in \mathcal{H} with a strongly cyclic vector ξ_0 . Here $\xi_0 \in \mathcal{D}$ is said to be strongly cyclic for \mathfrak{M} if $\overline{\mathfrak{M}\xi_0}^{\mathfrak{M}} = \mathcal{D}$. With no loss of generality we will assume that $\|\xi_0\| = 1$. Let \mathfrak{N} be an O^* -subalgebra of \mathfrak{M} . A map \mathcal{E} of \mathfrak{M} onto \mathfrak{N} is said to be a conditional expectation of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} if it satisfies the following conditions:

- (i) $\mathcal{E}(X)^\dagger = \mathcal{E}(X^\dagger)$, $\forall X \in \mathfrak{M}$, and $\mathcal{E}(A) = A$, $\forall A \in \mathfrak{N}$;
- (ii) $\mathcal{E}(XA) = \mathcal{E}(X)A$ and $\mathcal{E}(AX) = A\mathcal{E}(X)$, $\forall X \in \mathfrak{M}$, $\forall A \in \mathfrak{N}$;
- (iii) $\omega_{\xi_0}(\mathcal{E}(X)) = \omega_{\xi_0}(X)$, $\forall X \in \mathfrak{M}$, where ω_{ξ_0} is a state on \mathfrak{M} defined as $\omega_{\xi_0}(X) = \langle X\xi_0 | \xi_0 \rangle$, $X \in \mathfrak{M}$.

In the case of von Neumann algebras, Takesaki [7] characterized the existence of conditional expectations using Tomita’s modular theory. Thus a conditional expectation does not necessarily exist for a general von Neumann algebra. Hence, in [3, 4], Ogi, Takakura and the second author considered a linear map \mathcal{E} of a \dagger -invariant subspace $D(\mathcal{E})$ of \mathfrak{M} onto \mathfrak{N} satisfying the above conditions (i), (ii) and (iii) on $D(\mathcal{E})$. Such a map is called an *unbounded conditional expectation* of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} , and it was shown that there exists the largest unbounded conditional expectation $\mathcal{E}_{\mathfrak{N}}$ of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} . Furthermore, the existence of conditional expectation was characterized, using Takesaki’s result in the case of von Neumann algebras. But a deeper analysis is needed since, at that stage, the problem was not solved even in the case of GW^* -algebras. One of the reasons is that the reduction of a GW^* -algebra is not necessarily a GW^* -algebra, in contrast with the case of von Neumann algebras. For $\mathcal{E}_{\mathfrak{N}}$ to be a conditional expectation of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} (i.e., everywhere defined on \mathfrak{M}), the following result given in [4, Corollary 6.2] holds.

Lemma 3.1. *Let \mathfrak{M} be a closed O^* -algebra on \mathcal{D} in \mathcal{H} such that $\mathfrak{M}'_w\mathcal{D} \subset \mathcal{D}$. Let ξ_0 be a strongly cyclic and separating vector, in the sense that $\overline{\mathfrak{M}'_w\xi_0} = \mathcal{H}$. Suppose that \mathfrak{N} is a closed O^* -subalgebra of \mathfrak{M} satisfying*

- (i) $\mathfrak{N}'_w\mathcal{D} \subset \mathcal{D}$;
- (ii) $\mathfrak{N}\xi_0$ is essentially self-adjoint for \mathfrak{N} ;
- (iii) $\Delta''_{\xi_0}{}^{it}(\mathfrak{N}'_w)' \Delta''_{\xi_0}{}^{-it} = (\mathfrak{N}'_w)'$, $\forall t \in \mathbb{R}$, where Δ''_{ξ_0} is the modular operator of the left Hilbert algebra $(\mathfrak{M}'_w)' \xi_0$;
- (iv) $\mathfrak{N}_{P_{\mathfrak{N}}}$ is a GW^* -algebra on $P_{\mathfrak{N}}\mathcal{D}$, where $P_{\mathfrak{N}}$ is a projection of \mathcal{H} onto $\overline{\mathfrak{N}\xi_0}$.

Then $\mathcal{E}_{\mathfrak{N}}$ is a conditional expectation of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} .

By Lemma 3.1 and Theorem 2.5 we deduce the following.

Theorem 3.2. *Let \mathfrak{M} be a closed O^* -algebra on \mathcal{D} in \mathcal{H} with a strongly cyclic and separating vector ξ_0 such that $\mathfrak{M}'_w\mathcal{D} \subset \mathcal{D}$ and let \mathfrak{N} be an O^* -subalgebra of \mathfrak{M} .*

Suppose that \mathfrak{N} is a GW^ -algebra on a Fréchet domain \mathcal{D} satisfying*

- (i) $\mathfrak{N}\xi_0$ and $\langle \mathfrak{N}'_w P_{\mathfrak{N}}\mathcal{D} \rangle$ are essentially self-adjoint for \mathfrak{N} ;
- (ii) $\Delta''_{\xi_0}{}^{it}(\mathfrak{N}'_w)' \Delta''_{\xi_0}{}^{-it} = (\mathfrak{N}'_w)'$, $\forall t \in \mathbb{R}$.

Then $\mathcal{E}_{\mathfrak{N}}$ is a conditional expectation of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} .

By Lemma 3.1 and Theorem 2.7 we deduce the following.

Theorem 3.3. *Let (\mathfrak{M}, ξ_0) be as in Theorem 3.2 and \mathfrak{N} an O^* -subalgebra of \mathfrak{M} . Suppose \mathfrak{N} is a GW^* -algebra on \mathcal{D} satisfying:*

- (i) $t_{\mathfrak{N}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators in \mathfrak{N} whose spectral projections leave the domain \mathcal{D} invariant;
- (ii) $\Delta''_{\xi_0}{}^{it} (\mathfrak{N}'_w)' \Delta''_{\xi_0}{}^{-it} = (\mathfrak{N}'_w)', \forall t \in \mathbb{R}$.

Then $\mathcal{E}_{\mathfrak{N}}$ is a conditional expectation of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} .

Proof. By Lemma 2.6, $\mathfrak{N}\xi_0$ and $\langle \mathfrak{N}'_w P_{\mathfrak{N}} \mathcal{D} \rangle$ are essentially self-adjoint for \mathfrak{N} ; by Theorem 3.2 it follows that $\mathcal{E}_{\mathfrak{N}}$ is a conditional expectation of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} . \square

Corollary 3.4. *Let (\mathfrak{M}, ξ_0) be as in Theorem 3.2 and \mathfrak{N} an O^* -subalgebra of \mathfrak{M} . Suppose \mathfrak{N} is a GW^* -algebra on $\mathcal{D}^\infty(\overline{T})$, where T is an essentially self-adjoint operator in \mathfrak{N} , and $\Delta''_{\xi_0}{}^{it} (\mathfrak{N}'_w)' \Delta''_{\xi_0}{}^{-it} = (\mathfrak{N}'_w)', \forall t \in \mathbb{R}$. Then, $\mathcal{E}_{\mathfrak{N}}$ is a conditional expectation of (\mathfrak{M}, ξ_0) w.r.t. \mathfrak{N} .*

Corollary 3.5. *Let (\mathfrak{M}, ξ_0) be as in Theorem 3.2 and \mathfrak{N} an O^* -subalgebra of \mathfrak{M} . Suppose \mathfrak{N} is an EW^* -algebra on a Fréchet domain \mathcal{D} satisfying $\Delta''_{\xi_0}{}^{it} (\mathfrak{N}'_w)' \Delta''_{\xi_0}{}^{-it} = (\mathfrak{N}'_w)', \forall t \in \mathbb{R}$. Then $\mathcal{E}_{\mathfrak{N}}$ can be extended to a conditional expectation $\mathcal{E}_{\mathfrak{N}'_{wc}}$ of $(\mathfrak{M}'_{wc}, \xi_0)$ w.r.t. \mathfrak{N}'_{wc} .*

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