

A NOTE ON A RESULT OF M. GROSSI

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ABSTRACT. The purpose of this note is to present a fact complementary to a result in a recent paper of M. Grossi. Making use of an energy balance identity, it is shown that the sufficient conditions for existence of solutions proved in Grossi's paper are also almost necessary.

1. INTRODUCTION

Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a starshaped domain. It has been proved as early as 1965 by S. I. Pohozaev [18] that problems

$$(1) \quad -\Delta u = u^p \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

have no positive solutions for $p \geq \frac{N+2}{N-2}$. For $1 < p < \frac{N+2}{N-2}$ the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact and the existence of positive solutions can be proved by variational methods. For $p = \frac{N+2}{N-2}$ (i.e. $p+1 = 2^* = \frac{2N}{N-2}$ the critical Sobolev exponent) the embedding is not compact anymore, and for $p > \frac{N+2}{N-2}$ the space $H_0^1(\Omega)$ is not a subspace of $L^{p+1}(\Omega)$ and the variational methods used to prove existence of solutions break down.

An interesting fact was observed by Brezis and Nirenberg in [3] relative to the problem

$$(2) \quad -\Delta u = u^{2^*-1} + \lambda u, \quad u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B,$$

where B is the unit ball in \mathbb{R}^N with $N \geq 3$. The authors proved that (2) admits a solution for positive values λ contained in an interval whose endpoints depend on the dimension N in a somewhat unexpected way. This paper gave rise to a flurry of work on problems with critical nonlinearities. It has been noticed that for some dimensions N the branch of solutions which bifurcates from the trivial solution exists for all λ between λ_1 (the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions) and zero, while for other “critical” dimensions this branch is bounded away from $\lambda = 0$ (see [2], [4], [6], [7], [10], [11], [12], [13], [16], [19], and the references therein).

Positive solutions for the more general critical exponent problem

$$-\Delta u + a(x)u = u^{2^*-1} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

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were discussed in [1], [3], [9], [17], [21], among many other works. As was made explicit by Druet [9], and earlier by Schoen [21] and Rey [20] (among others), Green’s function of the operator $-\Delta u + a(x)u$ with zero Dirichlet boundary conditions on Ω plays an important role in the existence of solutions. For supercritical exponents, besides the early nonexistence results, there are few positive (existence) results (see e.g. [8] and the references therein).

In [15] the author looks at radial positive solutions for supercritical problems

$$(3) \quad -\Delta u + a(|x|)u = u^p, \text{ in } B, \quad u = 0 \text{ on } \partial B,$$

where $a(r) \geq 0$ is a radial smooth function. By symmetry reduction, (3) becomes the ODE problem

$$(4) \quad -u'' - \frac{N-1}{r}u' + a(r)u = u^p, \quad u > 0 \text{ in } (0, 1), \quad u'(0) = u(1) = 0.$$

Let $H(r, s)$ be the regular part of Green’s function for the operator on the left hand side, and with the boundary conditions, in (4). The central result of [15] states

Theorem 1. *If there exists a nondegenerate critical point $\bar{r} \in (0, 1)$ of the function $F(r) = \frac{H(r,r)}{r^{N-1}}$, then for large enough p there exists at least one solution u_p of (4).*

We prove a somewhat complementary result. One way to state it would be in the form of the following:

Theorem 2. *If the function $F_p(r) = \frac{H(r,r)}{r^{\frac{p-1}{p+3}(N-1)}}$ is monotonic, nonconstant, then problem (4) has no solution.*

As $p \rightarrow \infty$ we have that $F_p \rightarrow F$ uniformly in any closed interval $[r_0, 1]$ with $0 < r_0 < 1$. It is straightforward to see that if the hypotheses of Theorem 1 are met, then for sufficiently large p , the derivative $F'_p(r)$ has to change sign, and therefore F_p cannot be monotonic. In this sense, Theorems 1 and 2 complement each other.

The proof of Theorem 2 is based on an energy balance identity which has been developed in the paper of Catrina and Lavine [5]. The motivation in [5] was the refined Pohozaev identity used for the special case $N = 3$ by Brezis and Nirenberg in [3].

2. PROOF OF THEOREM 2

Note that problem (4) can be written as

$$(5) \quad -(r^{N-1}u')' + r^{N-1}a(r)u = r^{N-1}u^p, \quad u(r) > 0 \text{ in } (0, 1), \quad u'(0) = u(1) = 0.$$

Let ξ and ζ be linearly independent solutions of the homogeneous equation

$$(6) \quad -(r^{N-1}\xi')' + r^{N-1}a(r)\xi = 0 \quad \text{such that} \quad \xi'(0) = \zeta(1) = 0.$$

Observe that for $a \equiv 0$, a pair of such solutions is

$$(7) \quad \xi_0(r) \equiv \frac{1}{N-2}, \quad \zeta_0(r) = r^{-(N-2)} - 1.$$

According to our setting, we define the Wronskian of two functions to be

$$W[u, \xi](r) = r^{N-1} (u(r)\xi'(r) - u'(r)\xi(r)),$$

and note that

$$W[\xi_0, \zeta_0](r) \equiv -1.$$

The proof of the following lemma will be sketched in the Appendix.

Lemma 1. *For any bounded coefficient function $a \geq 0$, the problem (6) has a pair ξ, ζ of linearly independent solutions such that $\xi'(0) = \zeta(1) = 0$. Moreover, $W[\xi, \zeta](r) \equiv -1$ and the limits*

$$\lim_{r \rightarrow 0^+} r^{N-2} \zeta(r) \text{ and } \lim_{r \rightarrow 0^+} r^{N-1} \zeta'(r) \text{ exist and are finite.}$$

We remark that the conclusions of the lemma above are valid for more general coefficient functions a , but for our purposes the present variant suffices. The next lemma states the energy balance identity that is at the center of the nonexistence result.

Lemma 2. *With ξ and ζ as in Lemma 1, any solution u of (5) satisfies the identity*

$$\int_0^1 r^{2(N-1)\frac{p+1}{p+3}} u^{p+1} \left(r^{\frac{4(N-1)}{p+3}} \xi \zeta \right)' dr = 0.$$

Proof. By multiplying the equation (6) by $-u$ and equation (5) by ξ and adding, we obtain

$$\frac{d}{dr} W[u, \xi](r) = r^{N-1} u^p \xi.$$

Similarly we have

$$\frac{d}{dr} W[u, \zeta](r) = r^{N-1} u^p \zeta.$$

By combining the two equalities above it follows that

$$\frac{d}{dr} (W[u, \xi]W[u, \zeta]) = r^{N-1} u^p (\xi W[u, \zeta] + \zeta W[u, \xi]).$$

Therefore

$$\begin{aligned} \frac{d}{dr} (W[u, \xi]W[u, \zeta]) &= r^{2(N-1)} u^p (u(\xi\zeta)' - 2u'\xi\zeta) \\ &= r^{2(N-1)} u^{p+1} (\xi\zeta)' - 2r^{2(N-1)} u^p u' \xi\zeta \\ &= r^{2(N-1)} u^{p+1} (\xi\zeta)' - \frac{2}{p+1} r^{2(N-1)} (u^{p+1})' \xi\zeta = \frac{d}{dr} \left(-\frac{2}{p+1} r^{2(N-1)} u^{p+1} \xi\zeta \right) \\ &\quad + r^{2(N-1)} u^{p+1} (\xi\zeta)' + \frac{2}{p+1} \left(r^{2(N-1)} \xi\zeta \right)' u^{p+1}. \end{aligned}$$

We now obtain

$$\begin{aligned} &\frac{d}{dr} \left(W[u, \xi]W[u, \zeta] + \frac{2}{p+1} r^{2(N-1)} u^{p+1} \xi\zeta \right) \\ &= \frac{p+3}{p+1} r^{2(N-1)} u^{p+1} (\xi\zeta)' + \frac{4(N-1)}{p+1} r^{2N-3} u^{p+1} \xi\zeta \\ &= \frac{p+3}{p+1} r^{2(N-1)} u^{p+1} \left((\xi\zeta)' + \frac{4(N-1)}{p+3} r^{-1} \xi\zeta \right) \\ &= \frac{p+3}{p+1} r^{2(N-1)\frac{p+1}{p+3}} u^{p+1} \left(r^{\frac{4(N-1)}{p+3}} \xi\zeta \right)'. \end{aligned}$$

The lemma follows from the fact that the integral of the left hand side is zero. Indeed, from the boundary conditions and from Lemma 1 we have that

$$\lim_{r \rightarrow 0^+} W[u, \xi](r)W[u, \zeta](r) = W[u, \xi](1)W[u, \zeta](1) = 0$$

and

$$\lim_{r \rightarrow 0^+} r^{2(N-1)} u^{p+1} \xi \zeta = r^{2(N-1)} u^{p+1} \xi \zeta \Big|_{r=1} = 0.$$

□

We are thus naturally led to define on the interval $(0, 1]$ the function

$$(8) \quad F_p(r) = r^{\frac{4(N-1)}{p+3}} \xi(r) \zeta(r)$$

whose derivative appears in the energy balance identity. The following corollary of Lemma 2 is now immediate.

Corollary 1. *If the dimension N , the coefficient function a , and the exponent p in (4) are such that the function F_p defined in (8) is monotonic, nonconstant, then the problem (4) has no solution.*

Proof of Theorem 2. The only fact that needs to be checked is that the function F_p defined in Theorem 2 is the same function as (8). Indeed, it is well known that Green’s function of the operator

$$-u'' - \frac{N-1}{r} u' + a(r)u, \quad \text{with boundary conditions} \quad u'(0) = u(1) = 0$$

is given by

$$G(r, s) = s^{N-1} \xi(\min(r, s)) \zeta(\max(r, s)), \quad \text{for } r, s \in (0, 1),$$

where ξ and ζ are functions satisfying (6) with the additional requirement that

$$W[\xi, \zeta] \equiv -1.$$

We then observe that the regular part of Green’s function as defined in [15],

$$H(r, s) = G(r, s) - \Gamma(r, s), \quad \text{where} \quad \Gamma(r, s) = \begin{cases} 0 & \text{if } r \leq s, \\ \frac{s^{N-1}}{N-2} (r^{2-N} - s^{2-N}) & \text{if } r > s, \end{cases}$$

satisfies on the diagonal

$$H(r, r) = G(r, r) = r^{N-1} \xi(r) \zeta(r).$$

Therefore

$$F_p(r) = \frac{H(r, r)}{r^{\frac{p-1}{p+3}(N-1)}} = r^{\frac{4(N-1)}{p+3}} \xi(r) \zeta(r),$$

i.e. the function that appears in (8). □

It is instructive to analyze the case $a \equiv 0$ as $p > 1$ increases from subcritical to supercritical. We have $F_p(r) = r^{\frac{4(N-1)}{p+3}} \xi_0(r) \zeta_0(r)$ with ξ_0 and ζ_0 given by (7), i.e.

$$F_p(r) = r^{\frac{4(N-1)}{p+3}} \frac{r^{-(N-2)} - 1}{N-2}.$$

The critical nonlinearity is given by $p = \frac{N+2}{N-2}$, and in this case $F_p(r) = \frac{1-r^{N-2}}{N-2}$, which is monotonic. Similarly, F_p is monotonically decreasing for all p supercritical, and we recover in this way Pohozaev’s nonexistence result in the radial case. For $1 < p < \frac{N+2}{N-2}$ we have

$$\lim_{r \rightarrow 0^+} F_p(r) = F_p(1) = 0,$$

and F_p cannot be monotonic, which is consistent with the known fact that solutions of (4) with $a \equiv 0$ exist. In fact, in the subcritical case, solutions of (4) can be

obtained by variational methods for a wide variety of coefficient functions a . In the case $a \geq 0$ and bounded, from Lemma 1 it follows again that

$$\lim_{r \rightarrow 0^+} F_p(r) = F_p(1) = 0,$$

and thus there is no obstruction.

As was mentioned in the introduction, if $F(r) = \frac{H(r,r)}{r^{N-1}}$ has a nondegenerate critical point \bar{r} , then so will have the function F_p for p sufficiently large. Under these circumstances, F_p cannot be monotonic in $(0, 1)$ and the energy balance identity is not an obstruction to the existence of solutions anymore. This is the case for subcritical nonlinearities.

3. APPENDIX

Even though the proof of Lemma 1 can be worked out directly on the equation (6) along the same lines as the proof below, for simpler notation we prefer to work with a transformed equation. Let

$$s = \frac{1}{r^{N-2}}, \quad u(r) = v(s), \quad a(r) = b(s).$$

Then

$$\frac{d}{dr} = -(N-2) \frac{1}{r^{N-1}} \frac{d}{ds} = -(N-2) s^{\frac{N-1}{N-2}} \frac{d}{ds}.$$

Therefore

$$u'(r) = -(N-2) s^{\frac{N-1}{N-2}} v_s(s) \quad \text{and} \quad -(r^{N-1} u')' = -(N-2)^2 s^{\frac{N-1}{N-2}} v_{ss}.$$

Hence, u being a solution of the equation

$$-(r^{N-1} u')' + r^{N-1} a(r) u = 0, \quad \text{for } r \in (0, 1)$$

is the same as v being a solution of the equation

$$-(N-2)^2 s^{\frac{N-1}{N-2}} v_{ss} + s^{-\frac{N-1}{N-2}} b(s) v = 0, \quad \text{for } s \in (1, \infty),$$

or equivalently

$$(9) \quad v_{ss} = \frac{c(s)}{s^k} v, \quad \text{where } c(s) = \frac{b(s)}{(N-2)^2} \quad \text{and} \quad k = 2 \frac{N-1}{N-2} > 2.$$

Since we assumed that a , and therefore c , is a nonnegative bounded function, select $s_0 > 0$ such that

$$\int_{s_0}^{\infty} \frac{c(t)}{t^{k-1}} dt \leq \frac{1}{2}.$$

For $s \geq s_0$, we define iteratively $v_0(s) \equiv 1$, and

$$v_{n+1}(s) = 1 + \int_s^{\infty} \left(1 - \frac{s}{t}\right) \frac{c(t)}{t^{k-1}} v_n(t) dt.$$

By induction it follows that

$$1 \leq v_n(s) \leq 2 \quad \text{for all } n = 0, 1, 2, \dots, \quad \text{and all } s \in [s_0, \infty).$$

Also, note that for all $n \geq 1$ the functions v_n are decreasing, and $\lim_{s \rightarrow \infty} v_n(s) = 1$. On the other hand, for any $n \geq 1$ and for any $s \geq s_0$, we have that

$$0 \leq v_{n+1}(s) - v_n(s) \leq \max_{[s_0, \infty)} (v_n - v_{n-1}) \int_{s_0}^{\infty} \frac{c(t)}{t^{k-1}} dt \leq \frac{1}{2} \max_{[s_0, \infty)} (v_n - v_{n-1}) \leq \frac{1}{2^{n+1}}.$$

Therefore, for any $n \geq m \geq 0$ it follows that for all $s \geq s_0$,

$$0 \leq v_n(s) - v_m(s) \leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \dots + \frac{1}{2^n} \leq \frac{1}{2^m}.$$

It follows that $\{v_n\}_n$ converges uniformly on $[s_0, \infty)$ to a function φ that satisfies

$$\varphi(s) = 1 + \int_s^\infty \left(1 - \frac{s}{t}\right) \frac{c(t)}{t^{k-1}} \varphi(t) dt,$$

and therefore it solves (9). Since $1 \leq v_n(s) \leq 2$ for all $s \geq s_0$, it follows that φ is bounded, positive, decreasing on $[s_0, \infty)$, and $\lim_{s \rightarrow \infty} \varphi(s) = 1$. If $s_0 > 1$, we extend φ backwards on $[1, \infty)$ so that it remains a solution of (9). Because it satisfies the equation, φ remains decreasing on the whole interval $[1, \infty)$, and therefore $\varphi(1) \geq 1$.

We now define

$$\psi(s) = \varphi(s) \int_1^s \frac{1}{\varphi^2(t)} dt.$$

It is straightforward to check that ψ is also a solution of (9), that $\psi(1) = 0$, and that

$$\varphi\psi_s - \varphi_s\psi \equiv 1 \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{1}{s}\psi(s) = 1.$$

We also have

$$(10) \quad \lim_{s \rightarrow \infty} \psi_s(s) = 1.$$

Indeed,

$$\psi_s(s) = \frac{1}{\varphi(s)} + \frac{\varphi_s(s)}{\varphi(s)}\psi(s) = \frac{1}{\varphi(s)} \left(1 + s\varphi_s(s) \frac{\psi(s)}{s}\right),$$

and (10) follows from $\lim_{s \rightarrow \infty} \varphi(s) = 1$, $\lim_{s \rightarrow \infty} \frac{\psi(s)}{s} = 1$, and by l'Hôpital's rule,

$$\lim_{s \rightarrow \infty} s\varphi_s(s) = \lim_{s \rightarrow \infty} \frac{-\int_s^\infty \frac{c(t)}{t^k} \varphi(t) dt}{s^{-1}} = \lim_{s \rightarrow \infty} -\frac{c(s)}{s^{k-2}} \varphi(s) = 0.$$

Let

$$\xi(r) = \frac{1}{N-2}\varphi(s) \quad \text{and} \quad \zeta(r) = \psi(s)$$

be solutions of (6). It follows that

$$\varphi_s(s) = -r^{N-1}\xi'(r) \quad \text{and} \quad \psi_s(s) = -\frac{1}{N-2}r^{N-1}\zeta'(r).$$

We calculate

$$\lim_{r \rightarrow 0^+} \xi'(r) = -\lim_{s \rightarrow \infty} s^{\frac{N-1}{N-2}}\varphi_s(s) = \lim_{s \rightarrow \infty} s^{\frac{N-1}{N-2}} \int_s^\infty \frac{c(t)}{t^k} \varphi(t) dt,$$

and by l'Hôpital's rule,

$$\lim_{r \rightarrow 0^+} \xi'(r) = \lim_{s \rightarrow \infty} \frac{N-2}{N-1} \frac{c(s)}{s^{k-\frac{2N-3}{N-2}}} \varphi(s) = 0$$

because $k = \frac{2(N-1)}{N-2} > \frac{2N-3}{N-2}$ and c is bounded.

Hence ξ and ζ satisfy the boundary conditions in (6).

We also have

$$W[\xi, \zeta](r) = r^{N-1}(\xi\zeta' - \xi'\zeta) = -(\varphi\psi_s - \varphi_s\psi) \equiv -1$$

and

$$\lim_{r \rightarrow 0^+} r^{N-2} \zeta(r) = \lim_{s \rightarrow \infty} \frac{1}{s} \psi(s) = 1.$$

The last fact left to justify is

$$\lim_{r \rightarrow 0^+} r^{N-1} \zeta'(r) = -(N-2) \lim_{s \rightarrow \infty} \psi_s(s) = -(N-2),$$

because of (10).

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