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BICOMMUTANTS OF REDUCED UNBOUNDED OPERATOR ALGEBRAS

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Abstract. The unbounded bicommutant $(\mathfrak{M}_{E'})''_{\rm wc}$ of the *reduction* of an O*-algebra $\mathfrak{M}$ via a given projection E' weakly commuting with $\mathfrak{M}$ is studied, with the aim of finding conditions under which the reduction of a GW*-algebra is a GW*-algebra itself. The obtained results are applied to the problem of the existence of conditional expectations on O*-algebras.

1. Introduction and preliminaries

If $\mathfrak{M}$ is a von Neumann algebra and E' a projection on the commutant $\mathfrak{M}'$ of $\mathfrak{M}$, then it is well known that the *reduced* algebra $\mathfrak{M}_{E'} = \{XE' \ : \ X \in \mathfrak{M}\}$ is again a von Neumann algebra. The notion of a von Neumann algebra has been generalized to unbounded operator algebras by introducing several classes of O*-algebras, such as GW*-algebras, EW*-algebras, etc. It is then natural and important for applications to pose the question if the *reduced* algebra of a GW*-algebra via a projection E' picked in its weak bounded commutant is again a GW*-algebra. In order to answer this question we undertake a more general study of the reduction process for O*-algebras. We begin with reviewing the basic definitions and properties of O*-algebras needed in this paper and refer to [5, 6] for more details.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D})$ the set of all linear operators X defined in $\mathcal{D}$ such that $X\mathcal{D} \subset \mathcal{D}$, the domain $D(X^*)$ of its adjoint contains $\mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with the usual operations: $X+Y$, αX, XY and the involution $X \mapsto X^\dagger := X^* \lceil \mathcal{D}$. A *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ is called an O*-algebra on $\mathcal{D}$. In this paper we will assume that an O*-algebra always contains the identity I.

Let $\mathfrak{M}$ be an O*-algebra on $\mathcal{D}$. The *graph topology* $t_{\mathfrak{M}}$ on $\mathcal{D}$ is the locally convex topology defined by the family $\{\|\cdot\|_X \ : \ X \in \mathfrak{M}\}$ of seminorms: $\|\xi\|_X = \|X\xi\|$, $\xi \in \mathcal{D}$. If the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ is complete, then $\mathfrak{M}$ is said to be *closed*. In particular, if $\mathcal{D}[t_{\mathfrak{M}}]$ is a Fréchet space, then $\mathcal{D}$ is called the Fréchet domain of $\mathfrak{M}$. More in general, we denote by $\widetilde{\mathcal{D}}(\mathfrak{M})$ the completion of the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ and put

$$\widetilde{X} := \overline{X} \lceil \widetilde{\mathcal{D}}(\mathfrak{M}) \quad \text{ and } \widetilde{\mathfrak{M}} := \{\widetilde{X} : X \in \mathfrak{M}\}.$$

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Then $\widetilde{\mathcal{D}}(\mathfrak{M}) = \bigcap_{X\in\mathfrak{M}} D(\overline{X})$ and $\widetilde{\mathfrak{M}}$ is a closed O*-algebra on $\widetilde{\mathcal{D}}(\mathfrak{M})$ which is called the *closure* of $\mathfrak{M}$, since it is the smallest closed extension of $\mathfrak{M}$. We next recall the notion of self-adjointness of $\mathfrak{M}$. If $\mathcal{D} = \mathcal{D}^*(\mathfrak{M}) := \bigcap_{X\in\mathfrak{M}} D(X^*)$, then $\mathfrak{M}$ is said to be *self-adjoint*. If $\widetilde{\mathcal{D}}(\mathfrak{M}) = \mathcal{D}^*(\mathfrak{M})$, then $\mathfrak{M}$ is said to be *essentially self-adjoint*. It is clear that

$$\mathcal{D} \subset \widetilde{\mathcal{D}}(\mathfrak{M}) \subset \mathcal{D}^*(\mathfrak{M}),$$
$$X \subset \widetilde{X} \subset X^{\dagger *}, \quad \forall X \in \mathfrak{M}.$$

The *weak commutant* $\mathfrak{M}'_{\mathrm{w}}$ of $\mathfrak{M}$ is defined by

$$\mathfrak{M}'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H}) : \langle CX\xi|\, \eta\rangle = \langle C\xi|\, X^\dagger\eta\rangle\,,\ \forall X \in \mathfrak{M}, \forall \xi,\eta \in \mathcal{D}\},$$

where $\mathcal{B}(\mathcal{H})$ is the *-algebra of all bounded linear operators on $\mathcal{H}$. Then $\mathfrak{M}'_{\mathrm{w}}$ is a weak-operator closed *-invariant subspace of $\mathcal{B}(\mathcal{H})$ (but it is not, in general, a von Neumann algebra) and $(\widetilde{\mathfrak{M}})'_{\mathrm{w}} = \mathfrak{M}'_{\mathrm{w}}$. If $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$, as happens for self-adjoint $\mathfrak{M}$, then $\mathfrak{M}'_{\mathrm{w}}$ is a von Neumann algebra. In this paper we will use the following bicommutants of $\mathfrak{M}$:

$$(\mathfrak{M}'_{\mathrm{w}})' = \{A \in \mathcal{B}(\mathcal{H}) :\ AC = CA, \forall C \in \mathfrak{M}'_{\mathrm{w}}\},$$
$$\mathfrak{M}''_{\mathrm{wc}} := (\mathfrak{M}'_{\mathrm{w}})'_{\mathrm{c}} = \{X \in \mathcal{L}^\dagger(\mathcal{D}) :\ \langle CX\xi|\, \eta\rangle = \langle C\xi|\, X^\dagger\eta\rangle\,,\ \forall C \in \mathfrak{M}'_{\mathrm{w}}, \forall \xi,\eta \in \mathcal{D}\}.$$

Then, $(\mathfrak{M}'_{\mathrm{w}})'$ is a von Neumann algebra on $\mathcal{H}$ and $\mathfrak{M}''_{\mathrm{wc}}$ is a τ_{s^*}-closed O*-algebra on $\mathcal{D}$ such that $\mathfrak{M} \subset \mathfrak{M}''_{\mathrm{wc}}$ and $(\mathfrak{M}''_{\mathrm{wc}})'_{\mathrm{w}} = \mathfrak{M}'_{\mathrm{w}}$, where the strong*-topology τ_{s^*} is defined by the family $\{p^*_\xi(\cdot)\ :\ \xi \in \mathcal{D}\}$ of seminorms:

$$p^*_\xi(X) := \|X\xi\| + \|X^\dagger\xi\|, \quad X \in \mathcal{L}^\dagger(\mathcal{D}).$$

Furthermore, if $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$, then

$$\mathfrak{M}''_{\mathrm{wc}} = \{X \in \mathcal{L}^\dagger(\mathcal{D}) :\ \overline{X} \text{ is affiliated with } (\mathfrak{M}'_{\mathrm{w}})'\} = \overline{(\mathfrak{M}'_{\mathrm{w}})'\lceil\mathcal{D}}^{\tau_{s^*}} \cap \mathcal{L}^\dagger(\mathcal{D}).$$

An O*-algebra $\mathfrak{M}$ on $\mathcal{D}$ is called a *GW*-algebra* if $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$ and $\mathfrak{M} = \mathfrak{M}''_{\mathrm{wc}}$.

Let $\mathfrak{M}$ be a closed O*-algebra on $\mathcal{D}$ such that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$ and let E' be a projection in the von Neumann algebra $\mathfrak{M}'_{\mathrm{w}}$. Let $\mathfrak{M}_{E'}$ denote the *reduced algebra* $\mathfrak{M}_{E'} = \{XE' :\ X \in \mathfrak{M}\}$. Then it is known [1] that

$$(\mathfrak{M}_{E'})'_{\mathrm{w}} = (\mathfrak{M}'_{\mathrm{w}})_{E'},$$
$$((\mathfrak{M}_{E'})'_{\mathrm{w}})' = ((\mathfrak{M}'_{\mathrm{w}})')_{E'}.$$

But the following question is open.

Question. Does the equality $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$ hold?

If the answer to this question is affirmative, then the reduced algebra $\mathfrak{M}_{E'}$ of a given GW*-algebra is again a GW*-algebra. This question will be considered in Section 2, where we will show that if $\mathcal{D}$ is the Fréchet domain of $\mathfrak{M}$ and the linear span $\langle\mathfrak{M}'_{\mathrm{w}}E'\mathcal{D}\rangle$ of the $\mathfrak{M}$-invariant subspace $\mathfrak{M}'_{\mathrm{w}}E'\mathcal{D}$ is essentially self-adjoint, then $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$. An $\mathfrak{M}$-invariant subspace $\mathcal{M}$ of $\mathcal{D}$ is called *essentially self-adjoint* if the O*-algebra $\mathfrak{M}\lceil\mathcal{M} := \{X\lceil\mathcal{M}\ :\ X \in \mathfrak{M}\}$ of $\mathcal{M}$ is essentially self-adjoint.

Furthermore, we will show that, if the graph topology $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators of $\mathfrak{M}$, whose spectral projections leave the domain $\mathcal{D}$ invariant, then again $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$.

In Section 3 we shall apply the results of Section 2 to the study of conditional expectations for O*-algebras.

2. Main results

Let $\mathfrak{M}$ be a closed O*-algebra on $\mathcal{D}$ such that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$ and let E' be a projection in the von Neumann algebra $\mathfrak{M}'_{\mathrm{w}}$. In this section we look for conditions for the equality $(\mathfrak{M}''_{\mathrm{wc}})_{E'} = (\mathfrak{M}_{E'})''_{\mathrm{wc}}$ to hold. Since $(\mathfrak{M}_{E'})'_{\mathrm{w}} = (\mathfrak{M}'_{\mathrm{w}})_{E'}$, it follows that

$$(\mathfrak{M}''_{\mathrm{wc}})_{E'} \subset ((\mathfrak{M}'_{\mathrm{w}})_{E'})'_{\mathrm{c}} = (\mathfrak{M}_{E'})''_{\mathrm{wc}}. \tag{1}$$

Hence we need only to prove the converse inclusion: $(\mathfrak{M}_{E'})''_{\mathrm{wc}} \subset (\mathfrak{M}''_{\mathrm{wc}})_{E'}$. An element X of $(\mathfrak{M}_{E'})''_{\mathrm{wc}}$ is an operator on $E'\mathcal{D}$, and so to show that $X \in (\mathfrak{M}''_{\mathrm{wc}})_{E'}$, we need to extend X to an operator on $\mathcal{D}$. We will consider this extension problem below.

Let Z denote the *central support* of E'; i.e., Z is the projection onto the closure of the subspace $\langle \mathfrak{M}'_{\mathrm{w}} E'\mathcal{H}\rangle$. We define $\mathcal{E} := \langle \mathfrak{M}'_{\mathrm{w}} E'\mathcal{D}\rangle \oplus (I-Z)\mathcal{D}$. Then $\langle \mathfrak{M}'_{\mathrm{w}} E'\mathcal{D}\rangle \subset ZD$ and $\mathcal{E} \subset \mathcal{D}$.

For $X \in (\mathfrak{M}_{E'})''_{\mathrm{wc}}$, we put

$$X_e\left(\sum_k C_k E'\xi_k + (I-Z)\eta\right) := \sum_k C_k X E'\xi_k, \quad C_k \in \mathfrak{M}'_{\mathrm{w}}, \xi_k, \eta \in \mathcal{D}.$$

Then we have the following.

Lemma 2.1. *$X_e \in \mathcal{L}^\dagger(\mathcal{E})$ and $((\mathfrak{M}_{E'})''_{\mathrm{wc}})_e := \{X_e : X \in (\mathfrak{M}_{E'})''_{\mathrm{wc}}\}$ is an O*-algebra on $\mathcal{E}$.*

Proof. For any $X \in (\mathfrak{M}_{E'})''_{\mathrm{wc}}$, $C_k, K_j \in \mathfrak{M}'_{\mathrm{w}}$ and $\xi_k, \zeta_j, \eta, \eta_1 \in \mathcal{D}$ we have

$$\begin{aligned}
&\left\langle X_e\left(\sum_k C_k E'\xi_k + (I-Z)\eta\right) \middle| \sum_j K_j E'\zeta_j + (I-Z)\eta_1\right\rangle \\
&= \sum_{k,j} \langle C_k X E'\xi_k | \, K_j E'\zeta_j\rangle \\
&= \sum_{k,j} \langle E'K_j^* C_k E' X E'\xi_k | \, E'\zeta_j\rangle \\
&= \sum_{k,j} \langle E'K_j^* C_k E'\xi_k | \, X^\dagger E'\zeta_j\rangle \\
&= \left\langle \sum_k C_k E'\xi_k \middle| \sum_j K_j X^\dagger E'\zeta_j\right\rangle \\
&= \left\langle \sum_k C_k E'\xi_k + (I-Z)\eta \middle| (X_e)^\dagger\left(\sum_j K_j E'\zeta_j + (I-Z)\eta_1\right)\right\rangle,
\end{aligned}$$

which implies that X_e is well-defined and $(X_e)^\dagger = (X^\dagger)_e$. It is clear that X_e is a linear operator on $\mathcal{E}$. Hence $X_e \in \mathcal{L}^\dagger(\mathcal{E})$. Moreover, as is easily seen, $(XY)_e = X_e Y_e$. Hence, $e(\mathfrak{M}_{E'}) := ((\mathfrak{M}_{E'})''_{\mathrm{wc}})_e$ is an O*-algebra on $\mathcal{E}$. □

Since $e(\mathfrak{M}_{E'})$ is an O*-algebra on $\mathcal{E}$, its closure is an O*-algebra on $\widetilde{\mathcal{E}}$. It is easily seen that $\widetilde{\mathcal{E}} = \overline{\langle \mathfrak{M}'_{\rm w} E'\mathcal{D}\rangle}^{t_{e(\mathfrak{M}_{E'})}} \oplus (I-Z)\mathcal{D}$. On the other hand, $\mathcal{D} = Z\mathcal{D} \oplus (I-Z)\mathcal{D}$; hence, if $\overline{\langle \mathfrak{M}'_{\rm w} E'\mathcal{D}\rangle}^{t_{e(\mathfrak{M}_{E'})}} = Z\mathcal{D}$, we can extend any $X \in (\mathfrak{M}_{E'})''_{\rm wc}$ to $\mathcal{D}$.

Theorem 2.2. *Let $\mathfrak{M}$ be a closed O*-algebra on $\mathcal{D}$ such that $\mathfrak{M}'_{\rm w}\mathcal{D} \subset \mathcal{D}$ and let E' be a projection in the von Neumann algebra $\mathfrak{M}'_{\rm w}$. Suppose that $\widetilde{\mathcal{E}}((\mathfrak{M}_{E'})''_{\rm wc})_e) = \mathcal{D}$ or, equivalently, $\overline{\langle \mathfrak{M}'_{\rm w} E'\mathcal{D}\rangle}^{t_{e(\mathfrak{M}_{E'})}} = Z\mathcal{D}$. Then, $(\mathfrak{M}''_{\rm wc})_{E'} = (\mathfrak{M}_{E'})''_{\rm wc}$.*

Proof. By the assumption, for every $X \in (\mathfrak{M}_{E'})''_{\rm wc}$, X_e extends to an operator $\widetilde{X_e}$ on $\mathcal{D}$; that is, $\widetilde{X_e} = \overline{X_e}\lceil\mathcal{D}$ and $(\widetilde{X_e})_{E'} = X$. Furthermore, we have $\widetilde{X_e} \in \mathfrak{M}''_{\rm wc}$. Indeed, for every $C \in \mathfrak{M}'_{\rm w}$, we have

$$\begin{aligned} CX_e\left(\sum_k C_k E'\xi_k + (I-Z)\eta\right) &= C\sum_k C_k X E'\xi_k \\ &= \sum_k CC_k XE'\xi_k \\ &= X_e C\left(\sum_k C_k E'\xi_k + (I-Z)\eta\right), \end{aligned}$$

for every $C_k \in \mathfrak{M}'_{\rm w}$, $\xi_k, \eta \in \mathcal{D}$. Hence, $CX_e = X_e C$ on $\mathcal{E}$ and so $C\widetilde{X_e} = \widetilde{X_e}C$. Thus $\widetilde{X_e} \in \mathfrak{M}''_{\rm wc}$ and $X = (\widetilde{X_e})_{E'} \in (\mathfrak{M}''_{\rm wc})_{E'}$. In conclusion, $(\mathfrak{M}_{E'})''_{\rm wc} \subset (\mathfrak{M}''_{\rm wc})_{E'}$ and by (1) the equality follows. □

The next step consists, of course, in looking for situations where the conditions of Theorem 2.2 are satisfied. We begin with the following.

Lemma 2.3. *Let $\mathfrak{M}$ and E' be as above. Suppose that*

(i) *for every $X \in (\mathfrak{M}_{E'})''_{\rm wc}$ there exists an element Y of $\mathfrak{M}''_{\rm wc}$ such that*

$$\|X_e\xi\| \leq \|Y\xi\|, \quad \forall \xi \in \mathcal{E}$$

(equivalently, $X_e^\dagger X_e \leq Y^\dagger Y$ on $\mathcal{E}$);

(ii) $\overline{\langle \mathfrak{M}'_{\rm w} E'\mathcal{D}\rangle}^{t_{\mathfrak{M}''_{\rm wc}}} = Z\mathcal{D}$.

Then, $(\mathfrak{M}_{E'})''_{\rm wc} = (\mathfrak{M}''_{\rm wc})_{E'}$.

Proof. The conditions (i) and (ii) immediately imply that

$$Z\mathcal{D} \subset \overline{\langle \mathfrak{M}'_{\rm w} E'\mathcal{D}\rangle}^{t_{e(\mathfrak{M}_{E'})}}.$$

Hence

$$\overline{\langle \mathfrak{M}'_{\rm w} E'\mathcal{D}\rangle}^{t_{e(\mathfrak{M}_{E'})}} = Z\mathcal{D}.$$

By Theorem 2.2, we get $(\mathfrak{M}_{E'})''_{\rm wc} = (\mathfrak{M}''_{\rm wc})_{E'}$. □

Remark 2.4. A comment is in order for condition (ii) of Lemma 2.3. Let us in fact consider the following statements:

(i) The $\mathfrak{M}''_{\rm wc}$-invariant subspace $\langle \mathfrak{M}'_{\rm w} E'\mathcal{D}\rangle$ is essentially self-adjoint for $\mathfrak{M}''_{\rm wc}$.

(ii) $\overline{\langle \mathfrak{M}'_{\rm w} E'\mathcal{D}\rangle}^{t_{\mathfrak{M}''_{\rm wc}}} = Z\mathcal{D}$.

Then (i) ⇒ (ii). In particular if $\mathfrak{M}$ is self-adjoint, then the two conditions are equivalent. For the proof we refer to [5].

Theorem 2.5. *Let $\mathfrak{M}$ be a closed O^*-algebra $\mathcal{D}$ with $\mathcal{D}$ a Fréchet domain of $\mathfrak{M}$. Assume that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. Let E' be a projection in $\mathfrak{M}'_{\mathrm{w}}$ and assume that $\langle \mathfrak{M}'_{\mathrm{w}} E'\mathcal{D}\rangle$ is essentially self-adjoint for $\mathfrak{M}''_{\mathrm{wc}}$. Then, $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$.*

Proof. Since $\mathcal{D}[t_{\mathfrak{M}}]$ is a Fréchet space, the topology $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of elements of $\mathfrak{M}$ satisfying

$$\|\xi\| \leq \|T_1\xi\| \leq \|T_2\xi\|, \ldots, \quad \forall \xi \in \mathcal{D}.$$

Any $X \in \mathcal{L}^\dagger(\mathcal{D})$ is a closed linear operator on the Fréchet space $\mathcal{D}[t_{\mathfrak{M}}]$ into the Hilbert space $\mathcal{H}$, which implies, by the closed graph theorem, that the topology $t_{\mathfrak{M}}$ is equivalent to the graph topology $t_{\mathcal{L}^\dagger(\mathcal{D})}$ defined by $\mathcal{L}^\dagger(\mathcal{D})$. This in turn implies that for every $X \in \mathcal{L}^\dagger(\mathcal{D})$ there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that

$$\|X\xi\| \leq \gamma \|T_{n_0}\xi\|, \quad \forall \xi \in \mathcal{D}. \tag{2}$$

Now take an arbitrary $X \in (\mathfrak{M}_{E'})''_{\mathrm{wc}}$. Since $XE' \in \mathcal{L}^\dagger(\mathcal{D})$, it follows from (2) that

$$\|XE'\xi\| \leq \gamma \|T_{n_0}\xi\|, \quad \forall \xi \in \mathcal{D}. \tag{3}$$

Since $\mathfrak{M}'_{\mathrm{w}}$ is a von Neumann algebra, for every $C \in \mathfrak{M}'_{\mathrm{w}}$ and $\xi \in \mathcal{D}$, we have

$$\begin{aligned} \|X_e(CE'\xi)\|^2 &= \|CXE'\xi\|^2 \\ &= \langle E'C^*CE'XE'\xi |\, XE'\xi\rangle \\ &= \|(E'C^*CE')^{1/2}XE'\xi\|^2 \\ &= \|XE'(E'C^*CE')^{1/2}\xi\|^2 \\ &\leq \gamma^2\|T_{n_0}(E'C^*CE')^{1/2}\xi\|^2 \\ &= \gamma^2\|(E'C^*CE')^{1/2}T_{n_0}\xi\|^2 \\ &= \gamma^2\|CE'T_{n_0}\xi\|^2 \\ &= \gamma^2\|T_{n_0}CE'\xi\|^2. \end{aligned}$$

Hence, condition (i) of Lemma 2.3 is satisfied. By the same lemma we then get the equality $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$. □

Lemma 2.6. *Let $\mathfrak{M}$ be a closed O^*-algebra on a Fréchet domain $\mathcal{D}$ in $\mathcal{H}$ such that $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators of $\mathfrak{M}$ whose spectral projections belong to $\mathfrak{M}$.*

Let $\mathcal{M}$ be an $\mathfrak{M}$-invariant subspace of $\mathcal{D}$ such that the projection $P_{\mathcal{M}}$ onto $\overline{\mathcal{M}}$ belongs to $\mathfrak{M}'_w$.

Then $\mathcal{M}$ is essentially self-adjoint for $\mathfrak{M}$ or, equivalently, $P_{\mathcal{M}}\mathcal{D} = \overline{\mathcal{M}}^{t_{\mathfrak{M}}}$.

Proof. First we observe that $\overline{\mathcal{M}}^{t_{\mathfrak{M}}} \subset P_{\mathcal{M}}\mathcal{D}$. Hence we only need to show that the converse inclusion also holds. Take an arbitrary $\xi \in \mathcal{D}$. Then there exists a sequence $\{\xi_n\}$ in $\mathcal{M}$ such that $\lim_{n\to\infty} \|\xi_n - P_{\mathcal{M}}\xi\| = 0$. By (3), for every $X \in \mathfrak{M}$, there exists $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that

$$\|X\xi\| \leq \gamma \|T_{n_0}\xi\|, \quad \forall \xi \in \mathcal{D}. \tag{4}$$

By the assumption, $\overline{T_{n_0}}$ is self-adjoint. Let $\overline{T_{n_0}} = \int_{-\infty}^{\infty} \lambda dE_{T_{n_0}}(\lambda)$ be the spectral resolution of $\overline{T_{n_0}}$. We put $E_k := E_{T_{n_0}}(k) - E_{T_{n_0}}(-k)$, $k \in \mathbb{N}$. Then, by assumption $E_k\lceil\mathcal{D} \in \mathfrak{M}$, $\forall k \in \mathbb{N}$ and so, $E_k\xi_n \in \mathcal{M}$, $\forall k, n \in \mathbb{N}$ and $\lim_{n\to\infty} \|E_k\xi_n - E_kP_{\mathcal{M}}\xi\| = 0$. Furthermore, by (4) we have

$$\|XE_k\xi_n - XE_k\xi_m\| \leq \gamma\|T_{n_0}E_k(\xi_n - \xi_m)\| \to 0, \text{ as } n, m \to \infty.$$

Hence $E_k P_{\mathcal{M}}\xi \in D(\overline{X\lceil \mathcal{M}})$.

Furthermore,

$$\begin{aligned}\|XE_kP_{\mathcal{M}}\xi - XP_{\mathcal{M}}\xi\| &\leq \gamma\|T_{n_0}E_kP_{\mathcal{M}}\xi - T_{n_0}P_{\mathcal{M}}\xi\| \\ &= \gamma\|P_{\mathcal{M}}(E_kT_{n_0}\xi - T_{n_0}\xi)\| \\ &\leq \gamma\|E_kT_{n_0}\xi - T_{n_0}\xi\| \to 0 \text{ as } k \to \infty,\end{aligned}$$

which implies that $P_{\mathcal{M}}\xi \in D(\overline{X\lceil \mathcal{M}})$. Hence,

$$P_{\mathcal{M}}\xi \in \bigcap_{X\in\mathfrak{M}} D(\overline{X\lceil \mathcal{M}}) = \overline{\mathcal{M}}^{t_{\mathfrak{M}}}.$$

Thus $P_{\mathcal{M}}\mathcal{D} = \overline{\mathcal{M}}^{t_{\mathfrak{M}}}$. This is equivalent to $\mathcal{M}$ being essentially self-adjoint for $\mathfrak{M}$, since $\mathfrak{M}$ is clearly self-adjoint. □

By Theorem 2.5 and Lemma 2.6 we have the following.

Theorem 2.7. *Let $\mathfrak{M}$ be a closed O*-algebra such that $t_{\mathfrak{M}}$ is defined by a sequence $\{T_n\}$ of essentially self-adjoint operators of $\mathfrak{M}$ whose spectral projections leave the domain $\mathcal{D}$ invariant. Let E' be a projection in $\mathfrak{M}'_{\mathrm{w}}$. Then $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$.*

Proof. It can be shown that $\mathfrak{M}$ is a self-adjoint O*-algebra on the Fréchet domain $\mathcal{D}$, so that $\mathfrak{M}'_w\mathcal{D} \subset \mathcal{D}$, $t_{\mathfrak{M}''_{\mathrm{wc}}} = t_{\mathfrak{M}}$ and $\mathfrak{M}''_{\mathrm{wc}}$ is a self-adjoint O*-algebra on the Fréchet domain $\mathcal{D}$. Furthermore

$$P_{\langle \mathcal{M}'_w E'\mathcal{D}\rangle} = Z \in \mathfrak{M}'_w \cap (\mathfrak{M}'_w)' = (\mathfrak{M}''_{\mathrm{wc}})'_{\mathrm{w}} \cap ((\mathfrak{M}''_{\mathrm{wc}})'_{\mathrm{w}})'.$$

Moreover, since the spectral projections of the operators T_n leave the domain $\mathcal{D}$ invariant, they automatically belong to $\mathfrak{M}''_{\mathrm{wc}}$. Hence, the self-adjoint O*-algebra $\mathfrak{M}''_{\mathrm{wc}}$ and the $\mathfrak{M}''_{\mathrm{wc}}$-invariant subspace $\langle \mathcal{M}'_w E'\mathcal{D}\rangle$ of $\mathcal{D}$ satisfy all the conditions in Lemma 2.6, and so $Z\mathcal{D} = \overline{\langle \mathcal{M}'_w E'\mathcal{D}\rangle}^{t_{\mathfrak{M}''_{\mathrm{wc}}}}$. Then, by Theorem 2.5, $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$. □

Corollary 2.8. *Let T be an essentially self-adjoint operator in $\mathcal{H}$ and $\mathfrak{M}$ be a self-adjoint O*-algebra on $\mathcal{D}^\infty(\overline{T}) := \bigcap_{n\in\mathbb{N}} D(\overline{T}^n)$, containing T. Let E' be a projection in $\mathfrak{M}'_{\mathrm{w}}$. Then $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$.*

Proof. The spectral projections $E_T(\lambda)$, $\lambda \in \mathbb{R}$, of $\overline{T}$ satisfy:

- $E_T(\lambda) \in (\mathfrak{M}'_{\mathrm{w}})'$, $\forall\lambda \in \mathbb{R}$;
- $E_T(\lambda)\mathcal{H} \subset \mathcal{D}^\infty(\overline{T})$, $\forall\lambda \in \mathbb{R}$.

The statement then follows from Theorem 2.7. □

Apart from GW*-algebras, another unbounded generalization of von Neumann algebras is provided by the notion of *extended* W*-algebras, shortly EW*-algebras, defined as follows: A closed O*-algebra $\mathfrak{M}$ on $\mathcal{D}$ is said to be an *EW*-algebra* if $(I + X^\dagger X)^{-1}$ exists in $\mathfrak{M}_{\mathrm{b}} := \{A \in \mathfrak{M} : \overline{A} \in \mathcal{B}(\mathcal{H})\}$, for every $X \in \mathfrak{M}$ and $\overline{\mathfrak{M}_{\mathrm{b}}} := \{\overline{A} \in \mathfrak{M} : A \in \mathfrak{M}_{\mathrm{b}}\}$ is a von Neumann algebra [2]. It is easily shown that every EW*-algebra on a Fréchet domain satisfies the conditions of Theorem 2.7 (recall that every symmetric element of an EW*-algebra is essentially self-adjoint). Hence we have the following.

Corollary 2.9. *Let $\mathfrak{M}$ be an EW*-algebra on the Fréchet domain $\mathcal{D}$ in Hilbert space $\mathcal{H}$ and E' a projection in $\mathfrak{M}'_{\mathrm{w}}$. Then, $(\mathfrak{M}_{E'})''_{\mathrm{wc}} = (\mathfrak{M}''_{\mathrm{wc}})_{E'}$.*

3. Applications

In this section we show how to use the results of Section 2 in the analysis of the existence of conditional expectations for O*-algebras, which were first studied in [3, 4].

Let $\mathfrak{M}$ be a given O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with a strongly cyclic vector ξ_0. Here $\xi_0 \in \mathcal{D}$ is said to be strongly cyclic for $\mathfrak{M}$ if $\overline{\mathfrak{M}\xi_0}^{t_{\mathfrak{M}}} = \mathcal{D}$. With no loss of generality we will assume that $\|\xi_0\| = 1$. Let $\mathfrak{N}$ be an O*-subalgebra of $\mathfrak{M}$. A map $\mathcal{E}$ of $\mathfrak{M}$ onto $\mathfrak{N}$ is said to be a conditional expectation of $(\mathfrak{M}, \xi_0)$ w.r.t. $\mathfrak{N}$ if it satisfies the following conditions:

(i) $\mathcal{E}(X)^\dagger = \mathcal{E}(X^\dagger)$, $\forall X \in \mathfrak{M}$, and $\mathcal{E}(A) = A$, $\forall A \in \mathfrak{N}$;
(ii) $\mathcal{E}(XA) = \mathcal{E}(X)A$ and $\mathcal{E}(AX) = A\mathcal{E}(X)$, $\forall X \in \mathfrak{M}$, $\forall A \in \mathfrak{N}$;
(iii) $\omega_{\xi_0}(\mathcal{E}(X)) = \omega_{\xi_0}(X)$, $\forall X \in \mathfrak{M}$, where ω_{ξ_0} is a state on $\mathfrak{M}$ defined as $\omega_{\xi_0}(X) = \langle X\xi_0 | \xi_0\rangle$, $X \in \mathfrak{M}$.

In the case of von Neumann algebras, Takesaki [7] characterized the existence of conditional expectations using Tomita's modular theory. Thus a conditional expectation does not necessarily exist for a general von Neumann algebra. Hence, in [3, 4], Ogi, Takakura and the second author considered a linear map $\mathcal{E}$ of a $\dagger$-invariant subspace $D(\mathcal{E})$ of $\mathfrak{M}$ onto $\mathfrak{N}$ satisfying the above conditions (i), (ii) and (iii) on $D(\mathcal{E})$. Such a map is called an *unbounded conditional expectation* of $(\mathfrak{M}, \xi_0)$ w.r.t. $\mathfrak{N}$, and it was shown that there exists the largest unbounded conditional expectation $\mathcal{E}_{\mathfrak{N}}$ of $(\mathfrak{M}, \xi_0)$ w.r.t. $\mathfrak{N}$. Furthermore, the existence of conditional expectation was characterized, using Takesaki's result in the case of von Neumann algebras. But a deeper analysis is needed since, at that stage, the problem was not solved even in the case of GW*-algebras. One of the reasons is that the reduction of a GW*-algebra is not necessarily a GW*-algebra, in contrast with the case of von Neumann algebras. For $\mathcal{E}_{\mathfrak{N}}$ to be a conditional expectation of $(\mathfrak{M}, \xi_0)$ w.r.t. $\mathfrak{N}$ (i.e., everywhere defined on $\mathfrak{M}$), the following result given in [4, Corollary 6.2] holds.

Lemma 3.1. *Let $\mathfrak{M}$ be a closed O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathfrak{M}'_w\mathcal{D} \subset \mathcal{D}$. Let ξ_0 be a strongly cyclic and separating vector, in the sense that $\overline{\mathfrak{M}'_w\xi_0} = \mathcal{H}$. Suppose that $\mathfrak{N}$ is a closed O*-subalgebra of $\mathfrak{M}$ satisfying*

(i) *$\mathfrak{N}'_w\mathcal{D} \subset \mathcal{D}$;*
(ii) *$\mathfrak{N}\xi_0$ is essentially self-adjoint for $\mathfrak{N}$;*
(iii) *${\Delta''_{\xi_0}}^{it}(\mathfrak{N}'_w)'{\Delta''_{\xi_0}}^{-it} = (\mathfrak{N}'_w)'$, $\forall t \in \mathbb{R}$, where Δ''_{ξ_0} is the modular operator of the left Hilbert algebra $(\mathfrak{M}'_w)'\xi_0$;*
(iv) *$\mathfrak{N}_{P_{\mathfrak{N}}}$ is a GW*-algebra on $P_{\mathfrak{N}}\mathcal{D}$, where $P_{\mathfrak{N}}$ is a projection of $\mathcal{H}$ onto $\overline{\mathfrak{N}\xi_0}$.*

Then $\mathcal{E}_{\mathfrak{N}}$ is a conditional expectation of $(\mathfrak{M}, \xi_0)$ w.r.t. $\mathfrak{N}$.

By Lemma 3.1 and Theorem 2.5 we deduce the following.

Theorem 3.2. *Let $\mathfrak{M}$ be a closed O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with a strongly cyclic and separating vector ξ_0 such that $\mathfrak{M}'_w\mathcal{D} \subset \mathcal{D}$ and let $\mathfrak{N}$ be an O*-subalgebra of $\mathfrak{M}$.*

Suppose that $\mathfrak{N}$ is a GW-algebra on a Fréchet domain $\mathcal{D}$ satisfying*

(i) *$\mathfrak{N}\xi_0$ and $\langle\mathfrak{N}'_w P_{\mathfrak{N}}\mathcal{D}\rangle$ are essentially self-adjoint for $\mathfrak{N}$;*
(ii) *${\Delta''_{\xi_0}}^{it}(\mathfrak{N}'_w)'{\Delta''_{\xi_0}}^{-it} = (\mathfrak{N}'_w)'$, $\forall t \in \mathbb{R}$.*

Then $\mathcal{E}_{\mathfrak{N}}$ is a conditional expectation of $(\mathfrak{M}, \xi_0)$ w.r.t. $\mathfrak{N}$.

By Lemma 3.1 and Theorem 2.7 we deduce the following.

Theorem 3.3. *Let* $(\mathfrak{M}, \xi_0)$ *be as in Theorem* 3.2 *and* $\mathfrak{N}$ *an* O^**-subalgebra of* $\mathfrak{M}$. *Suppose* $\mathfrak{N}$ *is a* GW^**-algebra on* $\mathcal{D}$ *satisfying:*

(i) $t_{\mathfrak{N}}$ *is defined by a sequence* $\{T_n\}$ *of essentially self-adjoint operators in* $\mathfrak{N}$ *whose spectral projections leave the domain* $\mathcal{D}$ *invariant;*

(ii) ${\Delta''_{\xi_0}}^{it}(\mathfrak{N}'_w)'{\Delta''_{\xi_0}}^{-it} = (\mathfrak{N}'_w)'$, $\forall t \in \mathbb{R}$.

Then $\mathcal{E}_{\mathfrak{N}}$ *is a conditional expectation of* $(\mathfrak{M}, \xi_0)$ *w.r.t.* $\mathfrak{N}$.

Proof. By Lemma 2.6, $\mathfrak{N}\xi_0$ and $\langle \mathfrak{N}'_w P_{\mathfrak{N}} \mathcal{D}\rangle$ are essentially self-adjoint for $\mathfrak{N}$; by Theorem 3.2 it follows that $\mathcal{E}_{\mathfrak{N}}$ is a conditional expectation of $(\mathfrak{M}, \xi_0)$ w.r.t. $\mathfrak{N}$. □

Corollary 3.4. *Let* $(\mathfrak{M}, \xi_0)$ *be as in Theorem* 3.2 *and* $\mathfrak{N}$ *an* O^**-subalgebra of* $\mathfrak{M}$. *Suppose* $\mathfrak{N}$ *is a* GW^**-algebra on* $\mathcal{D}^\infty(\overline{T})$, *where* T *is an essentially self-adjoint operator in* $\mathfrak{N}$, *and* ${\Delta''_{\xi_0}}^{it}(\mathfrak{N}'_w)'{\Delta''_{\xi_0}}^{-it} = (\mathfrak{N}'_w)'$, $\forall t \in \mathbb{R}$. *Then,* $\mathcal{E}_{\mathfrak{N}}$ *is a conditional expectation of* $(\mathfrak{M}, \xi_0)$ *w.r.t.* $\mathfrak{N}$.

Corollary 3.5. *Let* $(\mathfrak{M}, \xi_0)$ *be as in Theorem* 3.2 *and* $\mathfrak{N}$ *an* O^**-subalgebra of* $\mathfrak{M}$. *Suppose* $\mathfrak{N}$ *is an* EW^**-algebra on a Fréchet domain* $\mathcal{D}$ *satisfying* ${\Delta''_{\xi_0}}^{it}(\mathfrak{N}'_w)'{\Delta''_{\xi_0}}^{-it} = (\mathfrak{N}'_w)'$, $\forall t \in \mathbb{R}$. *Then* $\mathcal{E}_{\mathfrak{N}}$ *can be extended to a conditional expectation* $\mathcal{E}_{\mathfrak{N}''_{\rm wc}}$ *of* $(\mathfrak{M}''_{\rm wc}, \xi_0)$ *w.r.t.* $\mathfrak{N}''_{\rm wc}$.

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