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A RESIDUAL RADIAL LIMIT ZERO SET

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ABSTRACT. We construct a nonconstant holomorphic function on the unit ball in $\mathbb{C}^n$ having radial limit zero on a certain residual subset of the unit sphere.

If f is a real or complex-valued function on the unit ball $B = B(0,1) \subset \mathbb{C}^n$, then f is said to have *radial limit* L at a point ω in the unit sphere S if

$$\lim_{r\to 1^-} f(r\omega) = L.$$

The goal of this paper is to construct a residual set E in S and a nonconstant holomorphic function on B having radial limit zero at each point of E.

In dimension one, the results of Lusin and Privalov [3], McMillan [4], and Berman [1] characterize the radial limit zero sets of nonconstant holomorphic functions on the disc. Stated loosely, this characterization is that the sets must be locally "small" either in the sense of Lebesgue measure or the sense of Baire category. In higher dimensions, no such characterization is known.

Rudin [6, p. 67] asked if a nonconstant holomorphic function can have radial limit zero at every point of a full measure subset of S. The work of Hakim and Sibony [2] reveals that such functions do indeed exist. It is also natural to ask whether there exists a nonconstant holomorphic function on $B \subset \mathbb{C}^n$ $(n \geq 2)$ having radial limit zero at each point of a *residual* subset of S (the complement of a first Baire category set).

When $n = 1$, Privalov [5, p. 214] showed that such functions exist. We now give a brief outline of Privalov's construction: Let E be residual and have measure zero in S. For each $t \in \mathbb{N}$, let E_t be an open subset of the unit circle that contains E and has measure less than $1/2^t$. Let $g(\omega) = \sum_{t=1}^{\infty} \chi_{E_t}(\omega)$, where χ denotes the indicator function. Applying the Poisson Integral Formula to g, we then obtain a nonnegative harmonic function u on the unit disc having radial limit $+\infty$ at each point of E. Let v be a harmonic conjugate for u. Then $f = e^{-(u+iv)}$ is the desired holomorphic function.

Privalov's construction does not work in $\mathbb{C}^n$ $(n \geq 2)$ because harmonic functions on B do not generally have "harmonic conjugates". We present here a new method of construction that works in all dimensions. This construction will, in fact, exhibit a function having *general limit* zero on the residual set E.

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Theorem 1. *There is a residual set $E \subset S$ and a nonconstant holomorphic function $f : B \to \mathbb{C}$ with $\lim_{B \ni \zeta \to \omega} f(\zeta) = 0$ for each $\omega \in E$.*

Proof. For $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, define $g_\theta : \overline{B} \to \mathbb{R} \cup \{+\infty\}$ by

$$g_\theta(z_1, \dots, z_n) = \mathrm{Re}\left(\sqrt{\frac{1}{1 - \sum_{j=1}^n (e^{-i\theta_j} z_j)^2}}\right).$$

We show in Lemma 2 that g_θ is positive and finite on the (open) unit ball. Since $g_\theta|_B$ is the real part of a holomorphic function, it is pluriharmonic. Define $C_\theta = \{z \in \overline{B} : \sum_{j=1}^n (e^{-i\theta_j} z_j)^2 = 1\}$. (It turns out that each C_θ is a unit sphere, centered at the origin, of real dimension $n-1$. But we will not need this fact.)

For $k, t \in \mathbb{N}$, let $M_{k,t} = \max\{g_\theta(z) : z \in \overline{B(0, 1 - 1/(k+t))}\}$. $M_{k,t}$ is independent of the choice of $\theta \in \mathbb{R}^n$, so it is well defined. For $r \in (0,1)$, define tube-like neighborhoods $A_\theta(r)$ of C_θ as follows:

$$A_\theta(r) = \{z \in \mathbb{C}^n : \mathrm{dist}(z, C_\theta) < r\}.$$

Now choose $r_{k,t} > 0$ small enough that

$$\frac{g_\theta(z)}{M_{k,t} 2^{k+t}} > 1$$

for $z \in B \cap A_\theta(r_{k,t})$. (That such a choice of $r_{k,t}$ can be made is a consequence of Lemma 3 below.) Again, $r_{k,t}$ does not depend on θ, so it is well-defined.

Having chosen $r_{k,t}$, choose $r'_{k,t} > 0$ small enough such that for $\theta \in \mathbb{R}^n$,

$$\bigcup \{C_\phi : \phi \in B_{\mathbb{R}^n}(\theta, r'_{k,t})\} \subset A_\theta(r_{k,t}).$$

Once again, this choice is independent of $\theta \in \mathbb{R}^n$.

Let $\{a_k\}_{k=1}^\infty$ be a dense sequence in $[0, \pi)^n$. For $t \in \mathbb{N}$, let

$$P_t = \{\theta : \theta \in B_{\mathbb{R}^n}(a_k, r'_{k,t}) \text{ for some } k \in \mathbb{N}\} \cap [0, \pi)^n,$$

and let

$$E = \bigcap_{t \in \mathbb{N}} \bigcup_{\theta \in P_t} C_\theta.$$

We show in Lemma 4 that E is residual in S.

Let

$$u(z) = \sum_{t \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{g_{a_k}(z)}{M_{k,t} 2^{k+t}}.$$

By the method in which $M_{k,t}$ was defined, the double sum converges uniformly on compact subsets of B. For fixed t, the inner sum is greater than 1 in a (relative) neighborhood of E (i.e., it is greater than 1 on $B \cap \bigcup_{k=1}^\infty A_{a_k}(r_{k,t})$, which is a relative neighborhood of $\bigcup_{\theta \in P_t} C_\theta \supset E$). The intersection of finitely many relative neighborhoods of E is again a relative neighborhood of E. So for any $j \in \mathbb{N}$, there is a relative neighborhood of E on which $u \geq j$. Thus u is a positive pluriharmonic function on B with $\lim_{B \ni \zeta \to \omega} u(\zeta) = +\infty$ for each $\omega \in E$. Let v be a pluriharmonic conjugate of u. Finally, define $f(z) = e^{-(u(z) + iv(z))}$, and note that $|f(z)| = e^{-u(z)}$. So f is a holomorphic function on B with $\lim_{B \ni \zeta \to \omega} f(\zeta) = 0$, $\omega \in E$. $\square$

Lemma 2. *The function g_θ is positive and finite on B for each fixed $\theta \in \mathbb{R}^n$.*

Proof. Let $z \in B$. Then

$$\left|\sum_{j=1}^{n} \left(e^{-i\theta_j} z_j\right)^2\right| \leq \sum_{j=1}^{n} \left|e^{-i\theta_j} z_j\right|^2 = \sum_{j=1}^{n} |z_j|^2 < 1.$$

So g_θ is finite on B. To see that g_θ is positive on B, we simply note that the Möbius transformation $1/(1-z)$ maps the unit disc to a set of points whose real part is greater than $1/2$, and thus $\sqrt{1/(1-z)}$ maps the unit disc to a set of points whose real part is greater than $\sqrt{1/2}$.

$\square$

Lemma 3. *Let $\theta \in \mathbb{R}^n$, and let $M > 0$ be given. There exists $\epsilon' > 0$ such that if $w \in B$ and $\operatorname{dist}(w, C_\theta) < \epsilon'$, then* $\operatorname{Re}\left(\sqrt{\frac{1}{1-\sum_{j=1}^{n}(e^{-i\theta_j} w_j)^2}}\right) > M$.

Proof. We assume without loss of generality that $\theta = \mathbf{0} = (0, \ldots, 0)$. Let $M > 0$ be given. We may assume that $M > \sqrt{1/2}$. Recall that $C_{\mathbf{0}} = \{z \in \overline{B} : \sum_{j=1}^{n} {z_j}^2 = 1\}$. So there exists $\epsilon > 0$ such that if $w \in B$ and $\operatorname{dist}(w, C_{\mathbf{0}}) < \epsilon$, then $\left|\frac{1}{1-\sum_{j=1}^{n} w_j^2}\right| > M^2$. Thus, for such a w, we have $\frac{1}{1-\sum_{j=1}^{n} w_j^2} \in \mathbb{C} \setminus D(0, M^2)$. Note also that since $w \in B$, then $\sum_{j=1}^{n} w_j^2 \in D$. Since the Möbius transformation $1/(1-z)$ maps the unit disc to a set of points whose real part is greater than $1/2$, we thus have $\operatorname{Re}\left(\frac{1}{1-\sum_{j=1}^{n} w_j^2}\right) > \frac{1}{2}$. So $\sqrt{\frac{1}{1-\sum_{j=1}^{n} w_j^2}} \in A \cap (\mathbb{C} \setminus D(0, M))$, where $A = \{z \in \mathbb{C} : z = \sqrt{\zeta}$ for some ζ with $\operatorname{Re}\zeta > 1/2\}$. A simple yet tedious calculation reveals that if $z \in A \cap (\mathbb{C} \setminus D(0, M))$, then $\operatorname{Re}(z) > M \cdot \cos\left(1/2 \cdot \arctan\left(2(M^4 - 1/4)^{1/2}\right)\right)$, which tends to $+\infty$ as $M \to +\infty$. So there exists $\epsilon' > 0$ such that if $w \in B$ and $\operatorname{dist}(w, C_{\mathbf{0}}) < \epsilon'$, then $\operatorname{Re}\left(\sqrt{\frac{1}{1-\sum_{j=1}^{n} w_j^2}}\right) > M$. $\square$

Lemma 4. *The set E in the proof of Theorem 1 is residual in S.*

Proof. We use the same notation as in the proof of Theorem 1.

To show that

$$E = \bigcap_{t \in \mathbb{N}} \bigcup_{\theta \in P_t} C_\theta$$

is residual in S, it suffices to show that for each $t \in \mathbb{N}$,

$$S \setminus \bigcup_{\theta \in P_t} C_\theta$$

is nowhere dense in S. But, by Claim 1, we have

$$S \setminus \bigcup_{\theta \in P_t} C_\theta \subset \bigcup_{\theta \in P_t^c} C_\theta.$$

(Here, we are defining $P_t^c := \{\theta \in [0, \pi)^n : \theta \notin P_t\}$. That is, we are taking the complement of P_t with respect to $[0, \pi)^n$.) So it suffices to show that, for each $t \in \mathbb{N}$,

$$\bigcup_{\theta \in P_t^c} C_\theta$$

is nowhere dense in S. Suppose not. Then there is a $t_0 \in \mathbb{N}$ such that $\bigcup_{\theta \in P_{t_0}^c} C_\theta$ is dense in some open set $U \subset S$. Without loss of generality, we may assume that for each $j = 1, \ldots, n$

$$U \cap \{(z_1, \ldots, z_{j-1}, x_j + 0i, z_{j+1}, \ldots, z_n)\} = \emptyset.$$

In other words, we assume that each coordinate of each point in U has a nonzero imaginary part.

Define $G := \{\theta \in [0, \pi)^n : C_\theta \cap U \neq \emptyset\}$. We show in Claim 2 that G is open in $[0, \pi)^n$. We now show that $P_{t_0}^c$ is dense in G. Suppose not. Then $\exists w \in G$ and $\epsilon > 0$ such that $B(w, \epsilon) \subset G$ and such that

$$B(w, \epsilon) \cap P_{t_0}^c = \emptyset.$$

But in Claim 3 we show that

$$U \cap \bigcup_{\theta \in B(w,\epsilon)} C_\theta$$

is open (and nonempty) in U. Hence, since $\bigcup_{\theta \in P_{t_0}^c} C_\theta$ is dense in U, we have

$$U \cap \left(\bigcup_{\theta \in B(w,\epsilon)} C_\theta \right) \cap \left(\bigcup_{\theta \in P_{t_0}^c} C_\theta \right) \neq \emptyset.$$

So $\exists \phi \in P_{t_0}^c$ and $\exists \psi \in B(w, \epsilon)$ such that $U \cap C_\phi \cap C_\psi \neq \emptyset$. Thus, by Claim 4, $\phi = \psi$. So

$$B(w, \epsilon) \cap P_{t_0}^c \neq \emptyset,$$

a contradiction of our earlier assertion that this intersection is empty. By Claim 4, $P_{t_0}^c$ is indeed dense in G. But, since G is open in $[0, \pi)^n$, this contradicts the fact that (by the way it was constructed) $P_{t_0}^c$ is nowhere dense in $[0, \pi)^n$. Thus, for each $t \in \mathbb{N}$,

$$\bigcup_{\theta \in P_t^c} C_\theta$$

is nowhere dense in S. This is what we wished to prove. $\square$

Claim 1. $\bigcup_{\theta \in [0,\pi)^n} C_\theta = S$.

Proof. Suppose $z \in S$. Then $z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$ for some $\theta_1, \ldots, \theta_n \in [0, 2\pi)$ and $r_1, \ldots, r_n \in [0, 1]$ with $r_1^2 + \ldots + r_n^2 = 1$. For $j = 1, \ldots, n$, let

$$\theta_j' = \begin{cases} \theta_j & \text{if } \theta_j \in [0, \pi), \\ \theta_j - \pi & \text{if } \theta_j \in [\pi, 2\pi). \end{cases}$$

Then $\theta' = (\theta'_1, \ldots, \theta'_n) \in [0,\pi)^n$. We will now show that $z \in C_{\theta'}$. To show this, it suffices to show that $\sum_{j=1}^n (e^{-i\theta'_j} z_j)^2 = 1$. We have

$$\begin{aligned}\sum_{j=1}^n (e^{-i\theta'_j} z_j)^2 &= \sum_{j=1}^n (e^{-i\theta'_j} r_j e^{i\theta_j})^2 \\ &= \sum_{j=1}^n r_j^2 \left(e^{2i(\theta_j - \theta'_j)}\right) \\ &= \sum_{j=1}^n r_j^2 \\ &= 1.\end{aligned}$$

The claim has thus been established. □

Claim 2. Let U be an open subset of S satisfying the assumption that each coordinate of each point of U has nonzero imaginary part. Then the set $G := \{\theta \in [0,\pi)^n : C_\theta \cap U \neq \emptyset\}$ is open in $[0,\pi)^n$.

Proof. Suppose not. Then there is a point $\theta \in G$ and a sequence of points $\{\phi_k\}_{k=1}^\infty$ in $[0,\pi)^n \setminus G$ such that $\lim_{k\to\infty} \phi_k = \theta$. Thus $C_\theta \cap U \neq \emptyset$, but $C_{\phi_k} \cap U = \emptyset$ for each $k \in \mathbb{N}$.

Let $m \in C_\theta \cap U$. We may write $m = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$, where $\theta = (\theta_1, \ldots, \theta_n)$ and $r_1^2 + \ldots + r_n^2 = 1$. Since U is open, there is an $\epsilon > 0$ such that $S \cap B(m,\epsilon) \subset U$. Let $k \in \mathbb{N}$ be such that $||\phi_k - \theta|| < \epsilon$. Let $w := (r_1 e^{i\phi_{k_1}}, \ldots, r_n e^{i\phi_{k_n}})$, where $\phi_k = (\phi_{k_1}, \ldots, \phi_{k_n})$. Note that $w \in C_{\phi_k}$. Then, making use of the fact that $\left|e^{ia} - e^{ib}\right| \leq |a - b|$ for every $a, b \in \mathbb{R}$, we have

$$\begin{aligned}||m - w||_{\mathbb{C}^n} &= \left(r_1^2 \left|e^{i\theta_1} - e^{i\phi_{k_1}}\right|^2 ||m - w|| + \ldots + r_n^2 \left|e^{i\theta_n} - e^{i\phi_{k_n}}\right|^2 ||m - w||\right)^{\frac{1}{2}} \\ &\leq \left(r_1^2 \left|\theta_1 - \phi_{k_1}\right|^2 + \ldots + r_n^2 \left|\theta_n - \phi_{k_n}\right|^2\right)^{\frac{1}{2}} \\ &\leq \left(\left|\theta_1 - \phi_{k_1}\right|^2 + \ldots + \left|\theta_n - \phi_{k_n}\right|^2\right)^{\frac{1}{2}} \\ &= ||\theta - \phi_k||_{\mathbb{R}^n} \\ &< \epsilon.\end{aligned}$$

Thus $w \in S \cap B(m,\epsilon) \subset U$. But recall also that $w \in C_{\phi_k}$. Thus $w \in U \cap C_{\phi_k}$, contradicting the assumption that $C_{\phi_k} \cap U = \emptyset$ for each $k \in \mathbb{N}$. □

Claim 3. Let U be an open subset of S satisfying the assumption that each coordinate of each point of U has a nonzero imaginary part. Given an open ball $B(w,\epsilon) \subset G := \{\theta \in [0,\pi)^n : C_\theta \cap U \neq \emptyset\}$, the set

$$A := U \cap \bigcup_{\theta \in B(w,\epsilon)} C_\theta$$

is open (and nonempty) in U.

Proof. It is clear that $A \neq \emptyset$. To show that A is open, we will show that for fixed $m \in A$, $\exists \epsilon''' > 0$ such that $(B(m,\epsilon''') \cap S) \subset A$. Let $m \in A$. Write $m = (r_1 e^{i\phi_1}, \ldots, r_n e^{i\phi_n})$, where $\phi_j \in (0,\pi)$ for each $j = 1, \ldots, n$ and where $r_1^2 + \ldots + r_n^2 = 1$ with none of the r_j's equal to zero (this assures that m is written *uniquely* as such). Note that $\phi = (\phi_1, \ldots, \phi_n)$ is in $B(w, \epsilon)$. Let $\epsilon' > 0$ be given such that $B(\phi, \epsilon') \subset B(w, \epsilon)$. Note that since $\epsilon \in (0, \pi/2)$, we also have that $\epsilon' \in (0, \pi/2)$. Let

$$\epsilon'' = \min_{j \in \{1,\ldots,n\}} r_j \sin(\epsilon'/\sqrt{n}),$$

and let

$$\begin{aligned}\epsilon''' &= \min\{\epsilon'', \operatorname{dist}(m, S \setminus U), \operatorname{Im}(r_1 e^{i\phi_1}), \ldots, \operatorname{Im}(r_n e^{i\phi_n})\} \\ &= \min\{\epsilon'', \operatorname{dist}(m, S \setminus U), r_1 \sin(\phi_1), \ldots, r_n \sin(\phi_n)\}.\end{aligned}$$

Suppose $y \in S$ such that $||m - y||_{\mathbb{C}^n} < \epsilon'''$ (i.e., $y \in B(m, \epsilon''')$). Write $y = (t_1 e^{i\psi_1}, \ldots, t_n e^{i\psi_n})$, where $\psi_j \in [0, \pi)$ for each $j = 1, \ldots, n$ and where $t_1^2 + \ldots + t_n^2 = 1$. Since $||m - y||_{\mathbb{C}^n} < \epsilon'''$, then $||m - y||_{\mathbb{C}^n} < \epsilon''$. Thus, for each $j = 1, \ldots, n$,

$$\left| r_j e^{i\phi_j} - t_j e^{i\psi_j} \right| \leq ||m - y||_{\mathbb{C}^n} < \epsilon''' \leq \epsilon'' \leq r_j \sin(\epsilon'/\sqrt{n}).$$

But also, since $\epsilon''' \leq r_j \sin(\phi_j)$ for each fixed j, we have

$$\left| r_j e^{i\phi_j} - t_j e^{i\psi_j} \right| \leq ||m - y||_{\mathbb{C}^n} < \epsilon''' \leq r_j \sin(\phi_j).$$

So the point $t_j e^{i\psi_j}$ is a point in the disc centered at $r_j e^{i\phi_j}$ with radius

$$r_j \cdot \min\{\sin(\epsilon'/\sqrt{n}), \sin(\phi_j)\}.$$

We claim that this implies that

$$|\phi_j - \psi_j| < \frac{\epsilon'}{\sqrt{n}}.$$

To see this, it suffices to show that if $se^{i\omega}$ (where $\omega \in [0, \pi)$) is *any* point in the closure of this disc, then

$$|\phi_j - \omega| < \frac{\epsilon'}{\sqrt{n}}.$$

So suppose $se^{i\omega}$ is a point in this closed disc that maximizes $|\phi_j - \omega|$. (There are, of course, two such points.) Let A be the origin, B be the point $se^{i\omega}$, and C be the point $r_j e^{i\phi_j}$. Then the angle at B in the triangle ABC is a right angle. The angle at A is $|\phi_j - \omega|$. (Here we have implicitly used our assumption that $\left| r_j e^{i\phi_j} - se^{i\omega} \right| \leq r_j \sin(\phi_j)$. That is, in order to avoid the problem with the argument function that occurs on the real axis, we have assumed that $r_j e^{i\phi_j}$ and $se^{i\omega}$ either both lie in the upper half-plane or both lie in the lower half-plane.) So $\sin(|\phi_j - \omega|) = \min\{\sin(\epsilon'/\sqrt{n}), \sin(\phi_j)\} \leq \sin(\epsilon'/\sqrt{n})$. But since $\epsilon'/\sqrt{n} \in (0, \pi/2)$ and since the sin function is increasing on $(0, \pi/2)$, we have that

$$|\phi_j - \omega| < \frac{\epsilon'}{\sqrt{n}}.$$

We have thus shown that

$$|\phi_j - \psi_j| < \frac{\epsilon'}{\sqrt{n}}$$

for each $j = 1, \ldots, n$. Hence

$$\begin{aligned}||\phi - \psi||_{\mathbb{R}^n} &= \left(|\phi_1 - \psi_1|^2 + \ldots + |\phi_n - \psi_n|^2 \right)^{1/2} \\ &< \left(\frac{n\epsilon'^2}{n} \right)^{1/2} \\ &= \epsilon'.\end{aligned}$$

So $\psi \in B(\phi, \epsilon') \subset B(w, \epsilon)$. But $y \in C_\psi$, so $y \in \bigcup_{\theta \in B(w,\epsilon)} C_\theta$. Also, $y \in U$ (since $||m - y||_{\mathbb{C}^n} < \epsilon''' \leq \operatorname{dist}(m, S \setminus U)$). Therefore, $y \in U \cap \bigcup_{\theta \in B(w,\epsilon)} C_\theta = A$. Thus, A is open in U. $\square$

Claim 4. Given the set $U \subset S$ (satisfying the assumption that each coordinate of each point of U has a nonzero imaginary part) and given $\phi, \psi \in [0, \pi)^n$ such that $U \cap C_\phi \cap C_\psi \neq \emptyset$, then $\phi = \psi$.

Proof. Suppose $U \cap C_\phi \cap C_\psi \neq \emptyset$, and let $z \in U \cap C_\phi \cap C_\psi$. Write $z = (z_1, \ldots, z_n) = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$ where $\theta_j \in (0, \pi)$ and $r_j \in (-1, 1)$. None of the r_j's are zero, by the assumption on U. Since $z \in C_\phi$, then

$$1 = \sum_{j=1}^{n} (e^{-i\phi_j} z_j)^2.$$

So

$$1 = \sum_{j=1}^{n} r_j^2 e^{2i(\theta_j - \phi_j)}.$$

But since $\sum_{j=1}^{n} r_j^2 = 1$ with none of the r_j's equal to zero, and since $\theta_j, \phi_j \in [0, \pi)$, we have that $\theta_j = \phi_j$ for each $j = 1, \ldots, n$. Similarly, $\theta_j = \psi_j$ for each $j = 1, \ldots, n$. Thus $\phi = \psi$. $\square$

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